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Well-Dominated and Approximately Well-Dominated Graphs

D. K. Thakkar[1], B. M. Kakrecha[2] and G. J. Vala[2]

[1]Department of Mathematics, Saurashtra University, Rajkot – 360 005.

[2]General Department, Government Engineering College, Rajkot – 360 005.

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ABSTRACT

We prove that the upper domination number of a graph does not increase when a vertex is removed from the graph. Moreover, we consider well-dominated graphs and their upper domination number. We also define approximately well-dominated graphs.

Introduction

A subset S of the vertex set V(G) of the graph G is said to be a dominating set if for every vertex v in V(G) – S, v is adjacent to some vertex in S. A dominating set with minimum cardinality is called a minimum dominating set and it is also called a γ –set of G. The cardinality of a minimum dominating set is called the domination number of the graph G and it is denoted as γ (G).

A dominating set S is said to be a minimal dominating set if for every vertex v in S, S – {v} is not a dominating set. A minimal dominating set with maximum cardinality is called an upper dominating set and it is denoted as Γ–set of G. The cardinality of an upper dominating set is called the upper domination number of the graph G and it is denoted as Γ(G). Obviously, $\gamma(G) \le \Gamma(G)$ for any graph G.

The above concept can be found in [3] and [4].

If we remove a vertex v from the graph G, the domination number of the new graph G – {v} may increase decrease or remains same. In [4] the necessary and sufficient conditions under which the domination number increases or decreases have been proved.

In this paper, we proved a necessary and sufficient condition under which the upper domination number of a graph does not change when a vertex is removed from the graph. Infect we prove that the upper domination number never increases when a vertex is removed from the graph. We also consider well-dominated graphs which have been introduced by [2].

Preliminaries: If G is a graph, V(G) will denote the vertex set of the graph G and E(G) will denote the edge set of the graph G. If $v \in V(G)$ then G – {v} will denote the graph obtained by removing the vertex v and all edges incident to v. The following sets will be useful [4].

$V^+ = \{ v \in V(G) \mid \gamma(G - \{v\}) > \gamma(G) \}$

$V^- = \{ v \in V(G) \mid \gamma(G - \{v\}) < \gamma(G) \}$

$V^0 = \{ v \in V(G) \mid \gamma(G - \{v\}) = \gamma(G) \}$

We will consider only simple graphs with finite vertex sets.

Definition 1.1: [4] Let S be a subset of V(G) and $v \in S$ then the private neighborhood of v with respect to the set S is the set

$pn[v, S] = \{ w \in V(G) \mid N[w] \cap S = \{v\} \}$

The following result is well-known and its proof can be found in [4].

Theorem 1.2: A dominating subset S of V(G) is a minimal dominating set of G if and only if $pn[v, S] \neq \emptyset$ for every vertex v in S. Equivalently, for every vertex v in S, one of the following conditions satisfied :

(1) v is not adjacent to any vertex of S.

(2) there is a vertex w in V(G) – S such that $N[w] \cap S = \{v\}$.

Vertex Removal and Upper Domination Number

When a vertex v is removed from the graph G, the upper domination number of G – {v} may remain same or it may change. First we prove that this number cannot increase.

Theorem 2.1: Let G be a graph and $v \in V(G)$ then $\Gamma(G - \{v\}) \le \Gamma(G)$.

Proof: Let S be a Γ–set of G – {v}. If v is adjacent to some vertex of S then S is a minimal dominating set of G also, and hence $\Gamma(G) \ge |S| = \Gamma(G - \{v\})$. Thus $\Gamma(G - \{v\}) \le \Gamma(G)$.

If v is not adjacent to any vertex of S then $S_1 = S \cup \{v\}$ is a minimal dominating set in G, therefore $\Gamma(G) = |S_1| > |S| = \Gamma(G - \{v\})$. Thus $\Gamma(G - \{v\}) < \Gamma(G)$. This proves the theorem.

Theorem 2.2: Let G be a graph and $v \in V(G)$ then $\Gamma(G - \{v\}) = \Gamma(G)$ if and only if there is a Γ–set S of G not containing v such that one of the following two conditions hold:

(1) v is adjacent to at least two vertices of S.

(2) there is a vertex w in S such that $pn[w, S]$ contains at least two vertices including v.

Proof: Suppose $\Gamma(G - \{v\}) = \Gamma(G)$. Let S be any Γ–set of G – {v}. If v is not adjacent to any vertex of S then $S \cup \{v\}$ is a minimal dominating set of G and therefore $\Gamma(G) \ge |S \cup \{v\}| > |S| = \Gamma(G - \{v\})$. Thus $\Gamma(G - \{v\}) < \Gamma(G)$. This contradicts our hypothesis. Thus S is a dominating set in G and therefore Γ–set of G not containing v.

Tele:
E-mail addresses: kakrecha.bhavesh2011@gmail.com

If v is adjacent to at least two vertices of S then condition (1) holds.

If v is adjacent to exactly one vertex w of S then since S is a minimal dominating set of G – {v}, there is a vertex u of G – {v} such that u ∊ *pn*[w, S]. Thus *pn*[w, S] contains u and v. Hence condition (2) holds.

Conversely, suppose there is a Γ–set S of G not containing v such that condition (1) or condition (2) holds.

If condition (1) holds then since S is a minimal dominating set in G, for every vertex w in S, *pn*[w, S] contains a vertex x of G. This vertex x cannot be equal to v because v is adjacent to at least two vertices of S. Thus private neighborhood of each vertex w of S contains a vertex of G – {v}. So S is a minimal dominating set in G – {v}. Therefore Γ(G – {v}) ≥ |S| = Γ(G). But Γ(G – {v}) > Γ(G) is impossible and hence Γ(G – {v}) = Γ(G).

Suppose condition (2) holds, then *pn*[w, S] contains a vertex of G – {V}, also for any other vertex u of S, v cannot be in *pn*[u, S], therefore *pn*[u, S] contains a vertex of G – {v}. Therefore S is a minimal dominating set of G – {v}, and therefore again by similar argument Γ(G) = Γ(G – {v}). This proves the theorem.

We introduce the following two notations.

$W^- = \{ v \in V(G) \mid \Gamma(G - \{v\}) < \Gamma(G) \}$

$W^0 = \{ v \in V(G) \mid \Gamma(G - \{v\}) = \Gamma(G) \}$

Well-Dominated graphs

The concept of a well-dominated graph was introduced by [2], one can refer to [5] also.

Definition 3.1: [2] A graph G is said to be well-dominated if all minimal dominating sets have the same cardinality, equivalently, γ (G) = Γ(G).

We now prove the following theorem.

Theorem 3.2: Let G be a well-dominated graph

(1) For this graph G, V^+ is an empty set.

(2) If v ∊ V^0, then G – {v} is a well-dominated graph.

(3) If v ∊ V^0 then v ∊ W^0, that is V^0 ∊ W^0.

Proof: (1) If v ∊ V^+, then Γ(G) = γ (G) < γ (G – {v}) ≤ Γ(G – {v}).

Thus Γ(G) < Γ(G – {v}). Which is not true by Theorem 2.1, therefore V^+ is an empty set.

(2) γ (G) = γ (G – {v}) ≤ Γ(G – {v}) ≤ Γ(G).

Since γ (G) = Γ(G), we have γ (G – {v}) = Γ(G – {v}). Thus G – {v} is a well-dominated graph.

(3) It is clear from (2) that γ (G) = γ (G – {v}) = Γ(G – {v}) = Γ(G).

Theorem 3.3: Let G be a well-dominated graph and v ∈ V(G) such that v ∊ V^- then either v ∊ W^0 or G – {v} is well-dominated.

Proof: Since G is a well-dominated graph, γ (G) = Γ(G) also from v ∊ V^-, γ (G – {v}) < γ (G). Note that γ (G – {v}) = γ (G) – 1.

Thus γ (G) – 1= γ (G – {v}) ≤ Γ(G – {v}) ≤ Γ(G) = γ (G).

Therefore either Γ(G – {v}) = γ (G – {v}), then G – {v} is well-dominated or Γ(G – {v}) = Γ(G) then v ∊ W^0.

Now we define a new concept called approximately well-dominated graph.

Definition 3.4: A graph G is said to be an approximately well-dominated graph, if Γ(G) = γ (G) + 1. It follows that in an approximately well-dominated graph, every minimal dominating set is either a γ –set or Γ–set.

Example 3.5: In the cycle graph C_6 with vertices 1, 2, 3, 4, 5, 6, γ (C_6) = 2 and Γ(C_6) = 3.

Corollary 3.6: If G is a well-dominated graph and v ∊ V^- then either G – {v} is a well-dominated graph or G – {v} is an approximately well-dominated graph.

Proof: Note that γ (G – {v}) = γ (G) – 1. Since G is a well-dominated graph and v ∊ V^-, γ (G) = Γ(G) and γ (G – {v}) ≤ γ (G).

Therefore γ (G) – 1 = γ (G – {v}) ≤ Γ(G – {v}) ≤ Γ(G) = γ (G).

Thus if Γ(G – {v}) = γ (G – {v}) + 1, then G – {v} is an approximately well-dominated graph and if Γ(G – {v}) = γ (G – {v}) then G – {v} is well-dominated graph.

Theorem 3.7: Suppose G is an approximately well-dominated graph and v is any vertex of G,

(1) If v ∊ V^+, then G – {v} is well-dominated graph.

(2) v ∊ W^0 if and only if Γ(G – {v}) = γ (G) + 1.

Proof: Since G is an approximately well-dominated graph, Γ(G) = γ (G) + 1.

(1) If v ∊ V^+, then γ (G) < γ (G – {v}),

that is, γ (G) < γ (G – {v}) ≤ Γ(G – {v}) ≤ Γ(G) = γ (G) + 1 . . . (1).

Therefore γ (G – {v}) = Γ(G – {v}). Thus G – {v} is well-dominated graph.

(2) v ∊ W^0 ⇔ Γ(G – {v}) = Γ(G)

⇔ Γ(G – {v}) = γ (G) + 1 (since inequality (1)).

Theorem 3.8: Suppose G is an approximately well-dominated graph. v is a vertex such that v ∊ V^0, then G – {v} is either an approximately well-dominated graph or it is a well-dominated graph.

Proof: Since G is an approximately well-dominated graph, Γ(G) = γ (G) + 1, also v ∊ V^0, we have γ (G – {v}) = γ (G).

We know that from Theorem 2.1, either v ∈ W^0 or v ∈ W^-.

Case 1: If v ∊ W^0 then Γ(G – {v}) = Γ(G), then from hypothesis, Γ(G – {v}) = Γ(G) = γ (G) + 1 = γ (G – {v}) + 1.

That is, Γ(G – {v}) = γ (G – {v}) + 1.

Thus G – {v} is an approximately well-dominated graph.

Case 2: If v ∊ W^- then Γ(G – {v}) < Γ(G), therefore from hypothesis γ (G – {v}) ≤ Γ(G – {v}) < Γ(G) = γ (G) + 1 = γ (G – {v}) + 1.

That is, Γ(G – {v}) = γ (G – {v}).

Thus G – {v} is a well-dominated graph.

Theorem 3.9: Let G be an approximately well-dominated graph and v is a vertex such that v ∊ V^-, then exactly one of the following three possibilities hold :

(1) G – {v} is well-dominated graph.

(2) G – {v} is an approximately well-dominated graph.

(3) v ∊ W^0.

Proof: Suppose G is an approximately well-dominated graph, that is Γ(G) = γ (G) + 1, and v is a vertex such that v ∊ V^-, that is γ (G – {v}) < γ (G), also note that γ (G – {v}) = γ (G) – 1.

Thus γ (G) – 1 = γ (G – {v}) ≤ Γ(G – {v}) ≤ Γ(G) = γ (G) + 1 . . . (1).

Now from inequality (1), if Γ(G – {v}) = γ (G – {v}) then G – {v} is well-dominated graph, otherwise if Γ(G – {v}) ≥ γ (G – {v}), that is Γ(G – {v}) = γ (G – {v}) + 1, then G – {v} is an approximately well-dominated graph, otherwise if $\Gamma(G - \{v\}) = \Gamma(G)$, then $v \in W^0$.

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