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Domatic number of just excellent subdivision graphs

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ABSTRACT

A set of vertices D in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set. A graph G is said to be Just excellent (JE), if it to each u ∈ V, there is a unique γ - set of G containing u. In this paper we have obtained the domatic number of the subdivision graph of a just excellent graph.

Introduction

We consider only simple connected undirected graphs G = (V, E). The subgraph of G induced by the vertices in D is denoted by < D>. C_n, K_n denotes the cycle and complete graph with, n vertices respectively, and Pn is a path of length n. The open neighborhood of vertex v ∈ V (G) is denoted by N (v) = { u ∈ V (G) | (u v) ∈ E (G) } while its closed neighborhood is the set N [v] = N (v) ∪ { v }. The private neighborhood of v ∈ D is denoted by pn [v, D], is defined by pn [v, D] = N (v) – N (D – { v }). We indicate that u is adjacent to v by writing u ⊥ v.

Let S be a given set and A = { A_1, A_2, …, A_m }. If each A_i, i = 1, 2, …, m is a subset of S, $A_1 \cup A_2 \cup \ldots \cup A_n = S$, $A_i \cap A_j = \phi$ i ≠ j, i, j = 1, 2, …, m, then the set A is called a partition of S. A subdivision of a graph G is a graph resulting from the subdivision of edges in G. The subdivision of some edge e with endpoints { u, v } yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, { u, w } and { w, v }.We shall denote the graph obtained by subdividing any edge uv of a graph G, by G_{sd} uv. Let w be a vertex introduced by subdividing uv. We shall denote this by G_{sd} uv = w.

A set of vertices D in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set — abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and it is denoted by γ (G). A γ - set denotes a dominating set for G with minimum cardinality. A vertex v is said to be a, down vertex if γ (G – u) < γ (G), level vertex if γ (G – u) = γ (G), up vertex if γ (G – u) > γ (G). A vertex in V – D is k – dominated if it is dominated by at least k – vertices in D.

The concept of the domatic number was introduced by Cockayne and Hedetniemi in 1977. A domatic partition of a graph G = (V, E) is a partition of V into disjoint sets V_1, V_2, ..., V_K such that each V_i is a dominating set for G. The domatic number is the maximum number of such disjoint sets and it is denoted by d (G).

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A vertex v is said to be good if there is a γ - set of G containing v. A graph G is said to be excellent if every vertex of G is good. In [6] M. Yamuna and N. Sridharan, had defined a graph G to be Just excellent (JE), if it to each u∈V, there is a unique γ - set of G containing u.

Example

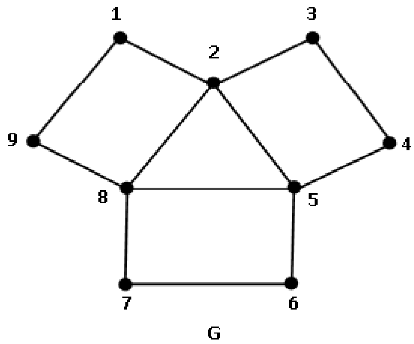

Fig. 1

In Fig. 1, G is a JE graph and { 1, 4, 7 }, { 2, 5, 8 } and { 3, 6, 9 } are the distinct γ-sets for G.

In [6], they have proved the following results

1. A graph G is JE if and only if,
   a. γ (G) divides n.
   b. d (G) = n / γ (G), where d (G) denotes the domatic partition of G.
   c. G has exactly n / γ (G) distinct γ - sets.

2.If G ≠ $\overline{K_n}$ is JE then | PN (u, D) | ≥ 2 for each vertex u ∈ D, where D is any γ - set of G.

If the condition (2) is not satisfied, then the graph is not JE.

In [4] the following result has been proved.

3. A JE graph has no down vertex.

In this paper we obtain the domatic number of subdivision graph of a just excellent graph. The domatic number of subdivision graph is denoted by d (G_{sd} uv), for all u, v ∈ V(G), u ⊥ v.

We already have proved the following result,

4. Let G be any graph. Then d (G_{sd} uv) ≤ 3.

Theorem 1

Let G be a JE graph such that d (G) ≤ 3. Then G has no 2 – dominated vertex.

Proof

Since G is JE, we know that,

i. γ (G) divides n.

ii. d (G) = n / γ (G).

iii. G has exactly n / γ (G) distinct γ - sets.

iv. | pn [u, D] | ≥ 2, for all u ∈ D.

Case i d (G) = 3

Subcase i

If γ (G) = 2, then by condition (ii), n = 6.

By condition (iii), d (G) = | { V_1 }, { V_2 }, { V_3 } |, | V_i | = 2, i = 1, 2, 3 and V_i , i = 1, 2, 3 are γ - sets. Let V_i = { x_i, y_i }, i = 1, 2, 3.

By condition (iv), pn [x_i] ≥ 2 and pn [y_i] ≥ 2, i = 1, 2, 3.

| V – { V_i } | = 4, i = 1, 2, 3, which implies every element in V – D is private neighborhood of some elements in V_i. In this case, G has no 2 – dominated vertex.

Subcase ii

If γ (G) = 3, then by condition (ii), n = 9.

As in subcase – i , d (G) = | { V_1 }, { V_2 }, { V_3 } |, where | V_i | = 3, i = 1, 2, 3 and V_i, i = 1, 2, 3 are γ - sets. Let V_i = { x_i, y_i, z_i }, i = 1, 2, 3.

By condition (iv), pn [x_i] ≥ 2, pn [y_i] ≥ 2 and pn [z_i] ≥ 2, i = 1, 2, 3.

| V – { V_i } | = 6, i = 1, 2, 3, which implies every element in V – D is private neighborhood of some elements in V_i. In this case, G has not 2 – dominated vertex.

By subcase – i and ii, we observe that, when d (G) = 3, as the value of γ is increased by 1, the value of n increases by 3. Also | V – V_i | = 2 | V_i |, i = 1, 2, 3, which implies, if G is JE such that d (G) = 3, then G has no 2 – dominated vertex.

Case ii d (G) = 2

Subcase i

If γ (G) = 2, then by condition (ii), n = 4.

By condition (iii), d (G) = | { V_1 }, { V_2 } |, | V_i | = 2, i = 1, 2.

By condition (iv), pn[u, D] ≥ 2, for any u ∈ D. Also if this condition is not satisfied, then G is not JE. | V – V_i | = 2, which implies pn [u, D] ≥ 2, for every u ∈ D is not possible.

If d (G) = γ (G) = 2, then G is not JE.

Subcase ii

If γ (G) = 3, then by condition (ii), n = 6.

By condition (iii), d (G) = | { V_1 }, { V_2 } |, | V_i | = 3, i = 1, 2. Since | V_1 | = | V_2 |, each u ∈ V_i can have atmost one private neighborhood, that is pn [u, D] < 2, for each vertex u ∈ D.

If d (G) = 2 and γ (G) = 3, then G is not JE.

From the above discussion, in case – ii, we observe that if d (G) = 2, as the value of γ is increased by 1, the value of n increases by 2. Also | V – V_i | = | V_i |, i = 1, 2. So when d (G) = 2, it is not possible that pn [u, D] ≥ 2. This implies that, if d (G) = 2, then G is not JE. Also for any graph G, d (G) ≥ 2. [Since d (G) = | V, V – D | is always possible for any γ - set D for G].

Hence we conclude that, if G is JE and d (G) ≤ 3, then G has no 2 – dominated vertex. □

Corollary 1

If G is JE, d (G) ≤ 3, then every vertex in V – D is a private neighborhood.

Remark

If G is JE, d (G) > 3, then it is possible that G has a 2 – dominated vertex.

Example

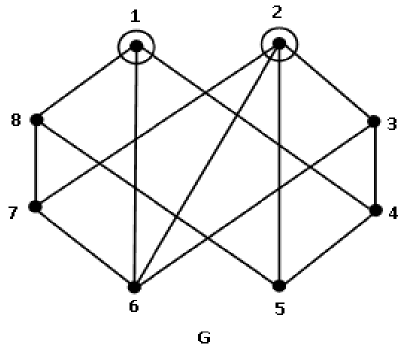

Fig. 2

In Fig. 2, G is a JE graph and d (G) = | { 1, 2 }, { 3, 8 }, { 4, 7 }, { 5, 6 } | = 4. Vertex 6 is 2 dominated vertex, with respect to { 1, 2 } as shown in the figure.

Corollary 2

There is no JE graph G such that d (G) = 2.

Proof

If possible assume that d (G) = 2, where G is JE.

If γ (G) = 2, then G is not JE as in subcase i of case ii.

If γ (G) = m, m ≥ 3, then n = 2m. Let d (G) = | { V_1 }, { V_2 } |, | V_i | = m, i = 1, 2. Since | V_1 | = | V_2 |, each u ∈ V_i can have atmost one private neighborhood as in subcase ii of case ii, that is pn[u, D] < 2, for each vertex u ∈ D, which implies G is not JE.

In general, there is no JE graph G such that d (G) = 2.

Theorem 2

In a JE graph every vertex is a level vertex.

Proof

Let G be a JE graph and let u ∈ V (G).

Case i u is not a down vertex

By result [3], we know that a JE graph does not contain a down vertex.

Case ii u is not an up vertex

Claim

If u is an up vertex for a graph G, then u must be included in every possible γ - set.

Proof

If there is a γ - set for G not containing u, then D is a dominating set for G – { u } also, that is γ (G – u) = γ (G), which is a contradiction as u is an up vertex [If u is an up vertex, then γ (G – u) > γ (G)]. This implies that, an up vertex must be included in every possible γ - set.

By the above claim, if u is an up vertex, then u must be included in every possible γ - set, a contradiction as G is JE.

Hence, from case – i and ii, we conclude that, in a JE graph every vertex is a level vertex. □

In [5], M. Yamuna and K. Karthika introduced the concept of domination subdivision stable graph. A graph G is said to be domination subdivision stable (DSS), if the γ - value of G does not change by subdividing any edge of G.

Example

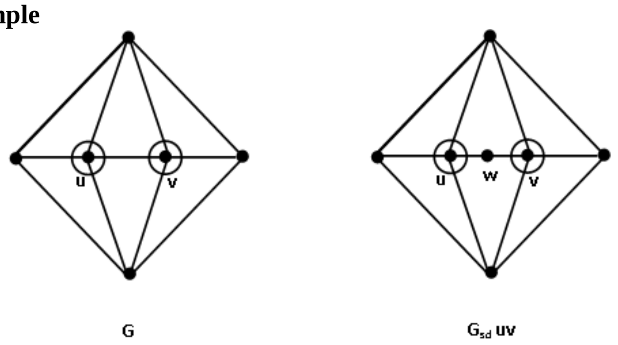

Fig. 3

In Fig. 3, γ (G) = γ (G_{sd} uv) = 2. This is true for each e = (a b)$\in$ E (G), which implies G is a DSS graph.

Theorem 3

Let G be a graph such that d (G) = | $V_1, V_2, \ldots, V_m$ |, where m $\geq$ 3 and | V_i | = γ (G), for all i = 1, 2, …, m. If G is not DSS, then d (G_{sd} uv) < m.

Proof

Let G be a graph and d (G) = | $V_1, V_2, \ldots, V_m$ | such that | V_i | = γ (G), for all i = 1, 2, …, m. In this case n = m γ (G). If G is not DSS, then there is atleast one u, v $\in$ V (G), u $\perp$ v such that γ (G_{sd} uv) = γ (G) + 1, that is

i.| V (G_{sd} uv) | = | V (G) | + 1.

ii.| γ (G_{sd} uv) | = γ (G) + 1.

If d (G_{sd} uv) = m, then | V (G_{sd} uv) | $\geq$ m (γ (G) + 1),

$\Rightarrow$ | V (G_{sd} uv) | $\geq$ m γ (G) + m.

$\Rightarrow$ | V (G_{sd} uv) | $\geq$ | V (G) | + m.

But by condition (i), we know that | V (G_{sd} uv) | $\neq$ | V (G) | + m, which implies d (G_{sd} uv) $\neq$ m. Hence d (G_{sd} uv) < m.

Theorem 4

If G is just excellent, then G is not DSS.

Proof

Let G be just excellent. Let D be a γ - set for G. Let u $\in$ D and let v$\in$ pn [u, D]. There is one such v, since by result [2], pn [u, D] $\geq$ 2 for each vertex u $\in$ D for a JE graph G. Let us assume that G is DSS. D does not dominates G_{sd} uv, since v is not dominated by D. If G_{sd} uv is DSS, then there is one γ - set D′ for G_{sd} uv, such that | D′ | = | D |.

Case 1 u$\in$ D′, w, v $\notin$ D′

D is a γ - set for G, where v $\in$ pn [u, D]. D′ is a γ - set for G, where v $\notin$ pn [u, D′], that is D and D′ are two distinct γ - set for G containing u, which is a contradiction.

Case 2 w $\in$ D′ and u, v $\notin$ D′

D″ = D′ – { w } $\cup$ { v } is a γ - set for G.

1. If pn [w, D′] = ϕ, then pn [v, D″] = ϕ.
2. If pn [w, D′] = 1 = v, then pn [v, D″] = ϕ.
3. If pn [w, D′] = 2, then pn [v, D″] = 1.

In all cases, we get a contradiction, by result [2], we know that, since G is JE graph, then pn [u, D] $\geq$ 2, for all u $\in$ V (G).

Case 3 v $\in$ D′, u, w $\notin$ D′

D″ and D′ are two distinct γ – sets for G containing v, which is a contradiction as G is JE.

Case 4 u, w $\in$ D′, v $\notin$ D′

D‴ = D′ – { w } $\cup$ { v } is a γ – set for G containing u. D and D‴ are two distinct γ - sets for G containing u, which is a contradiction as G is JE.

Case 5 v, w $\in$ D′, u $\notin$ D′

D^{iv} = D′ – { w } $\cup$ { u } is a γ – set for G. D and D^{iv} are two distinct γ - sets for G containing u, which is a contradiction as G is JE.

Case 6 u, v $\in$ D′, w $\notin$ D′

D′ is a γ – set for G also that is D and D′ are two distinct γ - sets for a containing u, which is a contradiction as G is JE.

In all cases we get a contradiction and hence G is not DSS.

Conclusion

By corollary [2] of theorem [1], we already know that there is no JE graph such that d (G) = 2. With this, and from the results so far proved we conclude the following about the domatic number of the subdivision of a just excellent graph.

Theorem 5

If G is JE, then d (G_{sd} uv) = 2, if d (G) = 3 and d (G_{sd} uv) $\leq$ 3, if d (G) $\geq$ 4.

Proof

If G is JE such that d (G) = 3. Let d (G) = | { V_1 }, { V_2 }, { V_3 } |, then by theorem [1], we know that,

i. G has no 2 – dominated vertex. Also since G is JE,

ii. | V_i | = γ (G).

iii. n = 3 γ (G).

iv. G is not DSS, by theorem [4].

By theorem [3], we conclude that d (G_{sd} uv) $\neq$ 3. Hence d (G_{sd} uv) = 2.

If d (G) $\geq$ 4, then by result [4], d (G_{sd} uv) $\leq$ 3.

References

1. Haynes, T. W., Hedetniemi, S. T., Slater, P. J, Fundamentals of Domination in Graphs, *Marcel Dekker*, New York, 1998.
2. Karthika, K, Domination Dot Stable – domatic dot stable – domination Subdivision Stable graphs, *M. Phil thesis*, VIT University, Vellore, India. (2011)
3. West, D. B, Introduction to Graph Theory, second ed., *Prentice-Hall,* Englewood Cliffs, NJ, 2001.
4. Yamuna, M., Karthika, K, Excellent – Domination Dot Stable Graphs, *International Journal of Engineering Science, Advanced Computing and Bio – Technology* Vol. 2, No. 4, pp. 209 – 216, 2011.
5. Yamuna, M., Karthika, K, Domination Subdivision Stable Graphs, *International Journal of Mathematical Archive* - 3(4), pp. 1467 – 1471, 2012.
6. Yamuna, M., Sridharan, N, Just Excellent Graphs, *International Journal of Engineering Science, Advance Computing and Bio – Technology* Vol. 1, No. 3, pp. 129 – 136, 2010.