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**Applied Mathematics** 



# On Identification of Distribution of Three Independent Markov Chains Subject to the Reliability Criterion

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#### ARTICLE INFO

Article history: Received: 6 October 2012; Received in revised form: 3 January 2013; Accepted: 15 January 2013;

## ABSTRACT

In this paper we solved the problem of identification of distribution one of many hypotheses for three independent objects by study of simple homogeneous stationary finite state of Markov chains.

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### Keywor ds

Markov chain, Identification, Reliability criterion.

#### Introduction

Ahlswede and Haroutunian (Ahlswede and Haroutunian, 2006) formulated an ensemble of problems on multiple hypotheses testing for many objects and on identification of hypotheses under reliability requirement.

The problem of many hypotheses testing on distributions of a finite state Markov chain is studied in (Yarmohammadi and Navaei, 2008) via large deviation techniques and also identification of distributions for one Markov chain is studied in (Haroutunian and Navaei, 2009). In this paper we solved the problem of identification of distribution of many hypotheses for three independent objects by study of simple homogeneous stationary finite state of Markov chains. We take known the definitions and results on many hypotheses Logarithmically Asymptotically Optimal (LAO) testing for the case of Markov chains and identification of distribution subject to the reliability criterion presented in (Yarmohammadi and Navaei, 2008), (Haroutunian, 1988), (Haroutunian and Navaei, 2009) that we introduce in continue.

Problem of LAO Identification of Distribution for Three Independent Markov Chains and Formulation of Results

Let 
$$x = (x_0, x_1, x_2, ..., x_N), \quad x_n \in \chi = \{1, 2, ..., I\},$$

 $x \in \chi^{I_{i+1}}, N = 0, 1, 2, \dots$  be vectors of observed states of simple homogeneous stationary Markov chain with finite number *I* of states. The *L* hypotheses concern the irreducible matrices of the transition probabilities

 $P_l = \{P_l(j \mid i), i, j = \overline{1, I}\}, l = 1, L.$ 

The stationarity of the chain provides existence for each  $l = \overline{1, L}$  of the unique stationary distributions  $Q_l = \{Q_l(i), i = \overline{1, I}\}, l = \overline{1, L}$ , such that

$$\sum_{i} Q_{i}(i) P_{i}(j \mid i) = Q_{i}(j), \qquad \sum_{i} Q_{i}(i) = 1, \quad i, j = \overline{1, I}.$$

We define the joint distributions  $Q_l o P_l$ ,  $Q_l o P_l = \{Q_l o P_l(i, j) = Q_l(i) P_l(j \mid i)\}$ ,  $i, j = \overline{1, I}$ ,  $l = \overline{1, L}$ .

The second order type of vector  $\mathcal{X}$  the square matrix of  $I^2$  relative frequencies of the simultaneous appearance on the pairs of neighbor places of the states i and j are  $\{N(i, j)N^{-1}, i, j = \overline{1, I}\}$ . It is clear that  $\sum_{i,j}^{N(i, j) = N} \dots$  Denote  $T_{QoP}^N$  the set of vectors from  $\mathcal{X}^{N+1}$  which have the type such that for some joint probability distribution QoP

 $N(i, j) = NQ(i)P(j \mid i), \quad i, j = 1, I.$ 

We shall use the following definition of the probability of the vector  $\overline{x} \in \chi^{N+1}$  of the Markov chain with transition probabilities  $P_l$  and stationary distribution  $Q_l$ ,

$$Q_l o P_l^N(\bar{x}) \equiv Q_l(x_0) \prod_{n=1}^N P_l(x_n \mid x_{n-1}), \quad l = \overline{1, L}$$
$$Q_l o P_l^N(A) \equiv \sum_{\bar{x} \in A} Q_l o P_l^N(\bar{x}), \quad A \subset \chi^{N+1}.$$

We expand the concept of identification for three independent stationary finite Markov chain. Let  $X_1, X_2$  and  $X_3$  be independent RV, taking values in the same finite state of Markov chain of set  $\mathcal{X}$  with one L, PD's, they are characteristics of corresponding independent objects. The random vector  $(X_1, X_2, X_3)$  assumes values  $(x^1, x^2, x^3) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$ .



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Let

$$\begin{aligned} & \left(x_1, x_2, x_3\right) = \left((x_0^1, x_0^2, x_0^3), \dots, (x_n^1, x_n^2, x_n^3), \dots, (x_N^1, x_N^2, x_N^3)\right), \\ & x^h \in \chi, \ h = 1, 2, 3, \ n = \{0, 2, \dots, N\}, \end{aligned}$$

be a sequence of results of N+1 independent observations of a simple homogeneses stationary Markov chain with finite number I of states. The statistic must define unknown PD's of objects on the basis of observed data. The selection for each object was done and it was denoted by  $\Phi_N$ . The objects independence test  $\Phi_N$  may be considered as the pair of tests  $\varphi_N^1$ ,  $\varphi_N^2$  and  $\varphi_N^3$  for the respective separate objects. We will show the whole compound test sequence by  $\Phi$ . The test  $\varphi_N^h$  is defined by a partition of the space  $\chi^{N+1}$  on the L sets and to every trajectory x the test  $\phi_N$  puts in one correspondence from L hypotheses. So the space  $\chi^{N+1}$  will be divided into L parts,  $g_{l,h}^N = \{x_i, \phi_N(x_i) = l\}, l = \overline{1, L}, h = 1, 2, 3$ . We define  $\alpha_{(l_1, 2, l_3)|(m_1, m_2, m_3)}(\Phi_N) = Q_{m_1} \circ P_{m_1}(g_{l_1, 1}^N)Q_{m_2} \circ P_{m_2}(g_{l_2, 2}^N) \circ Q_{m_3} \circ P_{m_3}(g_{l_3, 3}^N)$ , be the probability of the erroneous acceptance by the test

 $\Phi_{N \text{ of the hypotheses}} \begin{pmatrix} H_{l_1}, H_{l_2}, H_{l_3} \end{pmatrix}$  provided that  $(H_{m_1}, H_{m_2}, H_{m_3})$ 

where  $(m_1, m_2, m_3) \neq (l_1, l_2, l_3)$ ,  $m_h, l_h = \overline{1, L}, h = 1, 2, 3.$ 

The probability to reject true hypotheses  $(H_{m_1}, H_{m_2}, H_{m_3})$  is the following:

$$\alpha_{(m_1,m_2,m_3)|(m_1,m_2,m_3)}^{(N)}(\Phi_N) = \sum_{(l_1,l_2,l_3)\neq (m_1,m_2,m_3)} \alpha_{(l_1,l_2,l_3)|(m_1,m_2,m_3)}^{(N)}(\Phi_N).$$
(1)

We also study corresponding limits  $E_{(l_1,l_2,l_3)|(m_1,m_2,m_3)}(\Phi)$  of error probability exponents of the sequence of tests  $\Phi$ , called reliabilities:

$$E_{(l_1, l_2, l_3)|(m_1, m_2, m_3)}(\Phi) = \overline{\lim_{N \to \infty}} - \frac{1}{N} \log \alpha_{(l_1, l_2, l_3)|(m_1, m_2, m_3)}(\Phi),$$

$$m_i, l_i = \overline{1, L}, \ h = 1, 2, 3.$$
 (2)

We denote by  $E(\varphi)$  the reliability matrices of the sequences of test  $\varphi^h$ , h = 1, 2, 3, for each of the objects.

Applying (1) and (2), we obtain the following:

$$E_{(m_1,m_2,m_3)|(m_1,m_2,m_3)}(\Phi) = \min_{\substack{(l_1,l_2,l_3)\neq(m_1,m_2,m_3)}} E_{(l_1,l_2,l_3)|(m_1,m_2,m_3)}(\Phi)$$
(3)

In this section we use the following lemma.

$$E_{l|m}(\varphi^h), m, l = \overline{1, L}, h = 1, 2, 3$$

Lemma: If elements  $D_{l/m}(\varphi^{-1}, m, t^{-1}, t^{-1}, t^{-1}, t^{-2}, t^{-2})$  are strictly positive, then the following equalities hold for  $\Phi = (\varphi^{1}, \varphi^{2}, \varphi^{3})_{1}$ .

$$E_{(l_1, l_2, l_3)|(m_1, m_2, m_3)}(\Phi) = E_{l_1|m_1}(\phi^1) + E_{l_2|m_2}(\phi^2) + E_{l_3|m_3}(\phi^3), if \quad m_h \neq l_h, h = 1, 2, 3,$$
(4.a)

$$\begin{split} E_{(l_1, l_2, l_3)|(m_1, m_2, m_3)}(\Phi) &= \sum E_{l_h|m_h}(\varphi^h), \ if \ m_k = l_k \ m_h \neq l_h \ h \neq k, h, k = 1, 2, 3, \\ (4.b) \\ E_{(l_1, l_2, l_3)|(m_1, m_2, m_3)}(\Phi) &= E_{l_h|m_h}(\varphi^h), \ if \ m_k = l_k \ m_h \neq l_h \ h \neq k, k, h = 1, 2, 3. \\ (4.c) \end{split}$$

Consider for given positive elements  $E_{m,m,m|m,m,L}$  and  $E_{m,m,m|m,L,m}$ ,  $E_{m,m,m|L,m,m}$ ,  $m = \overline{1, L-1}$ , the family of regions:

$$\begin{split} R_m^{(1)} &\stackrel{\Delta}{=} \left\{ Q \circ P : D(Q \circ P \mid \mid Q \circ P_m) \leq E_{m,m,m|L,m,m} \right\} , \ m = \overline{1, L-1}, \\ R_m^{(2)} &\stackrel{\Delta}{=} \left\{ Q \circ P : D(Q \circ P \mid \mid Q \circ P_m) \leq E_{m,m,m|m,L,m} \right\} , \ m = \overline{1, L-1}, \\ R_m^{(3)} &\stackrel{\Delta}{=} \left\{ Q \circ P : D(Q \circ P \mid \mid Q \circ P_m) \leq E_{m,m,m|m,m,L} \right\} , \ m = \overline{1, L-1}, \\ R_L^{(1)} &\stackrel{\Delta}{=} \left\{ Q \circ P : D(Q \circ P \mid \mid Q \circ P_m) > E_{m,m,m|L,m,m}, m = \overline{1, L-1} \right\} , \\ R_L^{(2)} &\stackrel{\Delta}{=} \left\{ Q \circ P : D(Q \circ P \mid \mid Q \circ P_m) > E_{m,m,m|m,L,m}, m = \overline{1, L-1} \right\} , \\ R_L^{(3)} &\stackrel{\Delta}{=} \left\{ Q \circ P : D(Q \circ P \mid \mid Q \circ P_m) > E_{m,m,m|m,L,m}, m = \overline{1, L-1} \right\} . \end{split}$$

What is identification of the probability distributions for three independent objects? The answer for this question constitutes reply of the question whether or not the triple of distributions  $(r_1, r_2, r_3)_{\text{has occurred.}}$ 

 $\begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \text{There} \\ r_{h} = \overline{\mathbf{1}, L - 1}, \ h = \mathbf{1}, 2, 3, \ \text{the probabilities} \\ \end{array} & \text{for } each^{\left(r_{1}, r_{2}, r_{3}\right)}, \\ r_{h} = \overline{\mathbf{1}, L - 1}, \ h = \mathbf{1}, 2, 3, \ \text{the probability} \\ \end{array} \\ \begin{array}{c} \alpha_{(l_{1}, l_{2}, l_{3}) \neq (r_{1}, r_{2}, r_{3})}^{(N)} |_{(m_{1}, m_{2}, m_{3}) = (r_{1}, r_{2}, r_{3})} \\ \end{array} \\ \begin{array}{c} \text{to } accept \\ \left(l_{1}, l_{2}, l_{3}\right)_{\text{different from}} \\ \text{from } \left(r_{1}, r_{2}, r_{3}\right)_{\text{, when}} \\ \end{array} \\ \begin{array}{c} \text{to } accept \\ \left(r_{1}, r_{2}, r_{3}\right)_{\text{is in}} \\ \end{array} \\ \begin{array}{c} \alpha_{(N)}^{(N)} \\ \alpha_{(l_{1}, l_{2}, l_{3}) = (r_{1}, r_{2}, r_{3})} |_{(m_{1}, m_{2}, m_{3}) \neq (r_{1}, r_{2}, r_{3})} \\ \end{array} \\ \begin{array}{c} \text{the probability} \\ \end{array} \\ \begin{array}{c} \alpha_{(N)}^{(N)} \\ \end{array} \\ \begin{array}{c} \alpha_{(N)}^{(N)} \\ \alpha_{(l_{1}, l_{2}, l_{3}) = (r_{1}, r_{2}, r_{3})} |_{(m_{1}, m_{2}, m_{3}) \neq (r_{1}, r_{2}, r_{3})} \\ \end{array} \\ \begin{array}{c} \text{the probability} \\ \end{array} \\ \begin{array}{c} \alpha_{(N)} \\ \alpha_{(l_{1}, l_{2}, l_{3}) = (r_{1}, r_{2}, r_{3})} |_{(m_{1}, m_{2}, m_{3}) \neq (r_{1}, r_{2}, r_{3})} \\ \end{array} \\ \begin{array}{c} \text{the probability} \\ \end{array} \\ \begin{array}{c} \alpha_{(N)} \\ \alpha_{(l_{1}, l_{2}, l_{3}) = (r_{1}, r_{2}, r_{3})} |_{(m_{1}, m_{2}, m_{3}) \neq (r_{1}, r_{2}, r_{3})} \\ \end{array} \\ \begin{array}{c} \text{the probability} \\ \end{array} \end{array} \\ \begin{array}{c} \alpha_{(N)} \\ \alpha_{(l_{1}, l_{2}, l_{3}) = (r_{1}, r_{2}, r_{3})} |_{(m_{1}, m_{2}, m_{3}) \neq (r_{1}, r_{2}, r_{3})} \\ \end{array} \\ \begin{array}{c} \text{the probability} \\ \end{array} \end{array}$ 

when it is not correct.

The probability 
$$\alpha_{(l_1, l_2, l_3) \neq (r_1, r_2, r_3)}^{(l_1, l_2, l_3) \neq (r_1, r_2, r_3)} (m_1, m_2, m_3) = (r_1, r_2, r_3)$$
 is already  $\alpha^{(N)}$ .

known, it coincides with the probability  $(r_1, r_2, r_3)|(r_1, r_2, r_3)|(r_1, r_2, r_3)|$ Our aim is to determine the dependence of

$$\begin{array}{c} \chi^{(N)}_{(l_1, l_2, l_3) = (r_1, r_2, r_3)} |_{(m_1, m_2, m_3) \neq (r_1, r_2, r_3)} \text{ on given} \\ \alpha^{(N)}_{(r_1, r_2, r_3)} |_{(r_1, r_2, r_3)} \end{array}$$

We need to use the probabilities of different hypotheses. Let us that the hypotheses  $H_1: l = \overline{1,L}$ , have, say, probabilities  $P_r(r), r = \overline{1,L}$ . The only supposition we shall use is that  $P_r(r) > 0, r = \overline{1,L}$ . We demonstrate, the result formulated in the following theorem does not depend on values of  $P_r(r), r = \overline{1,L}$ , if they all are strictly positive. Thus, the following reasoning can be made for each  $r_h = \overline{1,L}, h = 1,2,3$ :

$$\alpha_{(l_1, l_2, l_3)=(r_1, r_2, r_3)}^{(N)} |_{(m_1, m_2, m_3) \neq (r_1, r_2, r_3)} = \frac{P_r^{(N)}((l_1, l_2, l_3) = (r_1, r_2, r_3), (m_1, m_2, m_3) \neq (r_1, r_2, r_3))}{P_r(m_1, m_2, m_3) \neq (r_1, r_2, r_3)}$$
for every LAO test  $\Phi$  from (3) and (4) we obtain the following :

For every LAO test 
$$\Psi$$
 from (5) and (4) we obtain the following :  

$$E_{(l_1, l_2, l_3) = (r_1, r_2, r_3) \mid (m_1, m_2, m_3) \neq (r_1, r_2, r_3)} = \min_{m_1 \neq r_1, m_2 \neq r_2} (E_{r_1 \mid m_1}^1, E_{r_2 \mid m_2}^2, E_{r_3 \mid m_3}^2).$$
(5)

 $E_{r_1|m_1}^1, E_{r_2|m_2}^2, E_{r_3|m_3}^2$  are determined in [6] for, where. corresponding, the first and second and third objects. For every LAO test  $\Phi$  from (3) and (4) we deduce that:

$$E_{(r_1, r_2, r_3)|(r_1, r_2, r_3)} = \min_{m_h \neq r_h, h=1, 2, 3} (E_{r_1|m_1}^1, E_{r_2|m_2}^2, E_{r_3|m_3}^2)$$
(6)

 $E_{r_1,r_1}^1, E_{r_2|r_2}^2, E_{r_3|r_3}^3$  satisfy the following and each conditions .

$$0 < E_{r_{1}|r_{1}}^{1} < \min\left[\min_{l=l,r_{1}-1} E_{l|l}^{*}(E_{l|l}^{1}), \min_{l=r_{1}+l,L} D(Q_{l} \circ P_{l} \| Q_{l} \circ P_{r_{1}})\right],$$

$$(7.a)$$

$$0 < E_{r_{2}|r_{2}}^{2} < \min\left[\min_{l=l,r_{2}-1} E_{l|m}^{*}(E_{l|l}^{2}), \min_{l=r_{2}+l,L} D(Q_{l} \circ P_{l} \| Q_{l} \circ P_{r_{2}})\right],$$

$$(7.b)$$

$$0 < E_{r_{3}|r_{3}}^{3} < \min\left[\min_{l=l,r_{3}-1} E_{l|m}^{*}(E_{l|l}^{2}), \min_{l=r_{3}+l,L} D(Q_{l} \circ P_{l} \| Q_{l} \circ P_{r_{3}})\right],$$

$$(7.c)$$

$$(7.c)$$

from (6), we know that the elements  $E_{l|m}^{*}(E_{l|l}^{1}), r_{1} = \overline{1, r_{1} - 1}, E_{l|m}^{*}(E_{l|l}^{1}), r_{3} = \overline{1, r_{3} - 1}$  are determined

by only  $E_{l|l}^{1}$ ,  $E_{l|l}^{2}$  and  $E_{l|l}^{3}$ . However, we are considering only elements  $E_{r_1|r_1}^1, E_{r_2|r_2}^2$  and  $E_{r_2|r_2}^2$ .

$$0 < E_{r_{1}|r_{1}}^{1} < \min\left[\min_{l=1,r_{1}-1} D(Q_{l} \circ P_{l} \| Q_{l} \circ P_{r_{1}}, \min_{l=r_{1}+1,L} D(Q_{l} \circ P_{l} \| Q_{l} \circ P_{r_{1}})\right],$$

$$(8.a)$$

$$0 < E_{r_2|r_2}^2 < \min\left[\min_{l=l,r_2-l} D(Q_l \circ P_l \| Q_l \circ P_{r_2}, \min_{l=r_2+l,L} D(Q_l \circ P_l \| Q_l \circ P_{r_2})\right],$$
(8.b)

$$0 < E_{r_3|r_3}^3 < \min\left[\min_{l=1,r_3-1} D(Q_l \circ P_l \| Q_l \circ P_{r_3}, \min_{l=r_3+1,L} D(Q_l \circ P_l \| Q_l \circ P_{r_3})\right],$$
(8.c)

 $r = \max(r_1, r_2, r_3)$  and  $k = \min(r_1, r_2, r_3)$ . let us denote From (6) we have that, when

$$\begin{split} E_{(r_1, r_2, r_3)} &= E_{r_1|r_1}^1 \\ & \text{then} \\ E_{r_1|r_1}^1 &\leq \min(E_{r_2|r_2}^2, E_{r_3|r_3}^2) \\ & \text{then} \\ E_{(r_1, r_2, r_3)} &= E_{r_2|r_2}^2 \\ & \text{then} \\ \min(E_{r_1|r_1}^1, E_{r_3|r_3}^1) &\geq E_{r_2|r_2}^2 \\ & \text{then} \\ E_{(r_1, r_2, r_3)} &= E_{r_3|r_3}^3 \\ & \text{when} \\ E_{(r_1, r_2, r_3)} &= E_{r_3|r_3}^3 \\ & \text{then} \\ & \min(E_{r_1|r_1}^1, E_{r_2|r_2}^1) &\geq E_{r_3|r_3}^2 \\ & \text{then} \\ & \min(E_{r_1|r_1}^1, E_{r_2|r_2}^1) &\geq E_{r_3|r_3}^2 \\ & \text{then} \\ & \min(E_{r_1|r_1}^1, E_{r_2|r_2}^1) &\geq E_{r_3|r_3}^2 \\ & \text{then} \\ & \min(E_{r_1|r_1}^1, E_{r_2|r_2}^1) &\geq E_{r_3|r_3}^2 \\ & \text{then} \\$$

thus ,it can be implied that given strictly positive elements

 $E_{(r_1,r_2,r_3)|(r_1,r_2,r_3)}$  must meet inequalities (8) and the combination of these restrictions gives

$$0 < E_{(r_1, r_2, r_3)|(r_1, r_2, r_3)} < \min\left[\min_{l=1, r-1} D(Q_l \circ P_l \| Q_l \circ P_r, \min_{l=r+1, L} D(Q_l \circ P_l \| Q_l \circ P_k)\right],$$
(9)
Using (9) we can determine

Using (9) we can deter  

$$E_{(l_1, l_2, l_3) = (r_1, r_2, r_3) \mid (m_1, m_2, m_3) \neq (r_1, r_2, r_3) \atop (m_1, m_2, m_3) = (r_1, r_2, r_3) \atop (m_1, m_2, m_3) \neq (r_1, r_2, r_3) \atop (m_1, m_2, m_3) = (r_1, r_2, r_3) = (r_1, r_2, r_3)$$
 (m\_1, m\_2, m\_3) = (r\_1, r\_2, r\_3) = (r\_1, r\_3, r\_3) = (r\_1, r\_3) = (r\_1, r\_3) = (r\_1, r\_3)

function

of

$$E_{(r_1, r_2, r_3) \mid (r_1, r_2, r_3)}$$

$$E_{(l_1, l_2, l_3) = (r_1, r_2) \mid (m_1, m_2) \neq (r_1, r_2)} (E_{(r_1, r_2) \mid (r_1, r_2)} =$$

 $= \min_{m_1 \neq r_1, m_2 \neq r_2} (E_{r_1 \mid m_1} (E_{(r_1, r_2) \mid (r_1, r_2)}), E_{r_2 \mid m_2} (E_{(r_1, r_2) \mid (r_1, r_2)})$ (10).

The obtained results can be summarized in the following theorem:

**Theorem.** If the distributions  $H_m, m = 1, L$ , are different and the given strictly positive numbers  $E_{(r_1, r_2, r_3)|(r_1, r_2, r_3)}$  satisfies

 $E_{(l_1, l_2, l_3) = (r_1, r_2) \mid (m_1, m_2) \neq (r_1, r_2)}$ is defined in condition (8) then (10).

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