Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 55A (2013) 13351-13355

ISSN: 2229-712X

A new approach to evaluate higher order differential equations by differential transformation method and homotopy perturbation method using boundary value problems

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ARTICLE INFO

Article history:

Received: 8 October 2012;
Received in revised form:
15 February 2013;
Accepted: 20 February 2013;

Keywords

Differential Transformation Method,
Homotopy Perturbation Method,
Higher order differential Equations.

ABSTRACT

We have to make comparison among differential transformation method (DTM) and Homotopy Perturbation Method (HPM). We provide two examples in order to compare our results and find exact solutions also. The numerical examples show that the DTM is a good method compared to the HPM since it is effective, uses less time in computation, easy to implement and achieve high accuracy. From the numerical results, DTM is suitable to apply for nonlinear problems. Numerical results show that DTM is a promising and powerful tool for solving the higher-order boundary value problems as compared to HPM.

Introduction

The differential transformation method is a numerical method based on a Taylor expansion. This method constructs an analytical solution in the form of a polynomial. Differential Transform Method (DTM) is one of the analytical methods for differential equations. The basic idea was initially introduced by Zhou [16] in 1986. Its main application therein is to solve both linear and nonlinear initial value problems in electrical circuit analysis. This method develops a solution in the form of a polynomial. Though it is based on Taylor series, still it is totally different from the traditional higher order Taylor series method. The DTM is an alternative procedure for getting Taylor series solution of the differential equations. This method reduces the size of computational domain and is easily applicable to many problems. Large list of methods, exact, approximate and purely numerical are available for the solution of differential equations. Most of these methods are computationally intensive because they are trial-and error in nature, or need complicated symbolic computations. The differential transformation technique is one of the numerical methods for ordinary differential equations. This method constructs a semi-analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming especially for high order equations. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The main advantage of this method is that it can be applied directly to nonlinear ODEs without requiring linearization, perturbation. This method will not consume too much computer time when applying to nonlinear or parameter varying systems. This method gives an analytical solution in the form of a polynomial. But, it is different from Taylor series method that requires computation of the high order derivatives.

The differential transform method is an iterative procedure that is described by the transformed equations of original functions for solution of differential equations. Chen and Liu have applied this method to solve two-boundary-value problems [2]. Jang, Chen and Liu apply the two-dimensional differential transform method to solve partial differential equations [3]. Yu and Chen apply the differential transformation method to the optimization of the rectangular fins with variable thermal parameters [4,5]. Jang, M.J., C.L. Chen and Y.C. Liy,[12-13-14] On solving the initial value problems using the differential transformation method Applied Mathematics and Computation. I.H. Hassan, N. Bildik, A. Konuralp, F. Bek, S. Kucukarslan [9,20] Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems and for PDE also. Unlike the traditional high order Taylor series method which requires a lot of symbolic computations, the differential transform method is an iterative procedure for obtaining Taylor series solutions.

The analytical results of the boundary value problems have been obtained in terms of a convergent series with easily computable components. The homotopy perturbation method (HPM) was introduced by J. H. He in 1998[11-14]. This method has been used by many mathematicians and engineers to solve various differential equations. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solution. This simple method has been applied to solve linear and non-linear equations. Some boundary value problems and many other subjects in different disciplines [6, 7, 18] afterwards it has been solved by Howarth [9] by means of some numerical methods. Since He's homotopy perturbation method (HPM) is a new technique, attempts have been conducted to apply this method for solving Blasius equation [19].Obtained solutions in comparison with previous HPM results provide the higher accuracy.

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In this paper, three higher order differential equation problems are considered by a differential transformation technique, a closed form series solution or an approximate solution can be obtained and the numerical solutions are compared with HPM method.

Description of the Method DTM

If $\omega(x)$ is a given function, its differential transform is defined as

$$\omega(n) = \frac{1}{n!}\left[\frac{d^n\omega(x)}{dx^n}\right]_{x=x_0} \quad (1)$$

and the differential inverse transformation of $\omega(x)$ is defined as

$$\omega(x) = \sum_{n=0}^{\infty} \omega(x)(x-x_0)^n \quad (2)$$

For practical application, the function $\omega(n)$ is expressed by a finite series

$$\omega(x) = \sum_{n=0}^{m} \omega(x)(x-x_0)^n \quad (3)$$

to put equation (2) into (3), we get

$$\omega(x) = \sum_{n=0}^{m}(x-x_0)^n \frac{1}{n!}\left[\frac{d^n\omega(x)}{dx^n}\right]_{x=x_0} \quad (4)$$

Which is the Taylor's series for $\omega(x)$ at $x = x_0$. Now, the fundamental operations of the DTM are given in table 1.

Function	Transformed function
$\omega(x) = n_1g(x) \pm n_2h(x)$	$\omega(n) = n_1G(n) \pm n_2H(n)$
$\omega(x) = \frac{d^n(x)}{dx^n}$	$\omega(n) = \frac{(n+m)!}{m!}G(n+m)$
$\omega(x) = g(x)h(x)$	$\omega(n) = \sum_{n_1}^{n} G(n_1)H(n-n_1)$
$\omega(x) = c\,h(x)$	$\omega(n) = c\,H(x)$, here c is constant
$\omega(x) = x^m$	$\omega(n) = \delta(n-m)$, where $\delta(n-m) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$
$\omega(x) = g_1(x)\, g_2(x) \ldots\ldots g_{m-1}(x)\, g_m(x)$	$\omega(n) = \sum_{n_{m-1}=0}^{n}\sum_{n_{m-2}=0}^{n_{m-1}}\cdots\sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2} G_2(n_1)G_1(n_2-n_1)\times G_{m-1}(n_{m-1}-n_{m-2})G_m(n-n_{m-1})$

Analysis of the Homotopy Perturbation Method

To clarify the basic ideas of the homotopy perturbation method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0,\ r \in \Omega \quad (5)$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0,\ r \in \Gamma \quad (6)$$

where A is a general differential operator, B is a boundary operator, u is a known analytical function, and Γ is the boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Therefore (5) can be rewritten as follows

$$L(u) + N(u) - f(r) = 0. \quad (7)$$

By the homotopy technique proposed by Liao [21], we can construct a homotopy

$V(r, p): \Omega\times[0, 1] \rightarrow R$ which satisfies

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0,\quad p\in[0, 1] \in \Omega \quad (8)$$

or

$$H(v, p) = L(v) - L(u_0) + p\,L(u_0) + p[N(v) - f(r)] = 0, \quad (9)$$

where $r \in \Gamma$ and $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of (5), which satisfies the boundary conditions. Obviously, from Equations (8) and (9) we will have:

$$H(v, 0) = L(v) - L(u_0) = 0, \quad (10)$$

$$H(v, 1) = A(v) - f(r) = 0, \quad (11)$$

And the changing process of p from zero to unity is just that of H(v, p) from $L(v)-L(u_0)$ to $A(v)-f(r)$. In topology, this is called deformation, $L(v) - L(u_0)$ and $A(v) - f(r)$ is called homotopic. The embedding parameter p is introduced much more naturally, unaffected by artificial factors. Furthermore, it can be considered as a small parameter for $0 < p \leq 1$. So it is very natural to assume that the solution of (8, 9) can be expressed as

$$v = v_0 + pv_1 + p^2v_2 + \cdots. \quad (12)$$

Therefore, when p=1, the approximate solution of (12) can be readily obtained as follows:

$$u = \lim_{p\rightarrow 1} v_0 + v_1 + v_2 + \cdots.$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages the series (12) is convergent for most cases.

We consider the general higher order boundary value problems of the type:

$$y^{2n}(x) == f(x, y, y'(x), \ldots\ldots, y^{(2n-1)}(x)),\ 0<x<1 \quad (13)$$

with the boundary conditions

$$y^{(2k)}(0) = a_{2k} \quad k = 0,1,2,3\ldots\ldots,(n-1) \quad (14)$$

$$y^{(2k)}(1) = b_{2k} \quad k = 0,1,2,3\ldots\ldots,(n-1) \quad (15)$$

where $f(x, y, y'(x), , y^{(2n-1)}(x))$ and y(x) are assumed real and as many as times differentiable as required for $x \in[0,1]$, α_{2k} and β_{2k}, k = 0,1,2,L,(n −1) are real finite constants Djidjedi, Twizell and Boutayeb [21], more over the constants α_{2k}, k = 0,1,2,3,…(n −1) describe the even order derivatives at the boundary x = 0 .

Using the transformation

$$y = y_1,\ \frac{dy}{dx} = y_2,\ \frac{d^2y}{dx^2} = y_3,\ldots\ldots,\ \frac{d^{2n-1}y}{dx^{2n-1}} = y_{2n}\ \frac{dy}{dx} = f(x,y,y_1,y_2,y_{2n-1}(x)) \quad (16)$$

Using the boundary conditions with the boundary conditions

$$y_1(a) = a_0,\ y_2(a) = a_1, y_3(a) = a_3,\ldots\ldots, y_{2n}(a) = a_{2n-1}, \quad (17)$$

$$y_1(b) = b_0,\ y_2(b) = b_1, y_3(b) = b_3,\ldots\ldots, y_{2n}(b) = b_{2n-1}, \quad (18)$$

which can be written as a system of integral equations:

$$y_1 = a + \int_0^x y_2(t)\,dt,$$

$$y_2 = b + \int_0^x y_3(t)\,dt,$$

$$y_3 = c + \int_0^x y_4(t)\,dt$$

$$\ldots\ldots\ldots\ldots$$

$y_{2n}=a_{2n-1}+\int_0^x f\{t,y_1(t),y_2(t),y_3(t),\ldots,y_{2n-1}(t)\}dt$ (19)

To explain (HPM), we consider (19) as follows :

$L(y_1, y_2, y_3,\ldots, y_{2n}) = L_1(y_1, y_2, y_3,\ldots,y_{2n}), L_2(y_1, y_2, y_3,\ldots,y_{2n}),\ldots\ldots,$

$\ldots\ldots, L_n(y_1, y_2, y_3,\ldots,y_{2n}) =0$ (20)

with solution $(f_1,f_2,\ldots,f_{2n})$ where

$L(y_1, y_2, y_3,\ldots, y_{2n}) = y_1 - a_0 - \int_0^x y_2(t)dt$,

……………………………………………………..

$L_{2n}(y_1, y_2, y_3,\ldots, y_{2n}) = y_{2n} - a_{2n-1} - \int_0^x f(t,y_1(t),y_2(t),\ldots,y_{2n-1}(t))dt$ (21)

We can define homotopy $H(y_1, y_2, y_3,\ldots, y_{2n},p)$ by

$H(y_1, y_2, y_3,\ldots, y_{2n},0)=F(y_1, y_2, y_3,\ldots, y_{2n}), H(y_1, y_2, y_3,\ldots, y_{2n},1)$

$=L(y_1, y_2, y_3,\ldots, y_{2n})$ (22)

Where

$F(y_1, y_2, y_3,\ldots, y_{2n}) =[F(y_1, y_2, y_3,\ldots, y_{2n}),\ldots, F_{2n}(y_1, y_2, y_3,\ldots, y_{2n})]^T$

$= [y_1 - a_0,\ldots y_{2n-1}-a_{2n-2}, y_{2n}-a_{2n-1}]^T$ (23)

$H(y_1, y_2, y_3,\ldots, y_{2n},p)=[H_1(y_1, y_2, y_3,\ldots, y_{2n},p),H_n(y_1, y_2, y_3,\ldots, y_{2n},p)]^T$ (24)

Typically we may choose a convex homotopy by:

$H(y_1, y_2, y_3,\ldots, y_{2n},p) = (1-p)F(y_1, y_2, y_3,\ldots, y_{2n}) + pL(y_1, y_2, y_3,\ldots, y_{2n}) = 0$ (25)

The convex homotopy (25) contiuously trace an implicitly defined carve from a starting point $H(y_1-a_0,\ldots y_{2n}-a_2, y_{2n}-a_{2n-1},0)$ be a solution $H(f_1, f_2, f_3,\ldots, f_{2n},1)$. The embedding parameter p monotonically increasing from zero to unite as trivial problem $F(y_1, y_2, y_3,\ldots, y_{2n}) = 0$ is continuously deformed to original problem $L(y_1, y_2, y_3,\ldots, y_{2n}) = 0$.

The uses the homotopy parameter p as an expanding parameter:

$y_1 = y_{10}+p\, y_{11}+p^2y_{12}+\ldots\ldots\ldots, y_{2n} = y_{2n0}+p\, y_{2n1}+p^2\, y_{2n2}+\ldots\ldots$

The approximate solution of Eq. (13), therefore, can be readily obtained:

$f_1= \lim_{p\to 1} y_1 = y_{10}+ y_{11}+ y_{12}+\ldots\ldots$

$f_{2n}= \lim_{p\to 1} y_{2n} = y_{2n0}+ y_{2n1}+ y_{2n2}+\ldots\ldots$ (26)

The convergence of the series (26) for the application of (HPM) to (20), we can write Eq. (25) as follows:

$H_1(y_1, y_2, y_3,\ldots, y_{2n},p)=(1-p)F_1(y_1, y_2, y_3,\ldots, y_{2n})+pL_1(y_1, y_2, y_3,\ldots, y_{2n}) = 0$ …………………………… (27)

$H_{2n}(y_1, y_2, y_3,\ldots, y_{2n},p)=(1-p)F_{2n}(y_1, y_2, y_3,\ldots, y_{2n})+pL_{2n}(y_1, y_2, y_3,\ldots, y_{2n}) = 0$ (27) substitution of (21), (22) and (24) into (27) yields

$(1-p)(y_1 - a_0)+ p(y_1 - a_0 - \int_0^x y_2(t)dt) =0,$

$(1-p)(y_{2n} - a_{2n-1})+ p(y_{2n} - a_{2n-1} - \int_0^x f(t,y_1(t),y_2(t),\ldots,y_{2n-1}(t))dt) =0.$ (28)

By equating the terms with identical powers of p, like p^0, p^1, p^2, p^{2k}.Combining all the terms of Equations give the solution of the problem, by using the boundary conditions (14) and (15) we can obtained all parameters.

Numerical examples

In this section, two numerical examples will be presented to assess the efficiency of the DTM. For the sake of comparison, we will use the absolute error defined as

Errors = |analytical solution − approximate solution|

The rest of this paper is organized as follows. In section 2 and 3, we give the analysis of the DTM and HPM. In section 4, we present numerical results to demonstrate the efficiency of DTM as compared to HPM with the help of two examples.

Example 1: We have to take a nonlinear sixth order boundary value problem

$e^x y^6(x) - y^2(x) = 0,\ 0 < x <1,$ (29)

with the conditions:

$y(0) =1, y^{(2)}(0) = 1, y^{(4)}(0) = 1, y^{(2)}(1) = e, y^{(4)}(1) = e$ (30)

Exact solution is: $y(x) = e^x$ Now applying DTM, The differential transform of (29) is:

$$Y(r+5) = \frac{1}{\prod_{i=1}^{6}(r+i)}\sum_{l=0}^{r}\sum_{m=0}^{l}\frac{(-1)^m}{m!}Y(l-m)Y(r-l)$$

Differential transformation of the boundary conditions (30) is:

$y(0) =1, y(2) = 0.5, y(2)= \frac{1}{4!}$

$$\sum_{r=0}^{N}Y(r) = \sum_{r=0}^{N}r(r-1)Y(r) = \sum_{r=0}^{N}\prod_{k=0}^{3}(r-k)\,Y(r) = e$$

Using (29), (30) and the inverse differential transform, the following series solution of the problem, up to higher terms obtained.

$y(x) = 1+ x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + 0.24802\times10^{-4}x^8+ 2.75573\times10^{-6}x^9+2.75573\times10^{-7}x^{10}+2.50523\times10^{-8}x^{11}+ 2.08768\times10^{-9}x^{12}+\ldots\ldots\ldots\ldots$(31)

Using the transformation $\frac{dy}{dx} = p(x), \frac{dp}{dx} = q(x), \frac{dq}{dx} = r(x), \frac{dr}{dx} = s(x), \frac{ds}{dx} = t(x),$

$\frac{dt}{dx} = e^{-x}y^2(x)$, we rewrite the above sixth-order boundary value problem as a system of differential equations with $y_0(x) =1$, $p_0(x) =A$, $q_0(x)= 1$, $r_0(x)=B$, $s_0(x) =1$, $t_0(x)= C$

Now, applying the homotopy perturbation method, we get, Using the transformation, we can rewrite the sixth-order boundary value problem (29) as the system of integral equations:

$y(x) = 1 +\int_0^x p(x)\,dx$, $p(x) = A +\int_0^x q(x)\,dx$, $q(x) = 1 +\int_0^x r(x)\,dx$ $r(x) = B +\int_0^x s(x)\,dx$, $s(x) = 1 +\int_0^x t(x)\,dx$, $t(x) = C +\int_0^x (e^{-x}y^2(x))\,dx$

Comparing the coefficients of like powers of p, we have:

Coefficients of p^0: $y_0(x) = 1$, $p_0(x)= A$, $q_0(x) = 1$, $r_0(x) = B$, $s_0(x) = 1$, $t_0(x) = C$,

Coefficients of p^1: $y_1(x) = Ax$, $p_1(x)= x$, $q_1(x) = Bx$, $r_1(x) = x$, $s_1(x) = Cx$, $t_1(x) = 1-e^{-x}$

Coefficients of p^2: $y_2(x) = \frac{x^2}{2}$, $p_2(x) = \frac{Bx^2}{2}$, $q_2(x) = \frac{x^2}{2}$, $r_2(x) = \frac{Cx^2}{2}$, $s_2(x) = -1+x+ e^{-x}$, $t_2(x) =2A-2Ae^{-x} -2Axe^{-x}$

Coefficients of p^3:$y_3(x) = \frac{Bx^3}{6}$, $p_3(x) = \frac{x^3}{6}$, $q_3(x) = \frac{Cx^3}{6}$, $r_3(x) =1- x+\frac{x^2}{2} - e^{-x}$, $s_3(x) = -4A+4Ae^{-x} +2Ax\ 2Axe^{-x}$, $t_3(x) = (1+A^2)(2-2e^{-x}-2xe^{-x}-x^2e^{-x})$

Coefficients of p^4: $y_4(x) = \frac{x^4}{24}$, $p_4(x) = \frac{Cx^4}{24}$, $q_4(x) = -1+ x- \frac{x^2}{2} + \frac{x^3}{6} + e^{-x}$, $r_4(x) = 6A- 4Ax+ \frac{Ax^2}{2} - 6xe^{-x} - 2Axe^{-x}$, $s_4(x) = (1+A^2)(-6+2x+-6e^{-x}+4xe^{-x} +x^2e^{-x})$, $t_4(x) = (0.33B+A)(6-6e^{-x}-6xe^{-x} -3x^2e^{-x} -x^3e^{-x})$

Above procedure is proceeding up to coefficient of p^6.Now, adding above all coefficient of p^0 to p^6, and then we obtain:

$$y^6(x) = 1+Ax+0.5x^2+\frac{B}{6}x^3 + \frac{1}{24}x^4+\frac{C}{120}x^5 + \frac{1}{720}x^6+\left(-\frac{1}{5040}+\frac{A}{2520}\right)x^7+ \left(\frac{1}{13440}-\frac{A}{10080}\right)x^8+\left(\frac{1}{60480}-\frac{1}{51840}+\frac{B}{181440}\right)x^9- \left(\frac{1}{241920}-\frac{A}{453600}-\frac{B}{453600}\right)x^{10}+\left(\frac{A}{3991680}+\frac{B}{1995840}-\frac{C}{19958400}-\frac{31}{39916800}\right)x^{11}+ \dots$$

The coefficients A, B, C, can be obtained using the boundary conditions at x=1,

A=1.0012526840, B=0.988483055, C= 1.085993892

The series solution can, thus, be written as

$$y^6(x) = 1+1.0012526840x + 0.5x^2+0.1647471759x^3- 0.04166666667x^4 -0.0090499491x^5 + 0.00138888889x^6 + 0.0001989097952x^7 + 0.00002492562x^8 +0.00000271296906x^9 - 0.000000252944751x^{10} + 0.0000000239041138x^{11}+\dots\dots$$

Table Absolute Error Estimation 1.1

X	Exact Solution	DTM Solution	HPM Solution	Error Estimate of DTM Solution	Error Estimate of HPM Solution
0	1.000000000	1.000000000	1.0000000000	-5.4×10^{-9}	0.00000000
0.1	1.105170918	1.105170918	1.1052942734	-1.0×10^{-8}	1.23×10^{-4}
0.2	1.221402758	1.221402758	1.2216381695	-1.4×10^{-8}	2.35×10^{-4}
0.3	1.349858808	1.349858808	1.3501845256	-1.7×10^{-8}	3.25×10^{-4}
0.4	1.491824698	1.491824698	1.4922102313	-1.8×10^{-8}	3.85×10^{-4}
0.5	1.648721271	1.648721271	1.6491298802	-1.8×10^{-8}	4.08×10^{-4}
0.6	1.822118800	1.822118800	1.8225106988	-1.5×10^{-8}	3.91×10^{-4}
0.7	2.013752707	2.013752707	2.014088799	-9.3×10^{-8}	3.36×10^{-4}
0.8	2.225540928	2.225540928	2.2257868159	-1.1×10^{-8}	2.45×10^{-4}

Example 2:

Consider the following nonlinear boundary value problem of sixth-order

$$y^{(6)}x = -6e^x + y(x),\ 0< x< 1 \qquad (32)$$

with boundary conditions as following

$$y(0) = 1, y^2(0) = -1,\ y^4(0) = -3,\ y(1) = 0,\ y^2(1) = -2e,\ y^4(1) = -4\,e \qquad (33)$$

The exact solution for this problem is y(x) □□□□□ x□e^x

Now apply the differential transform method:

Knowing that the differential transformation of e^x is 1/k! and applying the

Above given example, the following recurrence relation is obtained

$$Y(r+6) = \frac{1}{(r+1)(r+2)(r+3)(r+4)(r+5)(r+6)} \times \left(Y(r) - \frac{6}{r!}\right)$$

Differential transformations of the boundary conditions (33) are:

$$\sum_{r=0}^{N} Y(r) = 0, \sum_{r=0}^{N} r(r-1)Y(r) = -2e, \sum_{r=0}^{N}\prod_{k=0}^{3}(r-k)\,Y(r) = -4e$$

Using (32), (33) and the inverse differential transform, the following series solution of the problem, up to higher terms obtained.

$$y(x) = 1 - 6.66476\times10^{-7}x - \frac{x^2}{2} -0.33333x^3 - \frac{x^4}{8} -0.0333333x^5- \frac{x^6}{144} -0.00119048x^7 + \frac{x^8}{5780} - 0.0000220456x^9 - \frac{x^{10}}{403200} - 2.50523\times10^{-7}x^{11}+ \dots\dots \quad (34)$$

Now applying HPM,

Using the transformation $\frac{dy}{dx} = p(x), \frac{dp}{dx} = q(x), \frac{dq}{dx} = r(x), \frac{dr}{dx} = s(x), \frac{ds}{dx} = t(x), \frac{dt}{dx} = -6e^x t(x)$, we rewrite the above sixth-order boundary value problem as a system of differential equations with $y_0(x)=1$, $p_0(x) =A$, $q_0(x)= -1$, $r_0(x)=B$, $s_0(x) = -3$, $t_0(x)= C$

Now, applying the homotopy perturbation method, we get, Using the transformation, we can rewrite the sixth-order boundary value problem (33) as the system of integral equations:

$y(x) = 1 +\int_0^x p(x)\,dx$, $p(x) = A +\int_0^x q(x)\,dx$, $q(x) = -1 +\int_0^x r(x)\,dx$

$r(x) = B +\int_0^x s(x)\,dx$, $s(x) = -3 +\int_0^x t(x)\,dx$, $t(x) = C +\int_0^x (-6e^x + y(x))\,dx$

Comparing the coefficients of like powers of p, we have:

Coefficients of p^0: $y_0(x) = 1$, $p_0(x) = A$, $q_0(x) = -1$, $r_0(x) = B$, $s_0(x) = -3$, $t_0(x) = C$,

Coefficients of p^1: $y_1(x) = Ax$, $p_1(x)= -x$, $q_1(x) = Bx$, $r_1(x) = -3x$, $s_1(x) = Cx$, $t_1(x) = 6-e^x +x$

Coefficients of p^2: $y_2(x) = -\frac{x^2}{2}$, $p_2(x) =\frac{Bx^2}{2}$, $q_2(x)=\frac{-3x^2}{2}$, $r_2(x) = \frac{Cx^2}{2}$, $s_2(x) = 6-6e^x +6x+\frac{x^2}{2}$, $t_2(x) = \frac{Ax^2}{2}$

Coefficients of p^3: $y_3(x) =\frac{Bx^3}{6}$, $p_3(x) = \frac{-3x^3}{6}$, $q_3(x) = \frac{Cx^3}{6}$, $r_3(x) = 3x^2+6- 6e^x +6x+\frac{x^3}{6}$, $s_3(x) =\frac{Ax^3}{6}$, $t_3(x) = \frac{-x^3}{6}$

Coefficients of p^4: $y_4(x) =-\frac{x^4}{8}$, $p_4(x) =\frac{Cx^4}{24}$, $q_4(x) =3x^2+6- 6e^x +x^3+6x+\frac{x^4}{24}$, $r_4(x) = \frac{Ax^4}{24}$, $s_4(x) =\frac{-x^4}{24}$, $t_4(x) = \frac{Bx^4}{24}$

Coefficients of p^5: $y_5(x) = \frac{Cx^5}{120}$, $p_5(x) =3x^2+6- 6e^x +x^3+6x+\frac{x^4}{4} + \frac{x^5}{120}$, $q_5(x) =\frac{Ax^5}{120}$, $r_5(x)=\frac{-x^5}{120}$, $s_5(x) =\frac{Bx^5}{120}$, $t_5(x) =\frac{-x^5}{40}$

Above procedure is proceeding up to coefficient of p^6.Now, adding above all coefficient of p^0 to p^6, and then we obtain:

$$y^{10}(x) = 1+Ax+0.5x^2+\frac{B}{6}x^3-\frac{1}{8}x^4+\frac{C}{120}x^5 - \frac{1}{120}x^6+\left(\frac{1}{840}+\frac{A}{5040}\right)x^7-$$

$$\frac{1}{5760}x^8+\left(-\frac{1}{60480}+\frac{B}{362880}\right)x^9-\frac{1}{403200}x^{10}+\left(\frac{1}{6652800}+\frac{C}{39916800}\right)x^{11}-\frac{1}{39916800}x^{12}+\cdots \qquad (35)$$

The coefficients A, B,C, can be obtained using the boundary conditions at x=1,

A=0.004162270, B=-2.041623366, C= -3.50042504

The series solution can, thus, be written as

$$y^6(x) = 1+0.004162270x - 0.5x^2-0.3402705611x^3-0.125x^4-0.02917020956x^5\ 0.008333333x^6-0.00118965x^7-0.000187x^8+.0000022134x^9-0.0000024x^{10}-0.0000000238x^{11}$$

Table Absolute Error Estimation 1.2

X	Exact Solution	HPM Solution	DTM Solution	Error Estimate of HPM Solution	Error Estimate of DTM Solution
0	1.00000000	1.00000000	1.00000000	0.00000000	0.0000000
0.1	0.99465383	0.99506316	0.994653826	0.00040933	6.57×10^{-8}
0.2	0.97712221	0.97790041	0.977122081	0.00077820	1.26×10^{-8}
0.3	0.94490117	0.94597165	0.944900991	0.00107048	1.74×10^{-8}
0.4	0.89509482	0.89635269	0.895094611	0.00125787	2.08×10^{-7}
0.5	0.82436064	0.82568302	0.824360414	0.00132238	2.21×10^{-7}
0.6	0.72884752	0.73010539	0.728847306	0.00125787	2.14×10^{-7}
0.7	0.60412581	0.60519629	0.604125628	0.00125787	1.85×10^{-7}
0.8	0.44510819	0.44588638	0.445108205	0.00007782	1.36×10^{-7}
0.9	0.24596031	0.24636964	0.245960240	0.00040933	7.23×10^{-8}

Conclusion

In this paper, we have introduced a new technique to solve higher order boundary value problems by the Differential transform method and Homotopy perturbation method .The numerical results in the Tables [1.1-1.2], show that the Differential transform method provides highly accurate numerical results as compared to Homotopy perturbation method. It can be concluded that is Differential transform method a highly efficient method for solving higher order boundary value problems arising in various fields of engineering and science.

References

[1]. R.P. Agarwal, Boundary value problems for higher ordinary differential equations, World Scientific, Singapore,(1986).

[2] Chen, C.L. and Y.C. Liu, 1998. Differential transformation technique for steady nonlinear Heat conduction problems. Applied Mathematics and Computation, 95: 155-164.

[3] Chen, C.L. and Y.C. Liu, 1998. Solution of two point boundary value problems using the differential transformation method Journal of Optimization Theory and Applications, 99: 23-35.

[4] Chen, C.L., S.H. Lin and C.K. Chen, 1996. Application of Taylor transformation to Nonlinear predictive control problem. Applied Mathematical Modeling, 20: 699-710.

[5] Chen, C.K. and S.H. Ho, 1996. Application of differential transformation to eigenvalue Problems. Applied Mathematics and Computation, 79: 173-188.

[6] D. D. Ganji, M. Rafei : Solitary wave solutions for a generalized Hirota–Satsuma coupled KdV equation by homotopy perturbation method. Physics Letters A. 356:131 137(2006)

[7] D. D. Ganji, A. Sadighi: Application of homotopy-perturbation and variational iteration Methods to nonlinear heat transfer and porous media equations. Journal of Computational and Applied Mathematics.207:24-34(2007)

[8] Hassan, I.H.A., Differential transformation technique for solving higher-order Initial value problems Applied Mathematics and Computation, 154: 299-311,(2004)

[9] Howarth, L. On the Solution of the Laminar Boundary-Layer Equations. Proceedings of the Royal Society of London . A. 164:547-579(1983)

[10] I.H. Hassan, Comparison differential transformation technique with Adomian Decomposition method for linear and nonlinear initial value problems, Chaos Solitons Fractals, in press, doi:10.1016/j.chaos.2006.06.040.

[11] J. H. He: Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering, 178:257-262(1999)

[12] J.H. He, A coupling method of homotopy technique and perturbation technique for Nonlinear problems, International Journal of Non-Linear Mechanics 37-43(2000)

[13] J.H. He, Comparison of homotopy perturbation method and homotopy analysis method. Applied Mathematics and Computation.527-539(2004)

[14] J. H. He, Homotopy perturbation method: a new nonlinear analytical technique. Applied Mathematics and Computation.73-79 (2003)

[15] Jang, M.J., C.L. Chen and Y.C. Liy, On solving the initial value problems using the Differential transformation method Applied Mathematics and Computation, 115: 145-160. World Appl. Sci. J., 15 (12): 1774-1779, (2011)

[16] J.K. Zhou, Differential Transformation and Its Applications for Electrical Circuits, Huazhong University Press, Wuhan, China, 1986(in Chinese).

[17] K. Djidjedi, E. H. Twizell, A. Boutayeb, Numerical methods for special non linear boundary Value problems of order 2m., J. Comput. Appl. Math., 47 (1993) 35-45.

[18] M. Rafei, D. D. Ganji: Application of homotopy perturbation method to the RLW and generalized modified Boussinesq equations. Physics Letters A. 364;1-6(2007)

[19] M. Esmaeilpour, D. D. Ganji: Application of He's homotopy perturbation method to boundary layer flow and Convection heat transfer over a flat plate. Physics Letters A. 33 38(2007).

[20] N. Bildik, A. Konuralp, F. Bek, S. Kucukarslan, Solution of different type of the PDE byDifferential transform method and Adomian's decomposition method, Appl. Math. Computer, 172 , 551–567(2006).

[21] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear Problems, PhD thesis, Shanghai Jiao Tong University, (1992)