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Relation between infinite series and infinite sequence

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ABSTRACT

In this paper I try to prove some relation about convergence of infinite series and infinite sequence. And also try to prove that if $\sum u_n$ and $\sum v_n$ are any two positive term convergent series then $\sum u_n v_n$ are also convergent series. And also try to give some examples which gives the support to the comparison test.

Introduction

We begin with infinite series. Here we use $\sum_{n=1}^{\infty} u_n$ to denote infinite series whose n[th] term is u_n .I have given brief summary of definitions which are useful for the present investigation.

Prilimaries:

Definition1.1 [1] (Infinite sequence): Any function $f: \mathbf{N} \to X$ is called infinite sequence in X. We use $\{u_n\}_{n=1}^{\infty}$ to denote infinite sequence whose n^{th} term is u_n .

Definition1.2 [2] (Series): Series is the sum of each term of sequence.

Definition1.3 [2] (Infinite series): A series is called infinite series if it contains infinitely many terms.

Definition 1.4[2] (Convergent series): An infinite Series $\sum_{n=1}^{\infty} u_n$ is called convergent series if $\lim_{n\to\infty} s_n$ exist, where $s_n = \sum_{1}^{n} u_n$.

Definition 1.5 [2] (Divergent series): A infinite Series $\sum_{n=1}^{\infty} u_n$ is called convergent series if $\lim_{n\to\infty} s_n = \pm\infty$, where $s_n = \sum_{1}^{n} u_n$.

Definition 1.6 [1] (Convergence sequence): Let $\{u_n\}_{n=1}^{\infty}$ be any infinite sequence then it is called convergent to l if for $\forall \varepsilon > 0 \ \exists\, n_0 \in \mathbf{N}$ such that $if\ n > n_0\ then\ |u_n - l| < \epsilon.$

Definition 1.7 [1] (Divergent sequence): Let $\{u_n\}_{n=1}^{\infty}$ be any infinite sequence then it is called divergent if for $\forall\, m > 0 \ \exists\, n_0 \in \mathbf{N}$ such that $if\ n > n_0\ then\ |u_n| > m.$

Some Well-known Result

Ratio Test[2]:

Let $\sum_{n=1}^{\infty} u_n$ be any positive term series and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ then

(1) If $l > 1$ then $\sum u_n$ is convergent series

(2) If $l < 1$ then $\sum u_n$ is divergent series

(3) If $l = 1$ then test fail.

P-series [2]:

Let $\sum \frac{1}{n^p}$ be any positive term series then

(1) If $\sum \frac{1}{n^p}$ is convergent if $p > 1$

(2) If $\sum \frac{1}{n^p}$ is divergent series if $p \le 1$.

Direct Comparison test [2]:

Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be any two positive terms series, and $u_n \le v_n$ for every n after few terms. Then

(1) $\sum_{n=1}^{\infty} u_n$ is convergent if $\sum_{n=1}^{\infty} v_n$ is convergent.

(2) $\sum_{n=1}^{\infty} v_n$ is divergent if $\sum_{n=1}^{\infty} u_n$ is divergent.

Zero test [2]:

Let $\sum u_n$ be any positive term series and if $\lim_{n\to\infty} u_n \neq 0$ then $\sum u_n$ is divergent.

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Main Theorem:

Theorem 1. If $\{u_n\}_{n=1}^{\infty}$ be any divergent sequence where every u_n is positive for every n .Then $\sum_{n=1}^{\infty} u_n$ is also divergent series.

Proof:

Here, $\{u_n\}_{n=1}^{\infty}$ is divergent

So, for $\forall\, m > 0\ \exists\, n_0 \in \mathbb{N}$ such that $if\ n > n_0\ then\ |u_n| > m.$

In particular if we take m=1, then

$\exists\, n_0 \in \mathbb{N}$ Such that $if\ n > n_0\ then\ |u_n| > 1$

$\Rightarrow |u_n| > 1\ ,\forall\, n > n_0$

$\Rightarrow u_n > 1\ ,\forall\, n > n_0$ as all u_n are positive

Now, consider the series $\sum_{n=1}^{\infty} 1$

Then $\lim_{n\to\infty} u_n = \lim_{n\to\infty} 1$

=1

$\neq 0$

So, by Cauchy's test for divergent $\sum_{n=1}^{\infty} 1$ is divergent

And so, by comparison test $\sum_{n=n_0}^{\infty} u_n$ is divergent.

Hence, $\sum_{n=1}^{\infty} u_n$ is also divergent because we know that if we add some finite number of terms in series then its nature remains unaltered.

Remarks:

Converse of the following theorem may not be true in general. See the following example

Example:

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$

Then by p-test we have $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. But , $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent and infact it is convergent to 0.

Theorem 2. Let $\sum_{n=1}^{\infty} u_n$ be any positive terms convergent series then $\{u_n\}_{n=1}^{\infty}$ is also convergent and indeed it is convergent to 0.

Proof:

Here, $\sum_{n=1}^{\infty} u_n$ is convergent series,

So by Zero test we have

$\lim_{n\to\infty} u_n = 0$

So, by the definition of convergence of sequence we have $\{u_n\}_{n=1}^{\infty}$ is convergent and in fact it is convergent to 0.

Remarks: Converse of the theorem may not be true in general. See the following example

Example:

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$

Then by p-test we have $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. But, $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent and in fact it is convergent to 0.

Theorem 3.

Let $\sum u_n$ and $\sum v_n$ be any two positive term series and if $\sum u_n$ and $\sum v_n$ are convergent then $\sum u_n v_n$ is also convergent.

Proof:

Let $\sum a_n = \sum u_n v_n$

Now,

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$

$= \lim_{n\to\infty} \frac{u_n v_n}{u_{n+1} v_{n+1}}$

$= \lim_{n\to\infty} \frac{u_n}{u_{n+1}} \times \lim_{n\to\infty} \frac{v_n}{v_{n+1}}$

> 1 as $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$ and $\lim_{n\to\infty} \frac{v_n}{v_{n+1}} > 1$ Because $\sum u_n\ and\ \sum v_n$ are Convergent series.

So, by using D'Alembert's Ratio test

$\sum a_n = \sum u_n v_n$ is also convergent.

Question: Is above theorem is true for divergent series that means if $\sum u_n\ and\ \sum v_n$ are two divergent series then does it implies that $\sum u_n v_n$ is divergent.

Answer: It is not true in general, see below two example.

Example 1.

Consider $\sum u_n = \sum \frac{1}{n} = \sum v_n$ then $\sum u_n v_n = \sum \frac{1}{n^2}$ .then $\sum u_n$ and $\sum v_n$ are divergent series but $\sum u_n v_n$ is convergent series using P-test.

Example 2.

Consider $\sum u_n = \sum \frac{1}{\sqrt{n}} = \sum v_n$ then $\sum u_n v_n = \sum \frac{1}{n}$ .then $\sum u_n$ and $\sum v_n$ are divergent series and $\sum u_n v_n$ is also divergent series using P-test.

Remark: The following example shows that if $\sum u_n$ is convergent and $\sum v_n$ is divergent series then $\sum u_n v_n$ is convergent.

Example:

Consider $\sum u_n = \sum \frac{1}{n}$, $\sum v_n = \sum \frac{1}{n^2}$ then $\sum u_n v_n = \sum \frac{1}{n^3}$ .then $\sum u_n$ is divegent and $\sum v_n$ is convergent series and $\sum u_n v_n$ is convergent series using P-test.

Remark: The following example shows that if $\sum u_n$ is convergent and $\sum v_n$ is divergent series then $\sum u_n v_n$ is divergent.

Example 2.4 Consider $\sum u_n = \sum \frac{\ln n}{n^{\frac{5}{2}}}$, $\sum v_n = \sum n^{\frac{5}{2}}$ then $\sum u_n v_n = \sum \ln n$. And $\sum u_n$ is convergent series and $\sum v_n$ is divergent series and $\sum u_n v_n$ is divergent series using zero test.

Now, Let us first write the limit form of comparison test

Limit form of comparison test:

Let $\sum u_n \ and \ \sum v_n$ are two positive term series then

(1) If $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite and nonzero) then both series convergent and divergent together.

(2) if $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ and $\sum v_n$ is convergent then $\sum u_n$ is also convergent

(3) if $\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$ and $\sum v_n$ is divergent then $\sum u_n$ is also divergent.

Question: In case (2) in above comparison test if $\sum v_n$ is divergent does it implies that $\sum u_n$ is also divergent.

Answer is no. It is not true in general. See following example.

Example: consider the series $\sum u_n = \sum \frac{1}{n^2}$ and $\sum v_n = \sum \ln n$ .then using zero test we have $\sum v_n$ is divergent and using P-test we have $\sum u_n$ is convergent.

But, $\lim_{n\to\infty} \frac{u_n}{v_n}$

$= \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\ln n}$

$= 0.$

Question: In case (3) in above comparison test if $\sum v_n$ is convergent then does it implies that $\sum u_n$ is also convergent.

Answer is no. It is not true in general. See following example.

Example:

Consider the series $\sum u_n = \sum 1$ and $\sum v_n = \sum \frac{1}{n^2}$ .then using zero test we have $\sum u_n$ is divergent and using P-test we have $\sum v_n$ is convergent

But, $\lim_{n\to\infty} \frac{u_n}{v_n}$

$= \lim_{n\to\infty} \frac{1}{\frac{1}{n^2}}$

$= \infty$.

References:

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