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Applied Mathematics

Elixir Appl. Math. 55C (2013) 12947-12953

ISSN: 2229-712X

A Class of Multivalent Harmonic Functions with respect to k-Symmetric Points

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ARTICLE INFO

Article history:

Received: 6 December 2012;

Received in revised form:

2 February 2013;

Accepted: 7 February 2013;

Keywords

Multivalent harmonic functions,

Uniformly Starlike Functions,

Generalized hypergeometric functions,

Convolution.

ABSTRACT

A Goodman-Rønning type class of multivalent harmonic functions involving Dziok-Srivastava operators with respect to k-symmetric points is studied. An equivalent convolution class condition and a sufficient coefficient condition for this class is obtained. It is proved that this coefficient condition is necessary for its subclass. As an application of coefficient condition, a necessary and sufficient hypergeometric inequality is also given. Further, results on bounds, extreme points, a convolution property and a result based on the integral operator are obtained.

Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real-valued harmonic in D. In any simply connected domain $\mathrm{D} \subset \mathbb{C}$, f can be written in the form: $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [9]). Let H denote a class of harmonic functions $f = h + \overline{g}$, which are harmonic, univalent and orientation preserving in the open unit disc $\Delta = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$

Note that the family H reduces to the well-known class S of normalized univalent functions if the co-analytic part of f is identically zero, that is if $g = 0$.

The concept of multivalent harmonic complex valued functions by using argument principle, was given by Duren, Hengartner and Laugesen [10]. Using this concept, Ahuja and Jahagiri [1], [2] introduced a class $H(m)$ of m-valent harmonic and orientation preserving functions $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are m-valent functions of the form:

$$h(z) = z^m + \sum_{n=m+1}^{\infty} a_n z^n,\ g(z) = \sum_{n=m}^{\infty} b_n z^n, |b_m| < 1, m \in \mathbb{N}\{1,2,3...\} \quad (1.1)$$

which are analytic in $\Delta = \{z : |z| < 1\}$

For $p, q, r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, complex parameters α_i $(i = 1,2,..., p)$, $\gamma_i\ (i = 1,2,..., r)$ and $\beta_i\ (\neq -n; i = 1,2,.., q)$, $\delta_i\ (\neq -n; i = 1,2,.., s), n \in \mathbb{N}_0$, a linear operator :

$$\Theta_{r,s}^{p,q}\left((\alpha_i)_{1,p}, (\beta_i)_{1,q}; (\gamma_i)_{1,r}, (\delta_i)_{1,s}\right) \equiv \Theta_{r,s}^{p,q}\left[[\alpha_1];[\gamma_1]\right] : H(m) \to H(m)$$

is defined by

$$\Theta_{r,s}^{p,q}\left[[\alpha_1];[\gamma_1]\right] f(z) = H_m^{p,q}[\alpha_1] h(z) + \overline{H_m^{r,s}[\gamma_1] g(z)} \quad (1.2)$$

where the operators $H_m^{p,q}[\alpha_1]$, $H_m^{r,s}[\gamma_1]$ are the Dzoik-Srivastava operators [11] defined for the functions $h(z)$ and $g(z)$ of the form (1.1), respectively, by

$$H_m^{p,q}[\alpha_1] h(z) = z^m {}_pF_q\left((\alpha_i);(\beta_i);z\right) * h(z),\ H_m^{r,s}[\gamma_1] g(z) = z^m {}_rF_s\left((\gamma_i);(\delta_i);z\right) * g(z) \quad (1.3)$$

We have for $f(z) = h(z) + \overline{g(z)} \in H(m)$,

$$\Theta_{1,0}^{1,0}\left[[1];[1]\right] f(z) = h(z) + \overline{g(z)},\ \Theta_{1,0}^{1,0}\left[[2];[2]\right] f(z) = H(z) + \overline{G(z)}$$

where

$$H(z) = z\frac{d}{dz} h(z) - (m-1) h(z), G(z) = z\frac{d}{dz} g(z) - (m-1) g(z).$$

The function ${}_pF_q\left((\alpha_i);(\beta_i);z\right) = {}_pF_q(\alpha_1,...,\alpha_p,\beta_1,...,\beta_p;z)$ is the generalized hypergeometric function defined by

$${}_pF_q\left((\alpha_i);(\beta_i);z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ...(\alpha_p)_n}{(\beta_1)_n ...(\beta_q)_n\, n!} z^n \quad (p \le q+1; z \in \Delta) \quad (1.4)$$

which is analytic at $z = 1$ if (in case $p = q+1$) $\Sigma_{i=1}^{q}\beta_i - \Sigma_{i=1}^{p}\alpha_i > 0$, the symbol $(\lambda)_n$ is the Pochhammer symbol defined in terms of gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{Bmatrix} 1, & n = 0, \lambda \neq 0 \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1), & n \in \mathbb{N} \end{Bmatrix}.$$

The series expression of $\Theta_{r,s}^{p,q}\left[[\alpha_1];[\gamma_1]\right] f(z)$ defined by (1.2) is given by

$$\Theta_{r,s}^{p,q}\left[[\alpha_1];[\gamma_1]\right] f(z) = z^m + \sum_{n=m+1}^{\infty} \theta_n a_n z^n + \sum_{n=m}^{\infty} \overline{\phi_n b_n z^n} \quad (1.5)$$

where

$$\theta_n = \frac{\prod_{i=1}^{p}(\alpha_i)_{n-m}}{\prod_{i=1}^{q}(\beta_i)_{n-m}} \frac{1}{(n-m)!}, \phi_n = \frac{\prod_{i=1}^{r}(\gamma_i)_{n-m}}{\prod_{i=1}^{s}(\delta_i)_{n-m}} \frac{1}{(n-m)!}, n \ge m. \quad (1.6)$$

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Note that the Dziok-Srivastava operator $H_m^{p,q}[\alpha_1]$, defined by (1.3) contains several well-known linear operators such as the Hohlov linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator etc. whose references may be found in [11].

For some $k \in \mathrm{N}$ and for $\mu = 0,1,...,k-1$, the points: $z = \varepsilon^{\mu} z \quad \left(\varepsilon = e^{\frac{2\pi i}{k}}, z \in \Delta\right)$ are the symmetric points.

Corresponding to $f = h + \overline{g} \in H(m)$, (where $h(z)$ and $g(z)$ are of the form (1.1)), we consider $f_k = h_k + \overline{g_k} \in H(m)$, where

$$h_k(z) = \frac{1}{k}\sum_{\mu=0}^{k-1} \varepsilon^{-\mu m} h(\varepsilon^{\mu} z) = h(z) * \frac{z^m}{(1-z^k)} = z^m + \sum_{n=m+1}^{\infty} \Psi_n^k a_n z^n, z \in \Delta \qquad (1.7)$$

$$g_k(z) = \frac{1}{k}\sum_{\mu=0}^{k-1} \varepsilon^{-\mu m} g(\varepsilon^{\mu} z) = g(z) * \frac{z^m}{(1-z^k)} = \sum_{n=m}^{\infty} b_n \Psi_n^k z^n, z \in \Delta \qquad (1.8)$$

and for $n \ge m+1$,

$$\Psi_n^k = \begin{cases} 1, & n-m = lk, l \in \mathrm{N} \\ 0, & n-m = lk+j, l \in \mathrm{N} \cup \{0\}, j = 1,2,...,k-1, k \ge 2 \end{cases}, \Psi_m^k = 1. \qquad (1.9)$$

Two subclasses of S, namely, uniformly convex functions (UCV) and uniformly starlike functions (UST) were introduced by Goodman [12], [13], later Rønning [24] has given more applicable characterization of these classes.

For the purpose of this paper, on applying the linear operator $\Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]]$, we define a class $R_{m,k}([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha)$ of functions $f \in H(m)$ if it satisfy the Goodman-Rønning type class condition

$$\Re\left(\frac{z\left(\Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]] f(z)\right)'}{z' \Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]] f_k(z)}\right) > \rho\left|\frac{z\left(\Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]] f(z)\right)'}{z' \Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]] f_k(z)} - m\right| + m\alpha, \qquad (1.10)$$

with respect to k-symmetric points, where $f_k = h_k + \overline{g_k} \in H(m)$, $\rho \ge 0, 0 \le \alpha < 1, k \in \mathrm{N}$, $z = re^{i\theta}(r < 1, \theta \in \mathrm{R})$, $z' = \frac{\partial}{\partial\theta}(z)$, $\left(\Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]] f(z)\right)' = \frac{\partial}{\partial\theta}\left(\Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]] f(z)\right)$.

Observe that the functions $f \in H(m)$ satisfying (1.10) must satisfy the condition given by

$$\Re\left\{\left(1+\rho e^{i\phi}\right)\left(\frac{z\left(\Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]] f(z)\right)'}{z' \Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]] f_k(z)}\right) - m\rho e^{i\phi}\right\} > m\alpha, \phi \in \mathrm{R}. \qquad (1.11)$$

We also note that the Goodman-Rønning type class condition of the form (1.11) has been extensively studied earlier in several work. Some of them as follows:

(i) $R_{1,1}([\alpha_1]_{p,q}, [\alpha_1]_{p,q}; 1; \alpha) = G_H([\alpha_1], \alpha)$ studied by Murugusundaramoorthy et al. [22] (see also [18])

(ii) $R_{m,1}([\alpha_1]_{p,q}, [\alpha_1]_{p,q}; 0; \alpha) = S_H^*(m, \alpha_1, \alpha)$ studied by Omar and Halim [23]

(iii) $R_{1,1}([\alpha_1]_{p,q}, [\alpha_1]_{p,q}; 0; \alpha) = S_H^*(\alpha_1, \alpha)$ investigated by Al-Kharsani and Al-Khal [19]

(iv) $R_{1,k}([1]_{1,0}, [1]_{1,0}; 0; \alpha) = SH_s^{(k)}(\alpha)$ studied by Shaqsi and Darus [26]

(v) $R_{1,2}([1]_{1,0}, [1]_{1,0}; 0; \alpha) = SH_s(\alpha)$ studied by Ahuja and Jahangiri in [7] (see also [14])

(vi) $R_{m,1}([1]_{1,0}, [1]_{1,0}; \rho; \alpha) = R(m, \rho, \alpha)$ studied by El-Ashwah [8].

(vii) $R_{m,1}([1]_{1,0}, [1]_{1,0}; 1; \alpha) = G_H(m, \alpha)$ studied by Jahangiri et al. [16] (see also [17])

(viii) $R_{1,1}([1]_{1,0}, [1]_{1,0}; \rho; \alpha) = R_H(\rho, \alpha)$ studied by Ahuja et al. [3]

(ix) $R_{1,1}([1]_{1,0}, [1]_{1,0}; \rho; 0) = R_H(\rho)$ studied by Rosy et al. [29]

(x) $R_{m,1}([1]_{1,0}, [1]_{1,0}; 0; \alpha) = SH(m, \alpha)$ studied by Ahuja and Jahangiri [1] (see also [6], [5])

(xi) $R_{1,1}([1]_{1,0}, [1]_{1,0}; 0; \alpha) = SH(\alpha)$ studied by Jahangiri [15]

(xii) $R_{1,1}([1]_{1,0}, [1]_{1,0}; 0; 0) = SH$ studied by Silverman [27] (see also [28]).

Harmonic functions, associated with more generalized linear operators defined by convolutions, are also considered in [4], [20], [21], [25].

Let $\overset{\circ}{H}(m)$ be a subclass of $H(m)$ whose members $f = h + \overline{g}$, are such that, h and g are of the form

$$h(z) = z^m - \sum_{n=m+1}^{\infty} |a_n| z^n, \; g(z) = \sum_{n=m}^{\infty} |b_n| z^n, |b_m| < 1. \qquad (1.12)$$

We further let

$$\overset{\circ}{R}_{m,k}([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha) = R_{m,k}([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha) \cap \overset{\circ}{H}(m).$$

In this paper, an equivalent convolution class condition is derived and a coefficient inequality is obtained for the functions $f = h + \overline{g} \in H(m)$ to be in the class $R_{m,k}([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha)$. It is also proved that this inequality is necessary for $f = h + \overline{g}$, to be in $\overset{\circ}{R}_{m,k}([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha)$ class. As an application of coefficient inequality a necessary and sufficient hypergeometric inequality is also given. Further, based on the coefficient inequality, results on bounds, extreme points, convolution and convex combination and on an integral operator are obtained.

Throughout the paper, we consider that the parameters involved in the operator $\Theta_{r,s}^{p,q}[[\alpha_1];[\gamma_1]]$ such as $\alpha_i(i = 1, 2, ..p)$, $\gamma_i(i = 1,2,...,r)$ β_i $(i = 1,2,..,q)$, δ_i $(i = 1,2,..,s)$ are positive real and θ_n, ϕ_n given by (1.6) are such that $\theta_n \ge \frac{n}{m}$, $\phi_n \ge \frac{n}{m}(n \ge m)$, and Ψ_n^k is given by (1.9).

2 Convolution Class Condition

In this section, we obtain another class condition equivalent to (1.11) for functions belonging to the class $R_{m,k}([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha)$ with the use of convolutions.

Theorem 1 *Let* $\rho \ge 0, 0 \le \alpha < 1, m, k \in \mathrm{N}$ *and* $f = h + \overline{g} \in H(m)$, *(where* h *and* g *are of the form (1.1)). If* $f \in R_{m,k}([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha)$, *then*

$$\frac{H(z) * H_m^{p,q}[\alpha_1] h(z) - \overline{G(z) * H_m^{r,s}[\gamma_1] g(z)}}{z^m} \ne 0, \qquad (2.1)$$

where for $|\xi| = 1, \xi \neq -1, \phi \in \mathrm{R}, \left|\dfrac{(\xi+1)\rho e^{-i\phi}+\xi+\alpha}{1-\alpha}\right| \leq 1, z \in \Delta,$

$$H(z) = \frac{(\xi+1)(1+\rho e^{i\phi})A(z) - \left\{(\xi+1)\rho e^{i\phi}+\xi+2\alpha-1\right\}\dfrac{z^m}{1-z^k}}{2(1-\alpha)} = z^m + ...$$

$$G(z) = \frac{(\xi+1)(1+\rho e^{-i\phi})A(z) + \left\{(\xi+1)\rho e^{-i\phi}+\xi+2\alpha-1\right\}\dfrac{z^m}{1-z^k}}{2(1-\alpha)}$$

$$= \frac{(\xi+1)\rho e^{-i\phi}+\xi+\alpha}{1-\alpha} z^m + ...$$

Proof. Let $f \in R_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$, then we have

$$\Re\left\{(1+\rho e^{i\phi})\left(\frac{z\left(\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f(z)\right)'}{z'\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f_k(z)}\right) - m\rho e^{i\phi} - m\alpha\right\} > 0, \phi \in \mathrm{R}, z \in \Delta.$$

Since, at $z = 0$

$$\frac{1}{m(1-\alpha)}\left\{(1+\rho e^{i\phi})\left(\frac{z\left(\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f(z)\right)'}{z'\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f_k(z)}\right) - m\rho e^{i\phi} - m\alpha\right\} = 1,$$

we get an equivalent class condition

$$\frac{1}{m(1-\alpha)}\left\{(1+\rho e^{i\phi})\left(\frac{z\left(\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f(z)\right)'}{z'\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f_k(z)}\right) - m\left(\rho e^{i\phi}+\alpha\right)\right\} \neq \frac{\xi-1}{\xi+1} \qquad (2.2)$$

$(|\xi| = 1, \xi \neq -1, 0 < |z| < 1).$ Using (1.2), (1.7) and (1.8) for $f = h + \overline{g} \in H(m)$, we have

$$\frac{z\left(\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f(z)\right)'}{z'} = z\frac{d}{dz}H^{p,q}_m[\alpha_1]h(z) - \overline{z\frac{d}{dz}H^{r,s}_m[\gamma_1]g(z)},$$

$$\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f_k(z) = H^{p,q}_m[\alpha_1]h(z) * \frac{z^m}{1-z^k} + \overline{H^{r,s}_m[\gamma_1]g(z) * \frac{z^m}{1-z^k}}$$

and also,

$$z\frac{d}{dz}H^{p,q}_m[\alpha_1]h(z) = mz^m\left(\frac{1}{1-z} + \frac{z}{m(1-z)^2}\right) * H^{p,q}_m[\alpha_1]h(z),$$

$$z\frac{d}{dz}H^{r,s}_m[\gamma_1]g(z) = mz^m\left(\frac{1}{1-z} + \frac{z}{m(1-z)^2}\right) * H^{r,s}_m[\gamma_1]g(z).$$

Thus, by simple calculations and by denoting $z^m\left[\dfrac{1}{1-z} + \dfrac{z}{m(1-z)^2}\right] = A(z)$, from (2.2), we get

$$\frac{(\xi+1)(1+\rho e^{i\phi})}{2(1-\alpha)z^m}\left\{A(z) * H^{p,q}_m[\alpha_1]h(z) - \overline{A(z) * H^{r,s}_m[\gamma_1]g(z)}\right\}$$

$$-\frac{(\xi+1)(\rho e^{i\phi}+\alpha)+(\xi-1)(1-\alpha)}{2(1-\alpha)z^m}\left\{H^{p,q}_m[\alpha_1]h(z) * \frac{z^m}{1-z^k} + \overline{H^{r,s}_m[\gamma_1]g(z) * \frac{z^m}{1-z^k}}\right\} \neq 0$$

or,

$$\frac{\left((\xi+1)(1+\rho e^{i\phi})A(z) - \left\{(\xi+1)\rho e^{i\phi}+\xi+2\alpha-1\right\}\dfrac{z^m}{1-z^k}\right)}{2(1-\alpha)z^m} * H^{p,q}_m[\alpha_1]h(z)$$

$$-\frac{(\xi+1)(1+\rho e^{i\phi})\overline{A(z)} + \left\{(\xi+1)\rho e^{i\phi}+\xi+2\alpha-1\right\}\overline{\dfrac{z^m}{1-z^k}}}{2(1-\alpha)z^m} * \overline{H^{r,s}_m[\gamma_1]g(z)} \neq 0$$

which equivalently be expressed by (2.1). This proves the result.

3 Coefficient Inequality

Theorem 2 *Let* $\rho \geq 0, 0 \leq \alpha < 1, m,k \in \mathrm{N}$ *If the function* $f = h + \overline{g} \in H(m)$, *(where* h *and* g *are of the form (1.1)). satisfies*

$$\sum_{n=m+1}^{\infty}\frac{(1+\rho)n - m(\alpha+\rho)\Psi^k_n}{m(1-\alpha)}\theta_n|a_n| + \sum_{n=m}^{\infty}\frac{(1+\rho)n + m(\alpha+\rho)\Psi^k_n}{m(1-\alpha)}\phi_n|b_n| \leq 1, \qquad (3.1)$$

then f *is sense-preserving, harmonic multivalent in* Δ *and* $f \in R_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$, *The inequality (3.1) may equivalently be given by*

$$\sum_{l=1}^{\infty}\frac{(1+\rho)(m+lk) - m(\alpha+\rho)}{m(1-\alpha)}\theta_{m+lk}|a_{m+lk}| + \frac{(1+\rho)(m+lk) + m(\alpha+\rho)}{m(1-\alpha)}\phi_{m+lk}|a_{m+lk}| +$$

$$\frac{(1+\alpha+2\rho)}{(1-\alpha)}|b_m| + \sum_{j=1}^{k-1}\sum_{l=0}^{\infty}\frac{(1+\rho)(m+lk+j)}{m(1-\alpha)}\left(\theta_{m+lk+j}|a_{m+lk+j}| + \phi_{m+lk+j}|a_{m+lk+j}|\right) \leq 1. \qquad (3.2)$$

Proof. Under the given parametric constraints, we have

$$\frac{n}{m} \leq \left(\frac{(1+\rho)n - m(\alpha+\rho)\Psi^k_n}{m(1-\alpha)}\right)\theta_n, \ \frac{n}{m} \leq \left(\frac{(1+\rho)n + m(\alpha+\rho)\Psi^k_n}{m(1-\alpha)}\right)\phi_n, n \geq m. \qquad (3.3)$$

Thus, for $f = h + \overline{g} \in H(m)$, *where* h *and* g *are of the form (1.1)*, we get

$$|h'(z)| \geq m|z|^{m-1} - \sum_{n=m+1}^{\infty}n|a_n||z|^{n-1} \geq m|z|^{m-1}\left[1 - \sum_{n=m+1}^{\infty}\frac{n}{m}|a_n|\right]$$

$$\geq m|z|^{m-1}\left[1 - \sum_{n=m+1}^{\infty}\left(\frac{(1+\rho)n - m(\alpha+\rho)\Psi^k_n}{m(1-\alpha)}\right)\theta_n|a_n|\right]$$

$$\geq m|z|^{m-1}\left[\sum_{n=m}^{\infty}\left(\frac{(1+\rho)n + m(\alpha+\rho)\Psi^k_n}{m(1-\alpha)}\right)\phi_n|b_n|\right] > \sum_{n=m}^{\infty}n|b_n||z|^{n-1}$$

$$\geq |g'(z)|$$

which proves that $f(z)$ is sense preserving in Δ. Now to show that $f \in R_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$, we need to show (1.10), that is

$$\Re\left(\frac{\dfrac{z}{z'}\left(\Theta^{p,q}_{r,s}\left([\alpha_1],[\gamma_1]\right)f(z)\right)' - m\alpha\Theta^{p,q}_{r,s}\left([\alpha_1],[\gamma_1]\right)f_k(z)}{\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f_k(z)}\right) -$$

$$\Re\left(\frac{\rho e^{i\eta}\left|\dfrac{z}{z'}\left(\Theta^{p,q}_{r,s}\left[[\alpha_1],[\gamma_1]\right]f(z)\right)' - m\Theta^{p,q}_{r,s}\left[[\alpha_1],[\gamma_1]\right]f_k(z)\right|}{\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f_k(z)}\right) \geq 0, \ z \in \Delta, \qquad (3.4)$$

where $e^{i\eta} = \dfrac{\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f_k(z)}{\left|\Theta^{p,q}_{r,s}\left[[\alpha_1];[\gamma_1]\right]f_k(z)\right|}, \eta \in \mathrm{R}.$

On writing the corresponding series expansions, (3.4) is equivalent to

$$\Re\left(\frac{m(1-\alpha)z^m + \sum_{n=m+1}^{\infty}\left(n - m\alpha\Psi^k_n\right)\theta_n a_n z^n - \sum_{n=m}^{\infty}\left(n + m\alpha\Psi^k_n\right)\phi_n \overline{b_n z^n} - \rho e^{i\eta}K}{z^m + \sum_{n=m+1}^{\infty}\Psi^k_n\theta_n a_n z^n + \sum_{n=m}^{\infty}\Psi^k_n\phi_n\overline{b_n z^n}}\right) \geq 0 \qquad (3.5)$$

where

$$K = \left|\sum_{n=m+1}^{\infty}\left(n - m\Psi^k_n\right)\theta_n a_n z^n - \sum_{n=m}^{\infty}\left(n + m\Psi^k_n\right)\phi_n\overline{b_n z^n}\right|.$$

At $z = 0$, the condition (3.5) is trivial. For $0 \neq z \in \Delta$, the left hand side of (3.5) can be expressed as

$$\Re\left(\frac{m(1-\alpha) + A(z)}{1 + B(z)}\right) = m(1-\alpha)\Re\left(\frac{1+w(z)}{1-w(z)}\right) \qquad (3.6)$$

where

$$A(z) = \sum_{n=m+1}^{\infty}\left(n - m\alpha\Psi^k_n\right)\theta_n a_n z^{n-m} - \sum_{n=m}^{\infty}\left(n + m\alpha\Psi^k_n\right)\phi_n\overline{b_n z^n}z^{-m} - \rho e^{i\eta}Kz^{-m}$$

$$B(z) = \sum_{n=m+1}^{\infty}\Psi^k_n\theta_n a_n z^{n-m} + \sum_{n=m}^{\infty}\Psi^k_n\phi_n\overline{b_n z^n}z^{-m}.$$

Now, we only need to show $|w(z)| \le 1$. From (3.6) we get for $0 \ne z = r < 1$,

$$|w(z)| = \left| \frac{A(z) - m(1-\alpha)B(z)}{2m(1-\alpha) + A(z) + m(1-\alpha)B(z)} \right|$$

$$\le \frac{(1+\rho)\left[\sum_{n=m+1}^{\infty}\left(n - m\alpha\Psi_n^k\right)\theta_n |a_n| + \sum_{n=m}^{\infty}\left(n + m\alpha\Psi_n^k\right)\phi_n |b_n|\right]}{2m(1-\alpha) - \sum_{n=m+1}^{\infty}\left\{n + m\Psi_n^k - 2m\alpha\Psi_n^k + \rho\left(n - m\Psi_n^k\right)\right\}\theta_n |a_n| - \sum_{n=m}^{\infty}\left\{n - m\Psi_n^k + 2m\alpha\Psi_n^k + \rho\left(n + m\Psi_n^k\right)\right\}\phi_n |b_n|}$$

$$\le 1$$

if (3.1) holds. On putting values of $\Psi_n^k, n \ge m$ from (1.9), we get the equivalent coefficient condition (3.2). This proves Theorem 2.

We next show that the above sufficient coefficient condition is also necessary for functions in the class $\mathring{R}_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$

Theorem 3 *Let* $\rho \ge 0, 0 \le \alpha < 1, m, k \in \mathrm{N}$ *and let the function* $f = h + \overline{g} \in H(m)$, *be such that* h *and* g *are given by (1.12). Then* $f \in \mathring{R}_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$, *if and only if (3.1) holds. The inequality (3.1) is sharp for the function given by*

$$f(z) = z^m - \sum_{n=m+1}^{\infty} \frac{m(1-\alpha)}{(1+\rho)n - m(\alpha+\rho)\Psi_n^k} |x_n| z^n + \sum_{n=m}^{\infty} \frac{1-\alpha}{(1+\rho)n + m(\alpha+\rho)\Psi_n^k} |y_n| \overline{z^n}, \quad (3.7)$$

$\sum_{n=m+1}^{\infty} |x_n| + \sum_{n=m}^{\infty} |y_n| = 1.$

Proof. The if part, follows from Theorem 2. To prove the "only if part" let $f = h + \overline{g} \in H(m)$, be such that h and g are given by (1.12) and $f \in \mathring{R}_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$, then from (1.11) on writing the corresponding series expansions, we get

$$\Re\left\{ \frac{m(1-\alpha)z^m - \sum_{n=m+1}^{\infty}\left\{n - m\alpha\Psi_n^k\right\}\theta_n |a_n| z^n - \sum_{n=m}^{\infty}\left\{n + m\alpha\Psi_n^k\right\}\phi_n |b_n| \overline{z^n}}{z^m - \sum_{n=m+1}^{\infty}\Psi_n^k\theta_n |a_n| z^n + \sum_{n=m}^{\infty}\Psi_n^k\phi_n |b_n| \overline{z^n}} \right.$$
$$\left. -\rho e^{i\phi} \frac{\sum_{n=m+1}^{\infty}\left\{n - m\alpha\Psi_n^k\right\}\theta_n |a_n| z^n - \sum_{n=m}^{\infty}\left\{\left(n + m\alpha\Psi_n^k\right)\right\}\phi_n |b_n| \overline{z^n}}{z^m - \sum_{n=m+1}^{\infty}\Psi_n^k\theta_n |a_n| z^n + \sum_{n=m}^{\infty}\Psi_n^k\phi_n |b_n| \overline{z^n}} \right\} \ge 0, \ \phi \in \mathrm{R},$$

from which on choosing z to be real and $z \to 1^-$, and using the fact that $\Re\left(e^{i\phi}\right) \le \left|e^{i\phi}\right| = 1$, we obtain

$$\frac{m(1-\alpha) - \sum_{n=m+1}^{\infty}\left\{(1+\rho)n - m(\alpha+\rho)\Psi_n^k\right\}\theta_n |a_n| - \sum_{n=m}^{\infty}\left\{(1+\rho)n + m(\alpha+\rho)\Psi_n^k\right\}\phi_n |b_n|}{1 - \sum_{n=m+1}^{\infty}\Psi_n^k\theta_n |a_n| + \sum_{n=m+1}^{\infty}\Psi_n^k\phi_n |b_n|} \ge 0$$

which yields the required condition (3.1). Sharpness of the result can easily be verified for the function given by (3.7).

On applying coefficient inequality (3.1), we get a sufficient condition in the form of hypergeometric inequality for certain function $f = h + \overline{g} \in H(m)$, to be in $R_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$ class and it is proved that this inequality is necessary for certain $f \in \mathring{R}_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$, .

Corollary 1 *Let* $\rho \ge 0, 0 \le \alpha < 1, m, k \in \mathrm{N}$ *and let the function* $f = h + \overline{g} \in H(m)$, *where* h *and* g *are of the form (1.1) be such that*

$$|a_n| \le \frac{m(1-\alpha)}{(1+\rho)n - m(\alpha+\rho)\Psi_n^k}, n \ge m+1 \quad (3.8)$$

$$|b_n| \le \frac{m(1-\alpha)}{(1+\rho)n + m(\alpha+\rho)\Psi_n^k}, n \ge m. \quad (3.9)$$

If (in case $p = q+1$*)* $\sum_{i=1}^{q}\beta_i - \sum_{i=1}^{p}\alpha_i > 0$, *and (in case* $r = s+1$*)* $\sum_{i=1}^{s}\delta_i - \sum_{i=1}^{r}\gamma_i > 0$, *the hypergeometric inequality*

$${}_pF_q\left((\alpha_i);(\beta_i);1\right) + {}_rF_s\left((\gamma_i);(\delta_i);1\right) \le 2 \quad (3.10)$$

holds, then $f \in R_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$. *Further, if*

$$f(z) = z^m - \sum_{n=m+1}^{\infty} \frac{m(1-\alpha)}{(1+\rho)n - m(\alpha+\rho)\Psi_n^k} z^n + \sum_{n=m}^{\infty} \frac{m(1-\alpha)}{(1+\rho)n + m(\alpha+\rho)\Psi_n^k} \overline{z^n} \quad (3.11)$$

$\in \mathring{R}_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$, *then (3.10) holds.*

Proof. To prove the result, we need to show by Theorem 2 the inequality:

$$S_1 := \sum_{n=m+1}^{\infty} \frac{(1+\rho)n - m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n |a_n| + \sum_{n=m}^{\infty} \frac{(1+\rho)n + m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n |b_n| \le 1.$$

On using (3.8) and (3.9), and then (3.10), we get

$$S_1 \le \sum_{n=m+1}^{\infty}\theta_n + \sum_{n=m}^{\infty}\phi_n$$
$$= {}_pF_q\left((\alpha_i);(\beta_i);1\right) - 1 + {}_rF_s\left((\gamma_i);(\delta_i);1\right) \le 1$$

where, under the given conditions

$$\sum_{n=m+1}^{\infty}\theta_n = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\beta_i)_n} \frac{1}{n!} - 1 = {}_pF_q\left((\alpha_i);(\beta_i);1\right) - 1$$

$$\sum_{n=m}^{\infty}\phi_n = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(\gamma_i)_n}{\prod_{i=1}^{s}(\delta_i)_n} \frac{1}{n!} = {}_rF_s\left((\gamma_i);(\delta_i);1\right).$$

Further, (3.10) holds by Theorem 3, if $f(z)$ of the form (3.11) belongs to the class $\mathring{R}_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$. This proves the result.

4 Bounds

Our next theorems provide the bounds for the function in the class $\mathring{R}_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$ which are followed by a covering result for this class.

Theorem 4 *Let* $\rho \ge 0, 0 \le \alpha < 1, m, k \in \mathrm{N}$ *if* $f = h + \overline{g} \in \mathring{H}(m)$ *where* h *and* g *are of the form (1.12) belongs to the class* $\mathring{R}_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$ *then for* $|z| = r < 1$,

$$\left|\Theta_{r,s}^{p,q}\left[[\alpha_1];[\gamma_1]\right] f(z)\right| \le \left(1 + |b_m|\right) r^m + \frac{m}{m+1}\left(1 - \frac{1+2\rho+\alpha}{1-\alpha}|b_m|\right) r^{m+1}, \quad (4.1)$$

$$\left|\Theta_{r,s}^{p,q}\left[[\alpha_1];[\gamma_1]\right] f(z)\right| \ge \left(1 - |b_m|\right) r^m - \frac{m}{m+1}\left(1 - \frac{1+2\rho+\alpha}{1-\alpha}|b_m|\right) r^{m+1}. \quad (4.2)$$

The result is sharp.

Proof. Let $f \in \mathring{R}_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \rho; \alpha\right)$, then on using (3.1), related to (1.12), by (1.5), we get for $|z| = r < 1$,

$$\left|\Theta_{r,s}^{p,q}\left[[\alpha_1];[\gamma_1]\right] f(z)\right|$$
$$\le \left(1 + |b_m|\right) r^m + \sum_{n=m+1}^{\infty}\left(\theta_n |a_n| + \phi_n |b_n|\right) r^n \le \left(1 + |b_m|\right) r^m + r^{m+1} \sum_{n=m+1}^{\infty}\left(\theta_n |a_n| + \phi_n |b_n|\right)$$

$$\le (1+|b_m|)r^m + \frac{mr^{m+1}}{m+1}\left(\sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n|a_n| + \frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n|b_n|\right)$$

$$\le (1+|b_m|)r^m + \frac{mr^{m+1}}{m+1}\left(1-\frac{1+2\rho+\alpha}{1-\alpha}|b_m|\right)$$

which proves the result (4.1). The result (4.2) can similarly be obtained. The bounds (4.1) and (4.2) are sharp for the function given by

$$f(z) = z^m + |b_m|\overline{z^m} + \frac{m}{(m+1)\phi_{m+1}}\left(1-\frac{1+2\rho+\alpha}{1-\alpha}|b_m|\right)\overline{z^{m+1}}$$

for $\rho \ge 0, 0 \le \alpha < 1$, $|b_m| < \frac{1-\alpha}{1+2\rho+\alpha}$.

Corollary 2 *Let* $\rho \ge 0, 0 \le \alpha < 1, m,k \in \mathrm{N}$ *If* $f = h+\overline{g} \in \overset{\circ}{H}(m)$ *with* h *and* g *are of the form (1.12) belongs to the class* $\overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$, *then*

$$\left\{\omega : |\omega| < 1-\frac{m}{m+1}+\left(\frac{m(1+2\rho+\alpha)}{(m+1)(1-\alpha)}-1\right)|b_m|\right\} \subset f(\Delta).$$

Theorem 5 *Let* $\rho \ge 0, 0 \le \alpha < 1, m,k \in \mathrm{N}$ *and let* $\delta_{m+1} \le \min(\theta_n,\phi_n)$, $n \ge m+1$. *If* $f = h+\overline{g} \in \overset{\circ}{H}(m)$, *where* h *and* g *are of the form (1.12), belongs to the class* $\overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$ *then for* $|z| = r < 1$,

$$|f(z)| \le (1+|b_m|)r^m + \frac{m}{(m+1)\delta_{m+1}}\left(1-\frac{1+2\rho+\alpha}{1-\alpha}|b_m|\right)r^{m+1}, \quad (4.3)$$

$$|f(z)| \ge (1-|b_m|)r^m - \frac{m}{(m+1)\delta_{m+1}}\left(1-\frac{1+2\rho+\alpha}{1-\alpha}|b_m|\right)r^{m+1}. \quad (4.4)$$

The result is sharp.

Proof. Let $f \in \overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$, then on using (3.1), from (1.12), we get for $|z| = r < 1$,

$$|f(z)| \le (1+|b_m|)r^m + \sum_{n=m+1}^{\infty}(|a_n|+|b_n|)r^n \le (1+|b_m|)r^m + r^{m+1}\sum_{n=m+1}^{\infty}(|a_n|+|b_n|)$$

$$\le (1+|b_m|)r^m + \frac{r^{m+1}}{\delta_{m+1}}\sum_{n=m+1}^{\infty}(\theta_n|a_n|+\phi_n|b_n|)$$

$$\le (1+|b_m|)r^m + \frac{mr^{m+1}}{(m+1)\delta_{m+1}}\left(\sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n|a_n| + \frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n|b_n|\right)$$

$$\le (1+|b_m|)r^m + \frac{mr^{m+1}}{(m+1)\delta_{m+1}}\left(1-\frac{1+2\rho+\alpha}{1-\alpha}|b_m|\right)r^{m+1},$$

which proves (4.3). The result (4.4) can similarly be obtained. The bounds (4.3) and (4.4) are sharp for the function given by

$$f(z) = z^m + |b_m|\overline{z^m} + \frac{m}{(m+1)\delta_{m+1}}\left(1-\frac{1+2\rho+\alpha}{1-\alpha}|b_m|\right)\overline{z^{m+1}}$$

for $|b_m| < \frac{1-\alpha}{1+2\rho+\alpha}$.

Corollary 3 *Let* $\rho \ge 0, 0 \le \alpha < 1, m,k \in \mathrm{N}$ and $\delta_{m+1} \le \min(\theta_n,\phi_n)$, $n \ge m+1$. *If* $f = h+\overline{g} \in \overset{\circ}{H}(m)$ *with* h *and* g *are of the form (1.12) belongs to the class* $\overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$, *then*

$$\left\{\omega : |\omega| < 1-\frac{m}{(m+1)\delta_{m+1}}+\left(\frac{m(1+2\rho+\alpha)}{(m+1)\delta_{m+1}(1-\alpha)}-1\right)|b_m|\right\} \subset f(\Delta).$$

5 Extreme Points

In this section, we determine the extreme points for the class $\overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$

Theorem 6 *Let* $\rho \ge 0, 0 \le \alpha < 1, m,k \in \mathrm{N}$ *let* $f = h+\overline{g} \in \overset{\circ}{H}(m)$ *and*

$$h_m(z) = z^m,\ h_n(z) = z^m - \frac{m(1-\alpha)}{\left((1+\rho)n-m(\alpha+\rho)\Psi_n^k\right)\theta_n}z^n\ (n \ge m+1),$$

$$g_n(z) = z^m + \frac{m(1-\alpha)}{\left((1+\rho)n+m(\alpha+\rho)\Psi_n^k\right)\phi_n}\overline{z^n}\ (n \ge m),$$

then the function $f \in \overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$, *if and only if it can be expressed as* $f(z) = \sum_{n=m}^{\infty}\left(x_n h_n(z) + y_n g_n(z)\right)$ *where* $x_n \ge 0, y_n \ge 0$ *and* $\sum_{n=m}^{\infty}(x_n+y_n) = 1$. *In particular, the extreme points of* $\overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$ *are* $\{h_n\}$ *and* $\{g_n\}$

Proof. Suppose that $f(z) = \sum_{n=m}^{\infty}\left(x_n h_n(z) + y_n g_n(z)\right)$. Then,

$$f(z) = \sum_{n=m}^{\infty}(x_n+y_n)z^m - \sum_{n=m+1}^{\infty}\frac{m(1-\alpha)}{\left((1+\rho)n-m(\alpha+\rho)\Psi_n^k\right)\theta_n}x_n z^n + \sum_{n=m}^{\infty}\frac{m(1-\alpha)}{\left((1+\rho)n+m(\alpha+\rho)\Psi_n^k\right)\phi_n}y_n\overline{z^n}$$

$$= z^m - \sum_{n=m+1}^{\infty}\frac{m(1-\alpha)}{\left((1+\rho)n-m(\alpha+\rho)\Psi_n^k\right)\theta_n}x_n z^n + \sum_{n=m}^{\infty}\frac{m(1-\alpha)}{\left((1+\rho)n+m(\alpha+\rho)\Psi_n^k\right)\phi_n}y_n\overline{z^n}$$

$\in \overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$ by Theorem 3, since,

$$\sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k\theta_n}{m(1-\alpha)}\left(\frac{m(1-\alpha)}{\left((1+\rho)n-m(\alpha+\rho)\Psi_n^k\right)\theta_n}x_n\right)$$

$$+\sum_{n=m}^{\infty}\frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k\phi_n}{m(1-\alpha)}\left(\frac{m(1-\alpha)}{\left((1+\rho)n+m(\alpha+\rho)\Psi_n^k\right)\phi_n}y_n\right)$$

$= \sum_{n=m+1}^{\infty}x_n + \sum_{n=m}^{\infty}y_n = 1-x_m \le 1.$ Conversely, let $f \in \overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$, and let

$$|a_n| = \frac{m(1-\alpha)x_n}{\left((1+\rho)n-m(\alpha+\rho)\Psi_n^k\right)\theta_n},\ |b_n| = \frac{m(1-\alpha)y_n}{\left((1+\rho)n+m(\alpha+\rho)\Psi_n^k\right)\phi_n}$$

and $x_m = 1-\sum_{n=m+1}^{\infty}x_n - \sum_{n=m}^{\infty}y_n$, then, we get

$$f(z) = z^m - \sum_{n=m+1}^{\infty}|a_n|z^n + \sum_{n=m}^{\infty}|b_n|\overline{z}^n$$

$$= h_m(z) - \sum_{n=m+1}^{\infty}\frac{m(1-\alpha)}{\left((1+\rho)n-m(\alpha+\rho)\Psi_n^k\right)\theta_n}x_n z^n + \sum_{n=m}^{\infty}\frac{m(1-\alpha)}{\left((1+\rho)n+m(\alpha+\rho)\Psi_n^k\right)\phi_n}y_n\overline{z^n}$$

$$= h_m(z) + \sum_{n=m+1}^{\infty}(h_n(z)-h_m(z))x_n + \sum_{n=m}^{\infty}(g_n(z)-h_m(z))y_n$$

$$= h_m(z)\left(1-\sum_{n=m+1}^{\infty}x_n - \sum_{n=m}^{\infty}y_n\right) + \sum_{n=m+1}^{\infty}h_n(z)x_n + \sum_{n=m}^{\infty}g_n(z)y_n$$

$$= \sum_{n=m}^{\infty}\left(x_n h_n(z) + y_n g_n(z)\right).$$

This proves Theorem 6.

6 Convolution and Convex Combinations

In this section, we show that the class $\overset{\circ}{R}_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha\right)$ is invariant under convolution and convex combinations of its members.

Let the function $f = h+\overline{g} \in \overset{\circ}{H}(m)$ where h *and* g *are* of the form (1.12) and

$$F(z) = z^m - \sum_{n=m+1}^{\infty}\theta_n|A_n|z^n + \sum_{n=m}^{\infty}\phi_n|B_n|\overline{z^n} \in \overset{\circ}{H}(m). \quad (6.1)$$

The convolution between the functions of the class $\overset{\circ}{H}(m)$ is defined by

$$(f*F)(z) = f(z)*F(z) = z^m - \sum_{n=m+1}^{\infty}\theta_n|a_nA_n|z^n + \sum_{n=m}^{\infty}\phi_n|b_nB_n|\overline{z^n}$$

Theorem 7 *Let* $\rho \ge 0, 0 \le \alpha < 1, m,k \in \mathbb{N}$ *if* $f \in \overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$, *and* $F \in \overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$, *then* $f*F \in \overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$.

Proof. Let $f = h+\overline{g} \in \overset{\circ}{H}(m)$ where h *and* g *are* of the form (1.12) and $F \in \overset{\circ}{H}(m)$ of the form (6.1) be in $\overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$ class. Then by Theorem 3, we have

$$\sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n|A_n| + \sum_{n=m}^{\infty}\frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n|B_n| \le 1$$

which in view of (3.3), yields

$$|A_n| \le \frac{m(1-\alpha)}{((1+\rho)n-m(\alpha+\rho)\Psi_n^k)\theta_n} \le \frac{m}{n} \le 1, n \ge m+1$$

$$|B_n| \le \frac{m(1-\alpha)}{((1+\rho)n+m(\alpha+\rho)\Psi_n^k)\theta_n} \le \frac{m}{n} \le 1, n \ge m.$$

Hence, again by Theorem 3,

$$\sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n|a_nA_n| + \sum_{n=m}^{\infty}\frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n|b_nB_n|$$

$$\le \sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n|a_n| + \sum_{n=m}^{\infty}\frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n|b_n| \le 1$$

which proves that $f*F \in \overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$.

We prove next that the class $\overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$ is closed under convex combination of its members.

Theorem 8 *Let* $\rho \ge 0, 0 \le \alpha < 1, m,k \in \mathbb{N}$ *the class* $\overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$ *is closed under convex combination.*

Proof. Let $f_j \in \overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$, $j \in \mathbb{N}$ be of the form

$$f_j(z) = z^m - \sum_{n=m+1}^{\infty}|A_{j,n}|z^n + \sum_{n=m}^{\infty}|B_{j,n}|\overline{z^n}, j \in \mathbb{N}.$$

Then by Theorem 3, we have for $j \in \mathbb{N}$,

$$\sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n|A_{j,n}| + \sum_{n=m}^{\infty}\frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n|B_{j,n}| \le 1. \quad (6.2)$$

For some $0 \le t_j \le 1$, let $\sum_{j=1}^{\infty} t_j = 1$, the convex combination of $f_j(z)$ may be written as

$$\sum_{j=1}^{\infty}t_j\ f_j(z) = z^m - \sum_{n=m+1}^{\infty}\sum_{j=1}^{\infty}t_j|A_{j,n}|z^n + \sum_{n=m}^{\infty}\sum_{j=1}^{\infty}t_j|B_{j,n}|\overline{z^n}$$

Now by (6.2),

$$\sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n\sum_{j=1}^{\infty}t_j|A_{j,n}| + \sum_{n=m}^{\infty}\frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n\sum_{j=1}^{\infty}t_j|B_{j,n}|$$

$$= \sum_{j=1}^{\infty}t_j\left[\sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n|A_{j,n}| + \sum_{n=m}^{\infty}\frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n|B_{j,n}|\right] \le \sum_{j=1}^{\infty}t_j = 1$$

and so again by Theorem 3, we get $\sum_{j=1}^{\infty}t_j\ f_j(z) \in \overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$. This proves the result.

Remark 1 *Our results of Theorems 2-8 coincide with the results obtained in* [1] *and* [15] *for the classes* $SH(m,\alpha)$ *and* $SH(\alpha)$, *respectively. Also, by taking* $1-\frac{1}{m} \le \alpha < 1$, $\theta_n = A_n$, $\phi_n = B_n, n \ge m$ *in the Theorems 2-8, our results coincide with the results of Ahuja et al. obtained in* [4].

7 Integral Operator

Now we examine a closure property of the class $\overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$ involving the generalized Bernardi Libera-Livingston Integral operator $L_{m,c}$ which is defined for $f = h+\overline{g} \in \overset{\circ}{H}(m)$ by

$$L_{m,c}(f) = \frac{c+m}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+m}{z^c}\int_0^z t^{c-1}g(t)dt}, c > -m, z \in \Delta. \quad (7.1)$$

Theorem 9 *Let* $\rho \ge 0, 0 \le \alpha < 1, m,k \in \mathbb{N}$ *if* $f \in \overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$, *then* $L_{m,c}(f) \in \overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$.

Proof. Let $f = h+\overline{g} \in \overset{\circ}{H}(m)$ where h *and* g *are* of the form (1.12), belongs to the class $\overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$ Then, it follows from (7.1) that

$$L_{m,c}(f) = z^m - \sum_{n=m+1}^{\infty}\left(\frac{c+m}{c+n}\right)|a_n|z^n + \sum_{n=m}^{\infty}\left(\frac{c+m}{c+n}\right)|b_n|\overline{z^n} \in \overset{\circ}{R}_{m,k}([\alpha_1]_{p,q},[\gamma_1]_{r,s};\rho;\alpha)$$

by (3.1), since,

$$\sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\left(\frac{c+m}{c+n}\right)\theta_n|a_n| + \sum_{n=m}^{\infty}\frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\left(\frac{c+m}{c+n}\right)\phi_n|b_n|$$

$$\le \sum_{n=m+1}^{\infty}\frac{(1+\rho)n-m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\theta_n|a_n| + \sum_{n=m}^{\infty}\frac{(1+\rho)n+m(\alpha+\rho)\Psi_n^k}{m(1-\alpha)}\phi_n|b_n| \le 1.$$

This proves the result.

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