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Modelling effect of the depleting dissolved oxygen on the existence of interacting planktonic population

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ABSTRACT

In this paper, a mathematical model is proposed to study the effect of the depleting dissolved oxygen on the existence of interacting planktonic population. The mathematical model is formulated using the system of nonlinear ordinary differential equations. The model includes four state variables viz., nutrient concentration, density of algae, and density of the zooplankton population and concentration of dissolved oxygen. All the feasible equilibria of the system are obtained and the conditions for the existence of the interior equilibrium are determined. The local stability analyses of all the feasible equilibrium points are obtained. The non-linear stability analysis of the non trivial equilibrium point has been carried out and the criteria for the survival or extinction of the species have been obtained with numerical simulation.

Introduction

Excessive and indiscriminate uses of organic fertilizers often lead to accumulation of nitrates in water. The phosphate, when enters into water bodies support luxuriant growths of algal, resulting in the depletion of dissolved oxygen content and deterioration of water resources caused by eutrophication. Eutrophication is a process by which a waterbody becomes enriched in dissolved nutrients (e.g. nitrogen, phosphate etc.) that stimulate the growth of aquatic plant life and resulting in the depletion of dissolved oxygen (DO).

In the recent years several investigators have studied the effect of nutrients in aquatic system such as a lake causing eutrophication [1, 2, 4, 9, 10, 11, 12]. Arnold and Voss [6] studied the eutrophication in lakes with numerical behaviour. Khare, Misra and Dhar [14] have studied the effect of soil pollutant on the plant-herbivore interacting system by considering nutrients, plant, herbivore and soil pollutant as variables. Some other ecological modeling studies involving phytoplankton, zooplankton and nutrients relevant to our work, have also been conducted by many researcher but they have not considered the concentration of DO in the modeling process [5, 6, 7, 13, 14, 15, 16]. Many scientists [2, 3, 4, 9, 13, 17] have studied the depletion of dissolved oxygen on planktonic ecosystem. Naik and Manjapp [17] have studied the prediction of dissolved oxygen through mathematical modeling but they have not considered the effect of oxygen deficit on the algae and zooplankton population.

Keeping in view of the above, in this paper, we have studied the effect of the depleting dissolved oxygen on the existence of interacting planktonic population.

Mathematical Model

Let n be the cumulative concentration of various nutrients, a be the density of algae, P be the density of the zooplankton population, and C be the concentration of dissolved oxygen. We assume that the cumulative rate of discharge of nutrients into the aquatic system from outside in the water body is q, a constant which is depleted with rate (αn) due to natural factors. It is further assumed that the depletion of nutrients by algae is proportional to both the density of algae as well as the concentration of nutrient (i.e. na). It is further considered that the growth rate of algae is proportional to the terms $na/(\alpha_1+C_o-C)$. The natural depletion rate of algae and zooplankton are ν_1, ν_3 respectively. α_2 is rate of predation of algae by zooplankton. We consider that the rate of growth of dissolved oxygen by various sources is q_o assumed to be a constant and ν_2 is natural depletion rate of concentration C. It is further assumed that the growth rate of zooplankton is proportional to the terms $aP/(\alpha_4+C_o-C)$. α_1, α_4 are half saturation constants, C_o is DO saturation value and C_o - C is oxygen deficit.

In view of the above considerations, the system is governed by the differential equations:-

$$\frac{dn}{dt} = q - \alpha n - \beta_1 na \qquad (1)$$

$$\frac{da}{dt} = \frac{\beta_2 na}{(\alpha_1 + C_o - C)} - \nu_1 a - \alpha_2 aP \qquad (2)$$

$$\frac{dC}{dt} = q_0 - \nu_2 C \qquad (3)$$

$$\frac{dP}{dt} = \frac{\alpha_3 aP}{(\alpha_4 + C_0 - C)} - \nu_3 P \qquad (4)$$

With the initial conditions $n(0) = n_{10} > 0$, $a(0) = a_{10} > 0$, $C(0) = C_{10} > 0$, $P(0) = P_{10} > 0$.

Here α, ν_1, ν_2, ν_3 are depletion rate coefficients, β_1, β_2, α_2 and α_3 are proportionality constants which are positive.

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Boundedness and Equilibria of the System

In this section, we will establish that the system (2.2.1) – (2.2.4) is bounded. We begin with the following lemma.

Lemma 1: The set

$$\Omega = \left\{ (n,a,C,P) \in R_4^{\,+} : 0 \le n+a+P \le \frac{q}{\delta_m}, C \le \frac{q_0}{v_2} \right\}$$

is a region of attraction for all solutions initiating in the interior of positive octant, where, δ_m = Min {α, ν$_1$, ν$_3$}.

Proof: Let us consider the following function:

$$w(t) = n(t) + a(t) + P(t), \qquad (5)$$

$$\frac{dw}{dt} = \frac{dn}{dt} + \frac{da}{dt} + \frac{dP}{dt},$$

from model (1) – (4) and if δ_m = Min{α, ν$_1$, ν$_3$}, then we obtain the following expression:-

$$\frac{dw(t)}{dt} + \delta_m w(t) \le q$$

Now applying the theorem of differential inequalities [8], we obtain

As t ⟶ ∞, we have

$$0 \le w(t) \le \frac{q}{\delta_m}, \quad \Rightarrow \quad 0 \le n+a+P \le \frac{q}{\delta_m}$$

From equation (3), we have

$$\frac{dC}{dt} + v_2 C = q_0,$$

As t ⟶ ∞, we have

$$C(t) \le \frac{q_0}{v_2},$$

Hence, the solution of the system (1) – (4) is bounded in Ω.

The model (1) – (4) has three non-negative equilibria.

(i) $E_1\left(\frac{q}{\alpha}, 0, \frac{q_0}{v_2}, 0\right)$ always exist

(ii) $E_2(\bar{n}, \bar{a}, \bar{C}, 0)$, Where

$$\bar{n} = \frac{v_1}{\beta_2 v_2}\left(\alpha_1 v_2 + C_0 v_2 - q_0\right),$$

$$\bar{a} = \frac{q\beta_2 v_2 - \alpha v_1\left(\alpha v_2 + C_0 v_2 - q_0\right)}{\beta_1 v_1\left(\alpha_1 v_2 + C_0 v_2 - q_0\right)}, \quad \bar{C} = \frac{q_0}{v_2},$$

Thus, E$_2$ exist if

$\alpha_1 v_2 + C_0 v_2 - q_0 > 0$,

$q\beta_2 v_2 - \alpha v_1(\alpha_1 v_2 + C_0 v_2 - q_0) > 0$, $\alpha_1 + C_0 - \bar{C} > 0$, $q - \alpha\bar{n} > 0$,

$v_3\left(\alpha_4 + C_0 - \bar{C}\right) - \alpha_3\bar{a} > 0$

(iii) $E_3(n^*, a^*, C^*, P^*)$, Where

$$C^* = \frac{q_0}{v_2}, \; a^* = \frac{v_3}{\alpha_3 v_2}\left(\alpha_4 v_2 + C_0 v_2 - q_0\right),$$

$$n^* = \frac{q\alpha_3 v_2}{\left[\alpha\alpha_3 v_2 + \beta_1 v_3\left(\alpha_4 v_2 + C_0 v_2 - q_0\right)\right]},$$

$$P^* = \frac{q\alpha_3 v_2^2 \beta_2}{\alpha_2\left(\alpha_1 v_2 + C_0 v_2 - q_0\right)\left[\alpha\alpha_3 v_2 + \beta_1 v_3\left(\alpha_4 v_2 + C_0 v_2 - q_0\right)\right]} - \frac{v_1}{\alpha_2},$$

Thus, E$_3$ exist if

$\alpha_4 v_2 + C_0 v_2 - q_0 > 0$, $\beta_2 n^* - v_1(\alpha_1 + C_0 - C^*) > 0$, $\alpha_4 + C_0 - C^* > 0$, $\alpha_1 + C_0 - C^* > 0$

Dynamical Behaviour of the System

In this section, we will discuss the stability analysis of equilibria E$_1$, E$_2$ and E$_3$.

The variational matrix of the system (1) - (4) is given as follow:-

$$J_i = \begin{bmatrix} -\alpha - \beta_1 a & -\beta_1 n & 0 & 0 \\ \frac{\beta_2 a}{(\alpha_1 + C_0 - C)} & \frac{\beta_2 n}{(\alpha_1 + C_0 - C)} - v_1 - \alpha_2 P & \frac{\beta_2 na}{(\alpha_1 + C_0 - C)^2} & -\alpha_2 a \\ 0 & 0 & -v_2 & 0 \\ 0 & \frac{\alpha_3 P}{(\alpha_4 + C_0 - C)} & \frac{\alpha_3 aP}{(\alpha_4 + C_0 - C)^2} & \frac{\alpha_3 a}{(\alpha_4 + C_0 - C)} - v_3 \end{bmatrix}$$

Now, corresponding to the equilibrium point E$_1$, Jacobean J$_1$ is -

$$J_1 = \begin{bmatrix} -\alpha & -\frac{\beta_1 q}{\alpha} & 0 & 0 \\ 1\ 0 & \frac{\beta_2 q v_2}{\alpha\left(\alpha_1 v_2 + C_0 v_2 - q_0\right)} - v_1 & 0 & 0 \\ 0 & 0 & -v_2 & 0 \\ 0 & 0 & 0 & -v_3 \end{bmatrix}$$

J$_1$ has the Eigen-values λ$_1$ = -α, λ$_2$ = -ν$_2$, λ$_3$ = -ν$_3$ and $\lambda_4 = \frac{\beta_2 q v_2 - v_1\alpha(\alpha_1 v_2 + C_0 v_2 - q_0)}{\alpha(\alpha_1 v_2 + C_0 v_2 - q_0)}$ Hence, E$_1$ is stable if

$(\alpha\alpha_1 v_1 v_2 + C_0 v_1 v_2 \alpha) > (\beta_2 q v_2 + v_1 \alpha q_0)$.

Variation matrix corresponding to the equilibrium point E$_2$ is,

$$J_2 = \begin{bmatrix} -(\alpha + \beta_1\bar{a}) & -\beta_1\bar{n} & 0 & 0 \\ \frac{\beta_2\bar{a}}{(\alpha_1 + C_0 - \bar{C})} & \frac{\beta_2\bar{n}}{(\alpha_1 + C_0 - \bar{C})} - v_1 & \frac{\beta_1\bar{n}\bar{a}}{(\alpha_1 + C_0 - \bar{C})^2} & -\alpha_2\bar{a} \\ 0 & 0 & -v_2 & 0 \\ 0 & 0 & 0 & \frac{\alpha_3\bar{a}}{(\alpha_4 + C_0 - \bar{C})} - v_3 \end{bmatrix}$$

Using (1) – (4), above Jacobean Converts to

$$J_2 = \begin{bmatrix} -a_{11} & -\beta_1\bar{n} & 0 & 0 \\ a_{21} & 0 & a_{23} & -\alpha_2\bar{a} \\ 0 & 0 & -v_2 & 0 \\ 0 & 0 & 0 & \frac{\alpha_3\bar{a}}{(\alpha_4 + C_0 - \bar{C})} - v_3 \end{bmatrix}$$

Where, $a_{11} = \alpha + \beta_1\bar{a}, a_{23} = \frac{\beta_2\bar{n}\bar{a}}{(\alpha_1 + C_0 - \bar{C})^2}, a_{21} = \frac{v_1\bar{a}}{\bar{n}}$

Characteristic equation corresponding to the above Jacobean is

$$(-v_2 - \lambda)\left(\frac{\alpha_3\bar{a}}{(\alpha_4 + C_0 - \bar{C})} - v_3 - \lambda\right)\left[\lambda^2 + a_{11}\lambda + a_{21}\bar{n}\beta_1\right] = 0$$

and the Eigen-values corresponding to the above Jacobean J$_2$

$$\lambda_1 = -v_2, \lambda_2 = -\left[\frac{v_3\left(\alpha_4 + C_0 - \bar{C}\right) - \alpha_3\bar{a}}{\left(\alpha_4 + C_0 - \bar{C}\right)}\right]$$

$$\lambda = \frac{-a_{11} \pm \sqrt{a_{11}{}^2 - 4\beta_1 a_{21}\bar{n}}}{2}$$

$$\lambda_3 = -\left(\frac{a_{11} + \sqrt{a_{11}{}^2 - 4a_{21}\bar{n}\beta_1}}{2}\right)$$

$$\lambda_4 = -\left(\frac{a_{11} - \sqrt{a_{11}{}^2 - 4a_{21}\bar{n}\beta_1}}{2}\right)$$

Now, using Routh-hurwitz criterion we have shown that E_2 is asymptotically stable.

Now, we will examine the local behavior of the equilibrium point $E_3(n^*, a^*, C^*, P^*)$. The Jacobean matrix corresponding to the equilibrium point E_3 as,

$$J_3 = \begin{bmatrix} -(\alpha+\beta_1 a^*) & -\beta_1 n^* & 0 & 0 \\ \frac{\beta_2 a^*}{(\alpha_1+C_0-C^*)} & \frac{\beta_2 n^*}{(\alpha_1+C_0-C^*)}-\nu_1-\alpha_2 P^* & \frac{\beta_2 n^* a^*}{(\alpha_1+C_0-C^*)^2} & -\alpha_2 a^* \\ 0 & 0 & -\nu_2 & 0 \\ 0 & \frac{\alpha_3 P^*}{(\alpha_4+C_0-C^*)} & \frac{\alpha_3 a^* P^*}{(\alpha_4+C_0-C^*)^2} & \frac{\alpha_3 a^*}{(\alpha_4+C_0-C^*)}-\nu_3 \end{bmatrix}$$

using (1) – (4), above Jacobean converts to

$$J_3 = \begin{bmatrix} -a_{11} & -\beta_1 n^* & 0 & 0 \\ a_{21} & 0 & a_{23} & -\alpha_2 a^* \\ 0 & 0 & -\nu_2 & 0 \\ 0 & \frac{\nu_3 P^*}{a^*} & a_{43} & 0 \end{bmatrix}$$

Where,

$$a_{11} = \alpha + \beta_1 a^*, a_{21} = \frac{a^*(\nu_1+\alpha_2 P^*)}{n^*}, a_{23} = \frac{\beta_2 n^* a^*}{(\alpha_1+C_0-C^*)^2},$$

$$a_{43} = \frac{\alpha_3 a^* P^*}{(\alpha_4+C_0-C^*)^2}$$

Characteristic equation corresponding to the above Jacobean is –

$$(-\nu_2-\lambda)\left[-\lambda^3 - a_{11}\lambda^2 - (\alpha_2\nu_3 P^* + a_{21} n^* \beta_1)\lambda - a_{11}\alpha_2\nu_3 P^*\right] = 0$$

From the matrix J_3, it is easy to note that the one Eigen-value of J_3 is $-\nu_2$ and other three Eigen-values are obtained by the following equation

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \qquad (6)$$

where,

$b_1 = a_{11} > 0$, $b_2 = \alpha_2\nu_3 P^* + a_{21} n^* \beta_1 > 0$, $b_3 = a_{11}\alpha_2\nu_3 P^* > 0$, $b_1 b_2 - b_3 = a_{11} a_{21} n^* \beta_1 > 0$

Now, using the Routh-hurwitz criterion we have shown $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ and $b_1 b_2 - b_3 > 0$ are satisfied. Thus, equilibrium point E_3 is asymptotically stable.

Now, form the following theorem we will discuss the nonlinear stability analysis of the equilibrium E_3 which has been studied by Lyapunovs direct method.

Theorem 1: The equilibria E_3 is non linearly stable in Ω, if the following conditions are satisfied,

$$\left[\frac{m_1\beta_2\nu_2}{(\alpha_1\nu_2 + C_0\nu_2 - q_0)} - \beta_1 n^*\right]^2 < \frac{2m_1\alpha\alpha_2}{3}$$

$$\left[\frac{m_1 n^* \beta_2}{(\alpha_1\nu_2 + C_0\nu_2 - q_0)(\alpha_1 + C_0 - C^*)}\right]^2 \nu_2 < \frac{m_1 m_2 \alpha_2}{3}$$

$$\left[\frac{m_3\alpha_2\nu_2}{(\alpha_4\nu_2 + C_0\nu_2 - q_0)}\right]^2 < \frac{m_1^2\alpha_2^2}{6}$$

$$\left[\frac{m_3\alpha_3 a^*}{(\alpha_4\nu_2 + C_0\nu_2 - q_0)(\alpha_4 + C_0 - C^*)}\right]^2 \nu_2 < \frac{m_1 m_2 \alpha_2}{2}$$

Proof: We consider the following positive definite function:

$$V = \frac{1}{2}(n-n^*)^2 + m_1\left(a - a^* - a^* In\frac{a}{a^*}\right) + \frac{1}{2}m_2(C-C^*)^2 + m_3\left(P - P^* - P^* In\frac{P}{P^*}\right)$$

Where m_1, m_2 and m_3 are positive constants, to be chosen appropriately,

$$\frac{dV}{dt} = (n-n^*)\frac{dn}{dt} + m_1\frac{(a-a^*)}{a}\frac{da}{dt} + m_2(C-C^*)\frac{dC}{dt} + m_3\frac{(P-P^*)}{P}\frac{dP}{dt}$$

$$\frac{dV}{dt} = Z_1\frac{dn}{dt} + m_1\frac{Z_2}{a}\frac{da}{dt} + m_2 Z_3\frac{dC}{dt} + m_3\frac{Z_4}{P}\frac{dP}{dt}$$

We assume $Z_1 = (n - n^*)$, $Z_2 = (a - a^*)$, $Z_3 = (C - C^*)$, $Z_4 = (P - P^*)$

Using (1) – (4) and the inequality $a^2 + b^2 \geq 2ab$, then some algebraic manipulations $\frac{dV}{dt}$ reduces in the following form:

$$\frac{dV}{dt} \leq -\beta_1 a Z_1^2$$

$$-\frac{1}{2}2\alpha Z_1^2 + \left[\frac{m_1\beta_2}{(\alpha_1+C_0-C)} - \beta_1 n^*\right]Z_1 Z_2 - \frac{1}{2}\frac{m_1\alpha_2}{3}Z_2^2$$

$$-\frac{1}{2}\frac{m_1\alpha_2}{3}Z_2^2 + \frac{n^* m_1\beta_2}{(\alpha_1+C_0-C)(\alpha_1+C_0-C^*)}Z_2 Z_3 - \frac{1}{2}m_2\nu_2 Z_3^2$$

$$-\frac{1}{2}\frac{m_1\alpha_2}{3}Z_2^2 + \frac{m_3\alpha_3}{(\alpha_4+C_0-C)}Z_2 Z_4 - \frac{1}{2}\frac{m_1\alpha_2}{2}Z_4^2$$

$$-\frac{1}{2}m_2\nu_2 Z_3^2 + \frac{m_3\alpha_3 a^*}{(\alpha_4+C_0-C)(\alpha_4+C_0-C^*)}Z_3 Z_4 - \frac{1}{2}\frac{m_1\alpha_2}{2}Z_4^2$$

$$\frac{dV}{dt} \leq -\beta_1 a Z_1^2$$

$$-\frac{1}{2}P_{11}Z_1^2 + P_{12}Z_1 Z_2 - \frac{1}{2}P_{22}Z_2^2$$

$$-\frac{1}{2}P_{22}Z_2^2 + P_{23}Z_2 Z_3 - \frac{1}{2}P_{33}Z_3^2$$

$$-\frac{1}{2}P_{22}Z_2^2 + P_{24}Z_2 Z_4 - \frac{1}{2}P_{44}Z_4^2$$

$$-\frac{1}{2}P_{33}Z_3^2 + P_{34}Z_3 Z_4 - \frac{1}{2}P_{44}Z_4^2$$

Where,

$$P_{11} = 2\alpha, P_{22} = \frac{m_1\alpha_2}{3}, P_{33} = m_2\nu_2, P_{44} = \frac{m_1\alpha_2}{2}$$

$$P_{23} = \frac{n^* m_1\beta_2}{(\alpha_1+C_0-C)(\alpha_1+C_0-C^*)}, P_{24} = \frac{m_3\alpha_3}{(\alpha_4+C_0-C)},$$

$$P_{34} = \frac{m_3\alpha_3 a^*}{(\alpha_4+C_0-C^*)(\alpha_4+C_0-C)}, P_{12} = \left[\frac{m_1\beta_2}{(\alpha_1+C_0-C)} - \beta_1 n^*\right]$$

Thus, sufficient conditions for $\frac{dV}{dt}$ to be negative definite in Ω are that the following inequalities hold:

$$P_{12}^2 < P_{11}.P_{22}, P_{23}^2 < P_{22}.P_{33},$$

$$P_{24}^2 < P_{22}.P_{44}, P_{34}^2 < P_{33}.P_{44}.$$

Hence, V is a lyapunov's function with respect to E_3 whose domain contains the region of attraction Ω, proving the theorem.

Numerical Simulation

To check the feasibility of our analysis regarding stability conditions, we have conducted some numerical computation using MATLAB by choosing the following set of parameter values in model system (1) – (4).

$q = 3$, $\beta_1 = 0.5$, $\beta_2 = 0.35$, $\alpha_1 = 0.51$, $\alpha = 0.1$, $\nu_1 = 0.009$, $\alpha_2 = 0.41$, $\nu_2 = 3$, $\alpha_3 = 0.33$, $\alpha_4 = 0.3$, $q_0 = 24$, $C_0 = 30$, $\nu_3 = 0.01$.

It is found that under the above set of parameters, conditions for the existence of interior equilibrium $E_3(n^*, a^*, C^*, P^*)$ are satisfied and E_3 is given by

$n^* = 6.8191$, $a^* = 0.6799$, $C^* = 7.9994$,
$P^* = 0.2367$.

With the above values of parameters, we have seen that all the conditions of nonlinear stability analysis are satisfied.

In figure1, we observed that the interior equilibrium point is asymptotically stable. Form figure1, concentration of dissolved oxygen are fixed, nutrients increases, while density of algae and zooplankton population decreases, due to the oxygen deficit. It is further noted that all the stability conditions satisfied for the above values of parameters showing the local and nonlinear stability behavior of E_3.

Time Series Graph

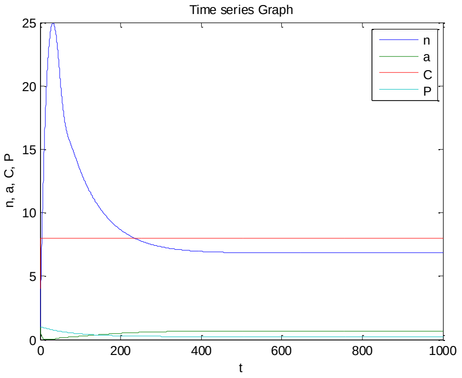

Figure 1

Conclusion

In this paper, we have proposed and analyzed the mathematical model of the algal bloom in aquatic system. The model exhibits three non-zero equilibria E_1, E_2 and E_3. From the stability analysis of E_1, we have seen that E_1 is locally stable if equilibrium point E_2 does not exist. From the stability analysis of the system (1) – (4), we have observed that all the feasible equilibria has been locally stable under certain conditions. We have studied the nonlinear stability analysis of interior equilibrium E_3 by Lyapunovs direct method.

By numerical solution of the model, It has been shown that concentration of dissolved oxygen are fixed, while the cumulative rate of input of nutrients increases. Due to the oxygen deficit, density of algae and zooplankton population will be decreases. Finally, dissolved oxygen, nutrients, algae and zooplankton will make a stable relationship.

From the figure 1, it can be seen that the concentration of nutrient, density of algae, concentration of DO and zooplankton populations all reach to their equilibrium values as time passes.

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