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Cubic graphs with equal complementary tree domination number and chromatic number

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ABSTRACT

A set D of vertices in a graph G = (V, E) is called a dominating set of G, if every vertex in V-D is adjacent to some vertex in D. The domination number γ (G) of G is the minimum cardinality of a dominating set in G. A dominating set D of a graph G is a complementary tree dominating set, if the induced subgraph <V-D> is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$.The chromatic number of G is the minimum number of colours assigned to the vertices of G such that no two adjacent vertices have the same colour and is denoted by χ (G).In this paper we investigate cubic graphs for which $\gamma_{ctd}(G) = \chi(G)$.

1. Introduction

Graphs considered here are nontrivial, simple, finite, connected and undirected. Let G be a graph with vertex set V(G) and edge set E(G). A graph with p vertices and q edges is denoted by G(p, q). In general, we use <X> to denote the subgraph induced by the set X of vertices. The concept of domination was first studied by Ore [4]. A set $D \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V(G) - D$ is adjacent to some vertex in D and D is said to be a minimal dominating set if D-{u} is not a dominating set for any $u \in D$. The domination number γ (G) of G is the minimum cardinality of a dominating set.

The concept of complementary tree domination was introduced by Muthammai, Bhanumathi and Vidhya[3]. A dominating set $D \subseteq V$ is called a complementary tree dominating set (ctd-set), if the sub graph $< V - D >$ induced by $V - D$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. We call a set of vertices a γ-set, if it is a dominating set with cardinality γ (G). Similarly a γ_{ctd}-set is defined. A colouring of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colour. An n-colouring of a graph G uses n colours. The chromatic Number χ (G) is defined to be the minimum n, for which G has an n-colouring.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. There are several papers in which graphs with equal parameters are investigated. In[5], Paulraj Joseph investigated cubic graphs whose domination number equals chromatic number. Motivated by the above, we now took the problem of characterizing the graphs for which complementary tree domination number equals to chromatic number. For terminology and notations not specifically defined here we refer reader to [1]. For more details about domination number and its related parameters, we refer to [2].

In this paper, we investigate cubic graphs whose complementary tree domination number equals chromatic number.

2. Prior Results

Theorem 2.1. [1] If G is neither a complete graph nor an odd cycle, then $\chi(G) \le \Delta(G)$

Theorem 2.2. [1] For any connected graph G, $\chi(G) \le \Delta(G) + 1$.

Theorem 2.3. [2]. If G is a graph of order p, with maximum degree Δ, then $\gamma(G) \le \left\lceil \frac{p}{\Delta+1} \right\rceil$

Theorem 2.4.[5].

For a connected graph G, $\gamma(G) = \chi(G) = 2$ if and only if

(i) G is bipartite with bipartition (X, Y) and

(ii) $|X| = 2$ (or)

there exist x in X, y in Y such that N(x) = Y and N(y) = X (or) there exist x in X and y in Y such that N(x) = Y – {y} and N(y) = X –{x}.

3. Main Results

In the following we find the cubic graphs with equal complementary tree domination number and chromatic number. In analogous to Theorem 2.4., we state the following theorem.

Theorem 3.1.

For a connected graph G, with at least 3 vertices, $\gamma_{ctd}(G) = \chi(G) = 2$ if and only if G is bipartite with bipartition (X, Y) such that there exist $x \in X$, $y \in Y$ with V(G) – {x, y} is a tree and either N(x) = Y and N(y) = X (or) N(x) = Y – {y} and N(y) = X – {x}.

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Let G = (V, E) be a connected cubic graph of order p with $\gamma_{ctd}(G) = \chi(G)$.

But $\chi(G) \leq \Delta(G)$ implies $\chi(G) \leq 3$. Clearly $\chi(G) \neq 1$. Theorem 3.1., characterizes graphs for with $\gamma_{ctd}(G) = \chi(G) = 2$. We consider the cubic graphs for which $\gamma_{ctd}(G) = \chi(G) = 3$.

But, $\gamma_{ctd}(G) \geq \left\lfloor \frac{p}{\Delta+1} \right\rfloor = \left\lfloor \frac{p}{4} \right\rfloor$. Since $\gamma_{ctd}(G) = 3$, $6 < p \leq 15$ and $p \neq 14$. Also, p is even, since G is cubic. Therefore, p = 8, 10 (or) 12.

Cubic Graphs of Order 8

Theorem 3.2.

Let G be a connected cubic graph on 8 vertices. Then $\gamma_{ctd}(G) = \chi(G) = 3$, if and only if G is isomorphic to any of the six graphs G_1, G_2, G_3, G_4 and G_5 given in Figure 1.

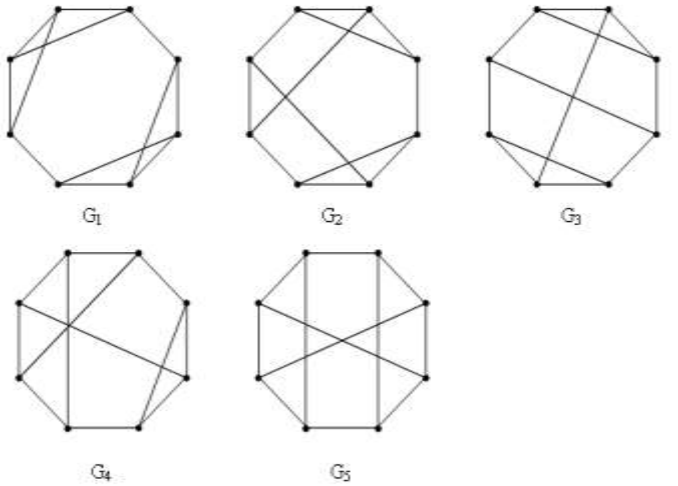


Figure 1

Proof:

Let G be a connected cubic graph on 8 vertices such that $\gamma_{ctd}(G) = \chi(G) = 3$.

Let D = {u, v, w} be a minimum ctd- set of G and let V – D = $\{x_1, x_2, x_3, x_4, x_5\}$. It is clear that $\langle D\rangle \ncong K_3$. If $\langle D\rangle \cong K_3$, then there exist atleast two vertices in V – D not adjacent to any of the vertices in D. Since $\langle V - D\rangle$ is a tree, either $\langle V - D\rangle$ is a path on 5 vertices or $\langle V - D\rangle \cong T$, where T is the tree obtained from the path on four vertices by attaching a pendant edge at any one of the supports. Second case is not possible, since there is a vertex of degree 3 in $\langle V - D\rangle$ and it cannot be adjacent to any of the vertices in D. Therefore, $\langle V - D\rangle$ is a path on 5 vertices. Let x_1, x_2, x_3, x_4, x_5 be the vertices of $\langle V - D\rangle$ and let

$E(\langle V - D\rangle) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5)\}$

Case 1. $\langle D\rangle \cong \overline{K}_3$

Let u be adjacent to x_1, x_2, x_3. Then v is adjacent to x_1, x_4, x_5 and w is adjacent to x_5 only. Hence, $\deg_G(w) = 1$, which is not possible. Similarly, if u is adjacent to x_2, x_3, x_4 (or) x_1, x_2, x_5, then at least one of the vertices v and w has degree less than 3 in G. Therefore, $\langle D\rangle \ncong \overline{K}_3$.

Case 2. $\langle D\rangle \cong K_2 \cup K_1$

Let the vertices of K_2 be u, v and let w be the vertex of K_1.

Subcase(a). u is adjacent to x_1, x_2.

Then v (or) w cannot be adjacent to x_2. If v is adjacent to x_1, x_3; x_1, x_4 (or) x_3, x_4, then degree of w in G will be 2, which is not possible.

Therefore v is adjacent x_1, x_5; x_3, x_5 (or) x_4, x_5 and w is adjacent to x_3, x_4, x_5; x_1, x_4, x_5 (or) x_1, x_3, x_5 respectively. Hence, G is isomorphic one of the graphs G_1, G_2 and G_3.

Sub case (b). Let u be adjacent to x_1 x_3.

Then v or w cannot be adjacent to x_3. As in Subcase(a), v is not adjacent to x_1, x_2; x_1, x_4 and x_2, x_4. Therefore, v is adjacent to x_1, x_5; x_2, x_5 (or) x_4, x_5. Correspondingly, w is adjacent to x_2, x_4, x_5; x_1, x_4, x_5; or x_1, x_2, x_5 respectively. Then G is isomorphic to one of the graphs G_2 and G_4.

Subcase (c). u is adjacent to x_1, x_4.

Then v or w cannot be adjacent to x_4. Since G is a cubic graph, v is not adjacent to x_1, x_2; x_1, x_4 and x_2, x_3. Therefore, v is adjacent to x_1, x_5; x_2, x_5 and x_3, x_5. Correspondingly, w is adjacent to x_2, x_3, x_5; x_1, x_3, x_5 (or) x_1, x_2, x_5 respectively. Then G is isomorphic to one of the graphs G_3, G_5 and G_4.

Subcase (d). u is adjacent to x_1, x_5.

Then v can be adjacent to any two vertices of <V-D> $\cong P_5$, a path on five vertices.

The following cases arise.

N(v) – {u}	N(w)	G is isomorphic to
x_1, x_2	x_3, x_4, x_5	G_1
x_1, x_3	x_2, x_4, x_5	G_2
x_1, x_4	x_2, x_3, x_5	G_3
x_1, x_5	x_2, x_3, x_4	G_1
x_2, x_3	x_1, x_4, x_5	G_2
x_2, x_4	x_1, x_3, x_5	G_6
x_2, x_5	x_1, x_3, x_4	G_3
x_3, x_4	x_1, x_2, x_5	G_2
x_3, x_5	x_1, x_2, x_4	G_2
x_4, x_5	x_1, x_2, x_3	G_1

For the graphs G_i, i = 1, 2, 3, 4, 5, $\chi(G_i) = 3$ and for the graph G_6, $\chi(G_6) = 2$

Therefore, $\gamma_{ctd}(G) = \chi(G) = 3$ if and only if $G \cong G_i$, i = 1, 2, 3, 4, 5.

Conversely if $G \cong G_i$, i = 1, 2, 3, 4, 5, then $\gamma_{ctd}(G) = \chi(G) = 3$.

Cubic Graphs of Order 10

Theorem 3.3.

Let G be a connected cubic graph of order 10. Then $\gamma_{ctd}(G) = \chi(G) = 3$ if and only if G is isomorphic to one of the seventeen graph G_i, i = 1, 2, ..., 17 given in Figure 2.

Proof:

Let D be a γ_{ctd}-set of G such that |D| = 3. Then |V – D| = 7 and $\langle V - D\rangle$ is a tree.

Let D = {u, v, w} and V – D = $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$

Since the graph G is cubic, $\langle V - D\rangle$ must be a path P_7 on 7 vertices. Let x_1, x_7 be the pendant vertices of P_7. Clearly, $\langle D\rangle \ncong K_3$ and P_3. Hence, $\langle D\rangle \cong K_2 \cup K_1$ (or) $\overline{K}_3$

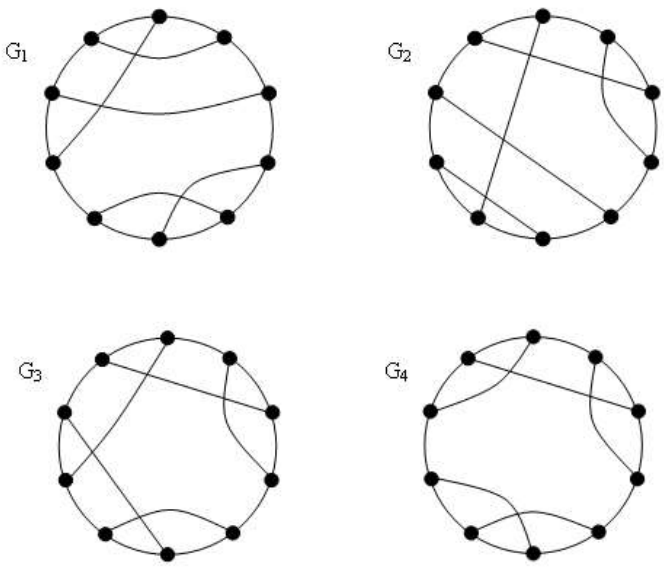
If $\langle D\rangle \cong K_2 \cup K_1$, then there exist atleast two vertices in G of degree atmost two and hence $\langle D\rangle \ncong K_2 \cup K_1$. Therefore, $\langle D\rangle \cong \overline{K}_3$

Then the following two cases arise

(i) Two vertices of D are adjacent to the both pendant vertices x_1 and x_7.

(ii) One vertex of D is adjacent to both the pendant vertices x_1, x_7 and each of the remaining two vertices of D are adjacent to exactly one pendant vertex (distinct)

Case (1): u and v are adjacent to both x_1 and x_7.
Then w is adjacent to none of x_1 and x_7.
Let u be adjacent x_1, x_7, x_2. Then v is adjacent to x_1, x_7, x_3 ; x_1, x_7, x_4 ; x_1, x_7, x_5 (or) x_1, x_7, x_6.
Correspondingly, w is adjacent to x_4, x_5, x_6; x_3, x_5, x_6; x_3, x_4, x_6 (or) x_3, x_4, x_5. Then G is isomorphic to one of the graphs G_1, G_2, G_3 and G_4.
Case (2): u is adjacent to both x_1 and x_7. v is adjacent to x_1 and w is adjacent to x_7.
Subcase (a): u is adjacent to x_1, x_2, x_7.
Then v is adjacent to x_1, x_3, x_4; x_1, x_3, x_5; x_1, x_3, x_6; x_1, x_4, x_5; x_1, x_4, x_6 (or) x_1, x_5, x_6. Correspondingly, w is adjacent to x_5, x_6, x_7; x_4, x_6, x_7; x_4, x_5, x_7; x_3, x_6, x_7; x_3, x_5, x_7 (or) x_3, x_4, x_7. Then G is isomorphic to one of the graphs G_5, G_3, G_6, G_7, G_8 and G_7.
Subcase(b): u is adjacent to x_1,x_3,x_7.
Then v is adjacent to x_1,x_2,x_4 ; x_1,x_2,x_5; x_1,x_2,x_6; x_1,x_4,x_5; x_1,x_4,x_6 or x_1,x_5,x_6. Correspondingly, w is adjacent to x_5, x_6, x_7; x_4, x_6, x_7; x_4, x_5, x_7; x_2, x_6, x_7; x_2, x_5, x_7 (or) x_2, x_4, x_7. Then G is isomorphic to one of the graphs G_1, G_6, G_9, G_{10} and G_{11}.
Subcase (c): u is adjacent to x_1, x_4, x_7.
Then v is adjacent to x_1, x_2, x_3; x_1, x_2, x_5; x_1, x_2, x_6; x_1, x_3, x_5; x_1, x_3, x_6 (or) x_1, x_5, x_6.
Correspondingly, w is adjacent to x_5, x_6, x_7; x_3, x_6, x_7; x_3, x_5, x_7; x_2, x_6, x_7; x_2, x_5, x_7 (or) x_2, x_3, x_7. Then G is isomorphic to one of the graphs G_{12}, G_{13}, G_{11}, G_{14}, G_{15} and G_{13}.
Subcase (d): u is adjacent to x_1, x_5, x_7.
Then v is adjacent to x_1, x_2, x_3; x_1, x_2, x_4; x_1, x_2, x_6; x_1, x_3, x_4; x_1, x_3, x_6 (or) x_1, x_4, x_6.
Correspondingly, w is adjacent to x_4, x_6, x_7; x_3, x_6, x_7; x_3, x_4, x_7; x_2, x_6, x_7; x_2, x_4, x_7 (or) x_2, x_3, x_7.Then G is isomorphic to one of the graphs G_1, G_6, G_9, G_{16} and G_{11}.
Subcase (e): u is adjacent to x_1, x_6, x_7.
Then v is adjacent to x_1, x_2, x_3; x_1, x_2, x_4; x_1, x_2, x_5; x_1, x_3, x_4; x_1, x_3, x_5 (or) x_1, x_4, x_5.
Correspondingly, w is adjacent to x_4, x_5, x_7; x_3, x_5, x_7; x_3, x_4, x_7; x_2, x_5, x_7; x_2, x_4, x_7 (or)x_2, x_3, x_7. Then G is isomorphic to one of the graphs G_5, G_3, G_7, G_{17}, G_8 and G_7.
From the above cases, $G \cong G_i$, $i = 1, 2, ..., 17$ and for these graphs $\chi(G_i) = 3$
Conversely, for the graphs G_i, $i = 1, 2, ..., 17$, $\gamma_{ctd}(G) = \chi(G) = 3$.



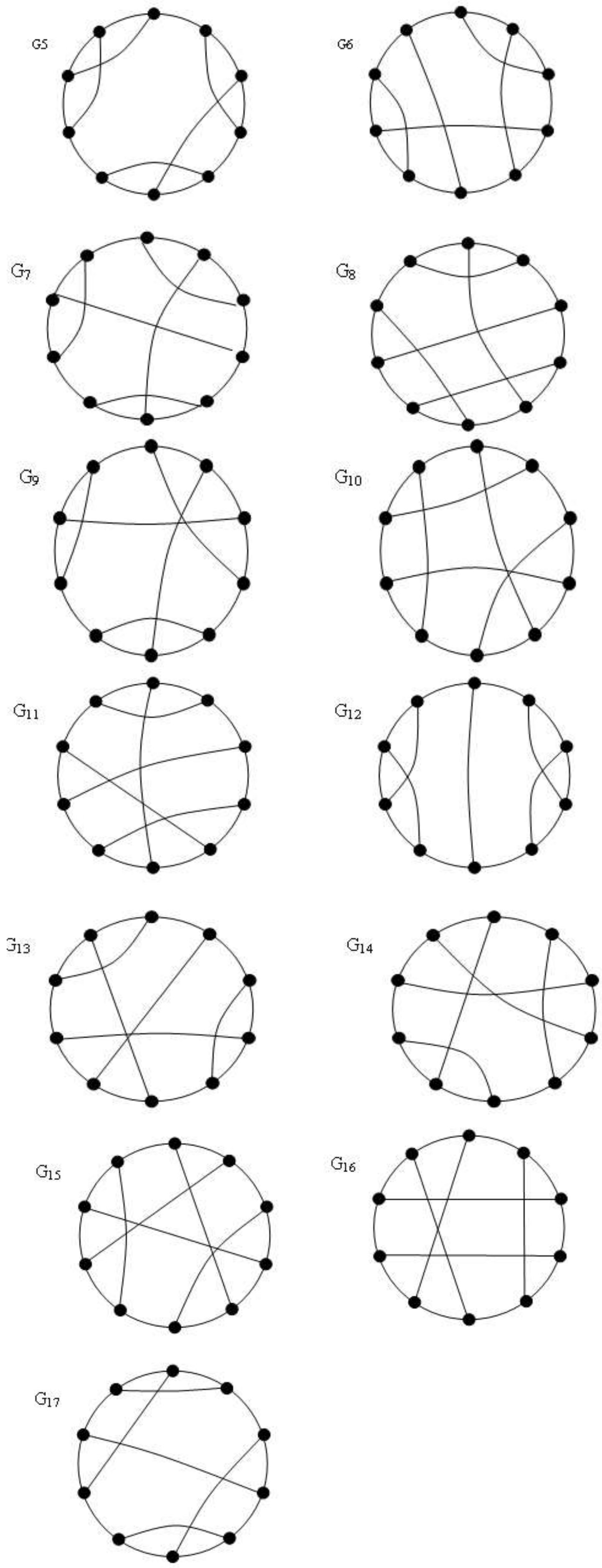


Figure 2

Cubic Graphs of Order 12

Theorem 3.4.

There exists no cubic graphs of order 12 such that $\gamma_{ctd}(G) = \chi(G) = 3$.

Proof:

Let G be a connected cubic graph of order 12, such that $\gamma_{ctd}(G) = \chi(G) = 3$.

Let D be a γ_{ctd} - set of G. Then $|D| = 3$ and $\langle V - D \rangle$ is a tree. Since each vertex in V – D is adjacent to atleast one vertex in D, each vertex in V – D has degree either one or two in $<V–D>$. That implies $\langle V - D \rangle$ is a path on nine vertices. Also, G has atleast one vertex of degree less than or equal to two, which is a contradiction, since G is a cubic graph. Therefore, there exists no cubic graphs of order 12 having $\gamma_{ctd}(G) = \chi(G) = 3$.

Thus, we have found the cubic graphs of order 8 and 10 for which $\gamma_{ctd}(G) = \chi(G) = 3$.

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