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A method for obtaining the n^th derivative of a function of the form $y = l_1(x)l_2(x)l_3(x)l_4(x)$

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ABSTRACT

In this work, we make use of the well known product rule and Leibnitz's theorem to generate a new method which can be used to obtain the higher order derivatives of any functions which depends on four variables. The new method does not require the knowledge of the preceding derivative before obtaining the succeeding ones.

Introduction

The newly emerging trends in engineering, sciences and technology demand the use of higher order derivatives of some functions. Importantly, in the implementation of some numerical integrators [1, 3, 4], the use of higher order derivatives of the interpolant involved is required.

Let $y = l_1(x)l_2(x)l_3(x)l_4(x)$ (1)

Then the first derivative of y using product rule is given as;

$$\frac{dy}{dx} = U\frac{dv}{dx} + V\frac{du}{dx} \qquad (2)$$

Such that (2) implies;

$$\frac{dy}{dx} = l_1(x)l_2(x)\frac{d}{dx}\{l_3(x)l_4(x)\}$$

$$+ l_3(x)l_4(x)\frac{d}{dx}\{l_1(x)l_2(x)\} \qquad (3)$$

If one replaces

$$\frac{d}{dx}\{l_3(x)l_4(x)\}\ and\ \frac{d}{dx}\{l_1(x)l_2(x)\}$$

with their Leibnitz's theorem expressions, one will obtain;

$$\frac{dy}{dx} = l_1(x)l_2(x)\left\{\sum_{i=0}^{n=1} C_i^1\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\right\} +$$

$$l_3(x)l_4(x)\left\{\sum_{i=0}^{n=1} C_i^1\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\right\} \qquad (4)$$

The second derivative of y is obtained from (2) as;

$$\frac{d^2y}{dx^2} = l_1(x)l_2(x)\frac{d^2}{dx^2}l_3(x)l_4(x) +$$

$$2\left\{\left(\frac{d}{dx}l(x)l_2(x)\right)\left(\frac{d}{dx}l_3(x)l_4(x)\right)\right\}$$

$$+l_3(x)l_4(x)\frac{d^2}{dx^2}l_1(x)l_2(x) \qquad (5)$$

By using the Leibnitz's theorem expression for the derivatives in (5), one obtains:

$$= l_1(x)l_2(x)\left\{\left\{\sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\right\}\right\} +$$

$$2\left\{\sum_{i=0}^{n=1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\right\}$$

$$\left\{\sum_{i=0}^{n=1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\right\}$$

$$+\{l_3(x)l_4(x)\}\left\{\sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\right\} \qquad (6)$$

For the 3rd, 4th, and 5th derivatives, one obtains the followings;

$$\frac{d^3y}{dx^3} = l_1(x)l_2(x)\frac{d^3}{dx^3}l_3(x)l_4(x)$$

$$+3\left\{\left(\frac{d}{dx}l_1(x)l_2(x)\right)\left(\frac{d^2}{dx^2}l_3(x)l_4(x)\right)\right\}$$

$$+3\left\{\left(\frac{d^2}{dx^2}l_1(x)l_2(x)\right)\left(\frac{d}{dx}l_3(x)l_4(x)\right)\right\}$$

$$+l_3(x)l_4(x)\frac{d^3}{dx^3}l_1(x)l_2(x) \qquad (7)$$

And from (7), one obtains; $\frac{d^3y}{dx^3}$

$$= l_1(x)l_2(x)\left\{\left\{\sum_{i=0}^{n=3} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\right\}\right\}$$

$$+3\left\{\sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\right\}$$

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$$\left\{\sum_{t=0}^{n=1} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+3\left\{\sum_{t=0}^{n=1} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\}$$
$$\left\{\sum_{t=0}^{n=2} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+l_3(x)l_4(x)\left\{\sum_{t=0}^{n=3} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\} \quad (8)$$

Likewise one obtains

$$\frac{d^4y}{dx^4} = l_1(x)l_2(x)\,\frac{d^4}{dx^4}l_3(x)l_4(x)$$
$$+4\left\{\left(\frac{d}{dx}l_1(x)l_2(x)\right)\left(\frac{d^3}{dx^3}l_3(x)l_4(x)\right)\right\}$$
$$+6\left\{\left(\frac{d^2}{dx^2}l_1(x)l_2(x)\right)\left(\frac{d^2}{dx^2}l_3(x)l_4(x)\right)\right\}$$
$$+4\left\{\left(\frac{d^3}{dx^3}l_1(x)l_2(x)\right)\left(\frac{d}{dx}l_3(x)l_4(x)\right)\right\}$$
$$+l_3(x)l_4(x)\,\frac{d^4}{dx^4}l_1(x)l_2(x) \quad (9)$$

And by Leibnitz's theorem, (9) comminutes unto;

$$\frac{d^4y}{dx^4} = l_1(x)l_2(x)\left\{\sum_{t=0}^{n=4} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+4\left\{\sum_{t=0}^{n=3} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\}$$
$$\left\{\sum_{t=0}^{n=1} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+6\left\{\sum_{t=0}^{n=2} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\}$$
$$\left\{\sum_{t=0}^{n=2} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+4\left\{\sum_{t=0}^{n=1} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\}$$
$$\left\{\sum_{t=0}^{n=3} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+l_3(x)l_4(x)\left\{\sum_{t=0}^{n=4} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\} \quad (10)$$

Below we have the expression for the 5th derivative:

$$\frac{d^5y}{dx^5} = l_1(x)l_2(x)\,\frac{d^5}{dx^5}l_3(x)l_4(x)$$
$$+5\left\{\left(\frac{d}{dx}l_1(x)l_2(x)\right)\left(\frac{d^4}{dx^4}l_3(x)l_4(x)\right)\right\}$$
$$+10\left\{\left(\frac{d^2}{dx^2}l_1(x)l_2(x)\right)\left(\frac{d^3}{dx^3}l_3(x)l_4(x)\right)\right\}$$
$$+10\left\{\left(\frac{d^3}{dx^3}l_1(x)l_2(x)\right)\left(\frac{d^2}{dx^2}l_3(x)l_4(x)\right)\right\}$$
$$+5\left\{\left(\frac{d^4}{dx^4}l_1(x)l_2(x)\right)\left(\frac{d}{dx}l_3(x)l_4(x)\right)\right\}$$
$$+l_3(x)l_4(x)\,\frac{d^5}{dx^5}l_1(x)l_2(x) \quad (11)$$

And by using Leibnitz's theorem, (11) becomes;

$$\frac{d^5y}{dx^5} = l_1(x)l_2(x)\left\{\sum_{t=0}^{n=5} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+5\left\{\sum_{t=0}^{n=4} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\}$$
$$\left\{\sum_{t=0}^{n=1} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+10\left\{\sum_{t=0}^{n=3} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\}$$
$$\left\{\sum_{t=0}^{n=2} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+10\left\{\sum_{t=0}^{n=2} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\}$$
$$\left\{\sum_{t=0}^{n=3} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+5\left\{\sum_{t=0}^{n=1} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\}$$
$$\left\{\sum_{t=0}^{n=4} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_3(x)\right)\left(\frac{d^t}{dx^t}l_4(x)\right)\right\}$$
$$+l_3(x)l_4(x)\left\{\sum_{t=0}^{n=5} C_t^n\left(\frac{d^{n-t}}{dx^{n-t}}l_1(x)\right)\left(\frac{d^t}{dx^t}l_2(x)\right)\right\} \quad (12)$$

Thus, for the nth order derivative, one obtains;

$$\sum_{r=0}^{n=m} C_r^m\left[\begin{array}{c}\left(\sum_{i=0}^{n=m-r} C_i^n\left[\frac{d^{n-i}l_3(x)}{dx^{n-i}}\right]\left[\frac{d^il_4(x)}{dx^i}\right]\right)\\ \left(\sum_{i=0}^{n=r} C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^il_2(x)}{dx^i}\right]\right)\end{array}\right] \quad (13)$$

Opeenoch's Theorem:

Let

$y = l_1(x)l_2(x)l_3(x)l_4(x), y = l_1(x)l_2(x)/l_3(x)l_4(x)$, $y = \frac{l_1(x)}{l_2(x)l_3(x)l_4(x)}$, $y = 1/l_1(x)l_2(x)l_3(x)l_4(x)$

or $y = l_1(x)l_2(x)l_3(x)/l_4(x)$.

Then the nth derivative of y is given as

$$\sum_{r=0}^{n=m} C_r^m\left[\begin{array}{c}\left(\sum_{i=0}^{n=m-r} C_i^n\left[\frac{d^{n-i}l_3(x)}{dx^{n-i}}\right]\left[\frac{d^il_4(x)}{dx^i}\right]\right)\\ \left(\sum_{i=0}^{n=r} C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^il_2(x)}{dx^i}\right]\right)\end{array}\right] \quad (14)$$

Proof:

Using Mathematical induction, it can be shown that for n=1, (3) and (4) hold.

So that it is true for n=1.

If it is true for n=1, it must also be true for n=2, such that (5) and (6) hold.

Now that it is true for n=1, 2, it must be true for n=k:

$$\frac{d^k y}{dx^k} = l_1(x)l_2(x)\,\frac{d^k}{dx^k}l_3(x)l_4(x) + k\left\{\left(\frac{d}{dx}l_1(x)l_2(x)\right)\left(\frac{d^{k-1}}{dx^{k-1}}l_3(x)l_4(x)\right)\right\} + \frac{k(k-1)}{2!}\left\{\left(\frac{d^2}{dx^2}l_1(x)l_2(x)\right)\left(\frac{d^{k-2}}{dx^{k-2}}l_3(x)l_4(x)\right)\right\} + \cdots + \frac{k(k-1)(k-2)\dots(3))}{(k-2)!}\left\{\left(\frac{d^{k-2}}{dx^{k-2}}l_1(x)l_2(x)\right)\left(\frac{d^2}{dx^2}l_3(x)l_4(x)\right)\right\} + \frac{k(k-1)(k-2)\dots(2)}{(k-1)!}\left\{\left(\frac{d^{k-1}}{dx^{k-1}}l_1(x)l_2(x)\right)\left(\frac{d}{dx}l_3(x)l_4(x)\right)\right\} + \frac{k!}{k!}\,l_3(x)l_4(x)\,\frac{d^k}{dx^k}l_1(x)l_2(x) \qquad (15)$$

The coefficients of the above expression are obtained by the binomial theorem. Thus;

$$\frac{d^k y}{dx^k} = l_1(x)l_2(x)\left\{\sum_{i=0}^{n=k} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\right\} + k\left\{\begin{array}{c}\sum_{i=0}^{n=1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\\ \left\{\sum_{i=0}^{n=k-1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\right\}\end{array}\right\} + \frac{k(k-1)}{2!}\left\{\sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\right\}\left\{\sum_{i=0}^{n=k-2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\right\} + \cdots + \frac{k(k-1)(k-2)\dots(3))}{(k-2)!}\left\{\begin{array}{c}\sum_{i=0}^{n=k-2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\\ \sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\end{array}\right\} + \frac{k(k-1)(k-2)\dots(2)}{(k-1)!}\left\{\begin{array}{c}\sum_{i=0}^{n=k-1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\\ \sum_{i=0}^{n=1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\end{array}\right\} + \frac{k!}{k!}\,l_3(x)l_4(x)\left\{\sum_{i=0}^{n=k} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\right\} \qquad (16)$$

This must be true for n=k+1 such that;

$$\frac{d^{k+1} y}{dx^{k+1}} = l_1(x)l_2(x)\sum_{i=0}^{n=k+1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right) + (k+1)\left\{\begin{array}{c}\sum_{i=0}^{n=1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\\ \sum_{i=0}^{n=k} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\end{array}\right\} + \frac{(k+1)(k)}{2!}\left\{\begin{array}{c}\sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\\ \sum_{i=0}^{n=k-1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\end{array}\right\} + \frac{(k+1)(k)(k-1)}{3!}\left\{\begin{array}{c}\sum_{i=0}^{n=3} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\\ \sum_{i=0}^{n=k-2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\end{array}\right\} + \cdots + \frac{(k+1)\dots(3))}{(k-1)!}\left\{\begin{array}{c}\sum_{i=0}^{n=k-1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\\ \sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\end{array}\right\} + \frac{(k+1)\dots(2)}{k!}\left\{\begin{array}{c}\sum_{i=0}^{n=k}\left(C_i^n\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right)\\ \sum_{i=0}^{n=1} C_i^n\left(\frac{d^{n-i}}{dx^{n-i}}l_3(x)\right)\left(\frac{d^i}{dx^i}l_4(x)\right)\end{array}\right\} + \frac{k!}{k!}\,l_3(x)l_4(x)\sum_{i=0}^{n=k+1}\left(C_i^n\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^i}{dx^i}l_2(x)\right) \qquad (17)$$

Thus, the opeenoch's theorem for the n^{th} derivative of y is given as

$$\sum_{r=0}^{n=m} C_r^m\left[\begin{array}{c}\left(\sum_{i=0}^{n-m-r} C_i^n\left[\frac{d^{n-i}l_3(x)}{dx^{n-i}}\right]\left[\frac{d^i l_4(x)}{dx^i}\right]\right)\\ \left(\sum_{i=0}^{n=r} C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^i l_2(x)}{dx^i}\right]\right)\end{array}\right] \qquad (18)$$

Conclusion:

The algorithm can easily be simulated by writing subroutines for the independent variables involved.

The following points are obvious concerning the new method:

(i) The superscript n decreases regularly by 1

(ii) The superscript i increases regularly by 1

(iii) The numerical coefficients are the normal binomial coefficients.

For increased accuracy in most numerical methods that involve the use of higher order derivatives, this new method can be used to obtain higher order derivatives of the functions involved. The labor involved in calculating and evaluating higher derivatives through the use of this new method is very minimal, since you can jump the process of obtaining the preceding derivatives to the point of obtaining desired derivative (order).

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