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On The Turning Point, Critical Line and the Zeros of Riemann Zeta Function

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ABSTRACT

The relationship among the turning point, the critical strip and the zeros of Riemann Zeta function is Investigated and established in this paper. A theorem to prove the link between the turning point and the real part of a quadratic function with complex roots is also presented, thereby showing that the real part of the roots will always be the critical point of the quadratic function, thus establishing Riemann's hypothesis. $\xi(t)$ is shown to be the analytic continuation formula of $\zeta(z)$ as presented by Riemann, amidst the various presentations of $\zeta(z)$ in his work. A proof of the nature of the zeros is found in this work.

Introduction

The statement made by B.Riemann that "*one now finds indeed approximately this number of real roots within these limits and it is very probable that all roots are real*" ,(November, 1859), has been of great interest to many notable mathematicians worldwide. This is because it is an excellent description of a good conjecture.

Our desire is to check if there is any link between the critical line (strip) and the critical point (turning point) of the continuation formula of $\zeta(z), \xi(t)$.

It is discovered $\zeta(z)$ that behaves as a polynomial of order four between $0 \leq z \leq 1$. but as $z \to \frac{1}{2}$, it reduces to a class of quadratic equations whose roots are a class of complex numbers.

2. Let

$$\xi(t) = \prod\left(\frac{z}{2}\right)(z-1)\pi^{-\frac{z}{2}}\zeta(z) \tag{1}$$

$$= \left[\frac{1}{2} + \frac{z(z-1)}{2}\int_1^\infty \psi(x)\left(x^{\frac{z}{2}-1} + x^{\frac{z-1}{2}}\right)dx\right] \tag{2}$$

$$= \left[\frac{1}{2} - \left(t^2 + \frac{1}{4}\right)\int_1^\infty \psi(x)\,x^{-\frac{3}{4}}\cos\left(\frac{t}{2}\log x\right)dx\right] \tag{3}$$

Differentiate $\xi(t)$ as defined in (1) to obtain

$$\frac{d\xi}{dt} = \frac{d}{dz}\left[\frac{z}{2}\Gamma\left(\frac{z}{2}\right)(z-1)\pi^{-\frac{z}{2}}\zeta(z)\right] \tag{4}$$

If we differentiate $\xi(t)$ as defined in (2) the result will be:

$$\frac{d}{dz}\left[\frac{1}{2} + \frac{z(z-1)}{2}\int_1^\infty \psi(x)\left(x^{\frac{z}{2}-1} + x^{\frac{z-1}{2}}\right)dx\right] \tag{5}$$

If we differentiate $\xi(t)$ as defined in (3) the result will be:

$$= \frac{d}{dz}\left[\frac{1}{2} - \left(t^2 + \frac{1}{4}\right)\int_1^\infty \psi(x)\,x^{-\frac{3}{4}}\cos\left(\frac{t}{2}\log x\right)dx\right] \tag{6}$$

We can further transform (4) into:

$$= \frac{d}{dz}\left[\frac{z}{2}\Gamma\left(\frac{z}{2}\right)(z-1)e^{-\frac{z}{2}\log\pi}\zeta(z)\right] \tag{7}$$

We proceed to differentiate (7) as follows and on further simplification obtain (8), (9);

$$\frac{d\xi}{dt} = \frac{z}{2}\Gamma\left(\frac{z}{2}\right)(z-1)\left(e^{-\frac{z}{2}\log\pi}\zeta'(z) + \left(\frac{-\log\pi}{2}\right)e^{-\frac{z}{2}\log\pi}\zeta(z)\right)$$

$$+\ e^{-\frac{z}{2}\log\pi}\zeta(z)\left(\frac{z}{2}\Gamma\left(\frac{z}{2}\right)(1) + (z-1)\left(\frac{1}{2}\Gamma\left(\frac{z}{2}\right) + \frac{z}{4}\Gamma'\left(\frac{z}{2}\right)\right)\right) \tag{8}$$

$$= \frac{z(z-1)}{2}\Gamma\left(\frac{z}{2}\right)e^{-\frac{z}{2}\log\pi}\zeta'(z) - \frac{z}{4}(z-1)\log\pi\,\Gamma\left(\frac{z}{2}\right)e^{-\frac{z}{2}\log\pi}\zeta(z)$$

$$+ \frac{z}{2}\Gamma\left(\frac{z}{2}\right)e^{-\frac{z}{2}\log\pi}\zeta(z) + \frac{(z-1)}{2}\Gamma\left(\frac{z}{2}\right)e^{-\frac{z}{2}\log\pi}\zeta(z)$$

$$+ \frac{z(z-1)}{4}\Gamma'\left(\frac{z}{2}\right)e^{-\frac{z}{2}\log\pi}\zeta(z) \tag{9}$$

$$\frac{d\xi}{dt} = \frac{z(z-1)}{2}\left[\Gamma\left(\frac{z}{2}\right)\zeta'(z) + \frac{1}{2}\Gamma'\left(\frac{z}{2}\right)\zeta(z)\right]e^{-\frac{z}{2}\log\pi}$$

$$+ \frac{1}{2}\left[z + (z-1) - \frac{z(z-1)}{2}\log\pi\right]\Gamma\left(\frac{z}{2}\right)\zeta(z)e^{-\frac{z}{2}\log\pi} \tag{10}$$

$$= \frac{z(z-1)}{2}\left[\Gamma\left(\frac{z}{2}\right)\zeta'(z) + \frac{1}{2}\Gamma'\left(\frac{z}{2}\right)\zeta(z)\right]e^{-\frac{z}{2}\log\pi}$$

$$+ \frac{1}{2}\left[(2z-1) - \frac{z(z-1)}{2}\log\pi\right]\Gamma\left(\frac{z}{2}\right)\zeta(z)e^{-\frac{z}{2}\log\pi} \tag{11}$$

To obtain the critical point of $\xi(t)$, we evaluate $\frac{d\xi}{dt} = 0$:

$$\left[\frac{z(z-1)}{2}\left[\frac{\Gamma\left(\frac{z}{2}\right)\zeta'(z)}{\Gamma\left(\frac{z}{2}\right)\zeta(z)} + \frac{\Gamma'\left(\frac{z}{2}\right)\zeta(z)}{2\Gamma\left(\frac{z}{2}\right)\zeta(z)}\right] + \frac{1}{2}\left[(2z-1) - \frac{z(z-1)}{2}\log\pi\right]\right]\Gamma\left(\frac{z}{2}\right)\zeta(z)e^{-\frac{z}{2}\log\pi} = 0 \tag{12}$$

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$$\left[\frac{z(z-1)}{2}\left[\frac{\zeta'(z)}{\zeta(z)}+\frac{\Gamma'\left(\frac{z}{2}\right)}{2\Gamma\left(\frac{z}{2}\right)}\right]+\frac{1}{2}\left[(2z-1)-\frac{z(z-1)}{2}\log\pi\right]\right]\Gamma\left(\frac{z}{2}\right)\zeta(z)e^{-\frac{z}{2}\log\pi}=0 \quad (13)$$

It can be shown that (13) results into (16) by using Corollary (14) and (15) as contained in [2]:

$$-\frac{\zeta'(z)}{\zeta(z)}=\frac{1}{2}+\frac{1}{(z-1)}+\frac{\Gamma'\left(\frac{z}{2}\right)}{2\Gamma\left(\frac{z}{2}\right)}-\frac{1}{2}\log\pi-(2z-1)\frac{\Gamma'\left(\frac{z}{2}\right)}{2\Gamma\left(\frac{z}{2}\right)} \quad (14)$$

And

$$\Gamma\left(\frac{z}{2}\right)\zeta(z)=\frac{1}{-z(1-z)\pi^{-\frac{z}{2}}}\prod_{\rho\in Z_+}\left[\left(1-\frac{z}{\rho}\right)\left(1-\frac{z}{(1-\rho)}\right)\right] \quad (15)$$

$$\frac{d\xi}{dt}=\left[\frac{z(z-1)}{2}\left(\frac{1}{2}\log\pi-(2z-1)\sum_{\rho\in Z_+}\left[\frac{1}{(Z-\rho)(Z-(1-\rho))}\right]-\frac{1}{z}-\frac{1}{(z-1)}-\frac{\Gamma'\left(\frac{z}{2}\right)}{2\Gamma\left(\frac{z}{2}\right)}+\frac{\Gamma'\left(\frac{z}{2}\right)}{2\Gamma\left(\frac{z}{2}\right)}\right)\right.$$
$$\left.+\frac{1}{2}\left[(2z-1)-\frac{z(z-1)}{2}\log\pi\right]\right]\Gamma\left(\frac{z}{2}\right)\zeta(z)e^{-\frac{z}{2}\log\pi}=0 \quad (16)$$

$=$

$$\left[\frac{z(z-1)}{4}\log\pi+\frac{z(z-1)(2z-1)}{2}\sum_{\rho\in Z_+}\left[\frac{1}{(Z-\rho)(Z-(1-\rho))}\right]-\frac{(z-1)}{2}-\frac{z}{2}+\frac{(2z-1)}{2}\right.$$
$$\left.-\frac{z(z-1)}{4}\log\pi\right]\Gamma\left(\frac{z}{2}\right)\zeta(z)e^{-\frac{z}{2}\log\pi}=0 \quad (17)$$

$=$

$$\left[\frac{(2z-1)}{2}-\frac{(z-1)}{2}-\frac{z}{2}+\frac{z(z-1)(2z-1)}{2}\sum_{\rho\in Z_+}\left[\frac{1}{(z-\rho)(z-(1-\rho))}\right]\right]\Gamma\left(\frac{z}{2}\right)\zeta(z)e^{-\frac{z}{2}\log\pi}=0 \quad (18)$$

$$\frac{(z^2-z)(2z-1)}{2}=\frac{2z^3-z^2-2z^2+z}{2}=\frac{2z^3-3z^2+z}{2}=0 \quad (19)$$

$=> (2z^2-3z+1)z=0 => \; z=0$ or $2z^2-3z+1=0, z=1 \; or \; z=\frac{1}{2}$

From the above, it can be shown that $\xi(t)=\Pi\left(\frac{z}{2}\right)(z-1)\pi^{-\frac{z}{2}}\zeta(z)$ exhibits some properties of a polynomial in order four in the interval [3]: $0\leq z\leq 1$, since it has three turning points. But on the critical line, the zeros of ζ (z) are complex roots in conjugates, thus at $z=\frac{1}{2}$, $\xi(t)=\Pi\left(\frac{z}{2}\right)(z-1)\pi^{-\frac{z}{2}}\zeta(z)$ shows some quadratic properties. The following steps show that as $z\Rightarrow\frac{1}{2}$, $\xi(t)$ is quadratic and has a critical point at $z=\frac{1}{2}$. In [2], a simpler form $\frac{d\xi}{dt}=0$ can be shown as:

$$\left[\frac{(2z-1)}{2}-\frac{(z-1)}{2}-\frac{z}{2}+\frac{z(z-1)(2z-1)}{2}\sum_{\rho\in Z_+}\left[\frac{1}{(Z-\rho)(Z-(1-\rho))}\right]\right]$$
$$\frac{\pi^{-\frac{z}{2}}}{(z(1-z))\pi^{-\frac{z}{2}}}\left[\prod_{\rho\in Z_+}\left[\left(1-\frac{z}{\rho}\right)\left(1-\frac{z}{(1-\rho)}\right)\right]\right]=0 \quad (20)$$

$$\left[\frac{z(z-1)(2z-1)}{2}\sum_{\rho\in Z_+}\left[\frac{1}{(Z-\rho)(Z-(1-\rho))}\right]\right]\left[\frac{1}{(z(1-z))}\prod_{\rho\in Z_+}\left[\left(1-\frac{z}{\rho}\right)\left(1-\frac{z}{(1-\rho)}\right)\right]\right]=0 \quad (21)$$

Obviously, (21) comminutes to;

$$\frac{z(z-1)(2z-1)}{2(z^2-z)}\left[\sum_{\rho\in Z_+}\left[\frac{1}{(Z-\rho)(Z-(1-\rho))}\right]\right]\left[\prod_{\rho\in Z_+}\left[\left(1-\frac{z}{\rho}\right)\left(1-\frac{z}{(1-\rho)}\right)\right]\right]=0 \quad (22)$$

$$\frac{(2z-1)}{2}\left[\sum_{\rho\in Z_+}\left[\frac{1}{(Z-\rho)(Z-(1-\rho))}\right]\right]\left[\prod_{\rho\in Z_+}\left[\left(1-\frac{z}{\rho}\right)\left(1-\frac{z}{(1-\rho)}\right)\right]\right]=0 \quad (23)$$

$$=>\frac{(z^2-z)(2z-1)}{2(z^2-z)}=0 \quad =>\frac{2\left(z-\frac{1}{2}\right)}{2}=0 \quad \therefore z=\frac{1}{2}$$

Note that the right hand sides of (1), (22) and (23) can be written as functions of t by substituting $z=\frac{1}{2}\pm it$ in any of these equations.

Yemienoch's Theorem: Let f (z) is a quadratic function of a complex variable z. If z= a±ib are the two Conjugate complex roots of f (z) then, a= $\frac{-B}{2A}$ will always be the turning point of f (z) provided $\frac{-B}{2A}$ is unique and b≠0.

Proof: Let $z=(a\pm ib)$ be the roots of $f(z)$ then,

$$z=(a+ib) \; or \; z=(a-ib)$$
$$\Rightarrow z-(a+ib)=0 \; or \; z-(a-ib)=0$$
$$(z-(a+ib))(z-(a-ib))=0$$
$$(z-a-ib)(z-a+ib)=0$$
$$z^2-za+zib-za+a^2-iab-zib+iab+b^2=0$$
$$f(z)=z^2-2za+(a^2+b^2)$$

We proceed to obtain the turning point of $f(z)$

$$\frac{df(z)}{dz}=2z-2a$$
$$\frac{df}{dz}=0\Rightarrow \quad 2z-2a=0$$
$$\Rightarrow 2z=2a \quad\Rightarrow\quad z=a$$

The above theorem is true for all quadratic functions.

Conclusion:

We have been able to show that the turning point of $\xi(t)$ is $z=\frac{1}{2}$ and $\xi(t)$ is the analytical continuation formula of $\zeta(z)$ and a quadratic function as $z\Rightarrow\frac{1}{2}$, with conjugate roots.

thus, since by Yemienoch's theorem every quadratic function with complex roots will always have the real part of its complex roots as its turning point, then $\zeta(z)$ also has its real part as the its critical point. And $z\Rightarrow\frac{1}{2}$ its turning point converges to $\frac{1}{2}$.

This is a way of establishing the Riemann's Hypothesis.

Riemann claimed that $\xi(t)$ is finite for all finite values of t *and that* $\xi(t)$ allows itself to be developed in power of t^2 as a very rapidly converging series are all true claims [1]. It is good to note at this point that t, will always be real roots and the first are $\frac{1}{2}\pm 14.13472524$. To investigating Riemann claims, the choice of $\varsigma(z)$ *and* $\xi(t)$ must be selectively selected as defined in his work [1].

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