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The Riemann zeta function and its extension into continuous optimization equation

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ABSTRACT

In this paper, the Riemann Zeta function is presented as a function with real and imaginary parts. Thus we are able to evaluate

$$\zeta(z)\overline{\zeta(z)} = \varphi^2(t) + \rho^2(t)$$

By writing $\zeta(z)\overline{\zeta(z)}$ as a bilinear function, and through the use of Sobolev space theorem, an optimization problem with a variable coefficient is derived. Some methods of solution are presented.

Introduction

Given that

$$\varepsilon(t) = 4\int_{1}^{\infty} \frac{d\left(x^{3/2}\phi^1\right)}{dx} x^{-1/4} \cos\left(\frac{t}{2}\log x\right) dx \quad (1)$$

Such that

$$\phi = \sum_{n=1}^{\infty} e^{-nn\pi x} \quad (2)$$

$$Thus\ \phi'(x) = -\sum_{n=1}^{\infty}\left[e^{-nn\pi x}\right] \quad (3)$$

$$for\ \frac{d}{dx}\left(x^{3/2}\phi'\right) = \frac{d}{dx}\left[-nn\pi x^{\frac{3}{2}}\sum_{n=1}^{\infty} e^{-nn\pi x}\right] \quad (4)$$

Equation (4) given

$$\sum_{n=1}^{\infty}\left((nn)^2\pi^2 x^{3/2} - \frac{3}{2}\pi x^{1/2} + n^2\right)e^{-nn\pi x} \quad (5)$$

This implies that (1) can be written as

$$\varepsilon(t) = 4\int_{1}^{\infty}\sum_{n=1}^{\infty}\left[n^4\pi^2 x^{3/2} - \frac{3}{2}n^2\pi x^{1/2}\right]e^{-nn\pi x} x^{-1/4}\cos\left(\frac{t}{2}\log x\right)dx \quad (6)$$

If one substitutes the Taylor's series expansions for $e^{-nn\pi x}$ and $\cos\left(\frac{t}{2}\log x\right)$ in (6), one will obtain $\varepsilon(t)$;

$$= \int_{1}^{\infty}\sum_{n=1}^{\infty}\left[4n^4\pi^2 x^{5/4} - 6n^2\pi x^{1/4}\right]\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n!}\left[\frac{1}{2}t\log x\right]^{2n}\right]\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(n^2\pi x)^n}{n!}\right]dx \quad (7)$$

On further simplification, it can be shown that $\varepsilon(t)$ gives

$$= \int_{1}^{\infty}\left(\sum_{n=1}^{\infty}\left[4n^4\pi^2 x^{5/4} - 6n^2 x^{1/4}\right] + \left(4n^4\pi^2 x^{5/4} - 6n^2\pi x^{1/4}\right)\left[(-1)^n\frac{(n^2\pi x)^n}{n!}\right]\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n!}\left[\frac{t}{2}\log x\right]^{2n}\right]\right)dx \quad (8)$$

The above equation (8) is also equivalent to (9) on using integrating by part;

$$\varepsilon(t) = \sum_{n=1}^{\infty}\left[\frac{24n^4\pi}{5} - \frac{16n^4\pi^2}{9} - \frac{(-1)^n}{n!}(n^2\pi)^n\left(\frac{16n^4\pi^2}{(9+4n)} - \frac{24n^4\pi}{(5+4n)}\right) + \frac{1}{2^{2n}}\left((n^2\pi)^{\ldots}\right.\right.$$

Recall that

$$\Pi\left(\frac{z}{2}\right)(z-1)\pi^{-z/2}\zeta(z) = \varepsilon(t) \quad (10)$$

Thus:

$$\zeta(z) = \frac{\pi^{z/2}}{\Pi\left(\frac{z}{2}\right)(z-1)}\left[4\int_{1}^{\infty}\frac{d\left(x^{3/2}\psi^1\right)}{dx}x^{-1/4}\cos\left(\frac{t}{2}\log x\right)dx\right] \quad (11)$$

$$\zeta(z) = \frac{\pi^{z/2}}{\Pi(z/2)(z-1)}\left[\sum_{n=1}^{\infty}\left[\frac{24n^4\pi}{5} - \frac{16n^4\pi^2}{9} - \frac{(-1)^n}{n!}(n^2\pi)^n\left(\frac{16n^4\pi^2}{(9+4n)} - \frac{24n^4\pi}{(\ldots}\right.\right.\right.$$

If we replace $\Pi\left(\frac{z}{2}\right)$ by $z\Gamma\left(\frac{z}{2}\right)$,the resulting function will be;

$$\zeta(z) = \frac{2\pi^{z/2}}{(z^2 - z)\Gamma(z/2)}\left[\sum_{n=1}^{\infty}\left[\frac{24n^4\pi}{5} - \frac{16n^4\pi^2}{9} - \frac{(-1)^n}{n!}(n^2\pi)^n\left(\frac{16n^4\pi^2}{(9+4n)} - (24n\ldots\right.\right.\right.$$

Riemann presented in [Riemann (1859)] that;

$$\frac{d}{dz}\left(\frac{1}{z}\log\Pi\left(\frac{z}{2}\right)\right) = \sum_{n=1}^{\infty}\frac{d}{dz}\left(\frac{1}{z}\log\left(1+\frac{z}{2n}\right)\right) \quad (14)$$

It follows that

$$\Pi\left(\frac{z}{2}\right) = \frac{z}{2}\Gamma\left(\frac{z}{2}\right) = \sum_{n=1}^{\infty}\left(1+\frac{z}{2n}\right) \quad (15)$$

Thus (13) can be written as

$$\zeta(z) = \frac{2\pi^{z/2}}{z(z-1)\prod_{n=1}^{\infty}(1+z/2n)}\left[\sum_{n=1}^{\infty}\left(\frac{24n^4\pi}{5} - \frac{16n^4\pi^2}{9} - \frac{(-1)^n}{n!}(n^2\pi)^n\left(\frac{16n^4\pi^2}{\ldots}\right.\right.\right.$$

If one substitutes $z = \frac{1}{2} + it$ into (16) and rationalizes the emerging equation, this will lead to;

$$\zeta(z) = \sum_{n=1}^{L=\infty} B\left\{\frac{2^3 nAC}{(n-1)!\,D} + \frac{1}{2^{2n}}\left(\frac{2^3 n\,EC}{D}\right)t^{2n} + i\left[\left(\frac{2^6 n^2 A}{D}\right)t + \frac{1}{2^{2n}}\left(\frac{2^6 n^2 E}{D}\right)t^{2n+1}\right]\right\}$$

Where ;

$$A = \left(\frac{24n^4\pi}{5} - \frac{16n^4\pi^2}{9} - \frac{(-1)^n}{n!}(n^2\pi)^n\left[\frac{16n^4\pi^2}{(9+4n)} - \frac{24n^4\pi}{(5+4n)}\right]\right) \quad (18)$$

$$B = \left(\frac{1}{4}\log\pi + \frac{it}{2}\log\pi\right)^{n-1} \quad (19)$$

$$C = 4t^2 + 4n + 1 \quad (20)$$

$$D = -\left\{(4t^2 + 4n + 1)^2 + 64n^2t^2\right\} \quad (21)$$

$$E = \left(\frac{(n^2\pi)^n}{(2n-1)!\,n!}\left\{\frac{64n^4\pi^2}{(9+4n)^2} - \frac{96n^4\pi}{(5+4n)^2}\right\} + \frac{(-1)^n}{n!}\left\{\frac{64n^4\pi^2}{81} - \frac{96n^4\pi}{25}\right\}\right) \quad (22)$$

Using binomial theorem on equation (19), we obtain

$$B = \left(\frac{1}{4}\log\pi + \frac{it}{2}\log\pi\right)^{n-1} = \left(\frac{\log\pi}{2}\right)^{n-1}\left(\frac{1}{2}+it\right)^{n-1} \quad (23)$$

If we choose $k = n - 1$ then B becomes

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$$\left(\frac{\log\pi}{2}\right)^k\left(\frac{1}{2}+it\right)^k=\left(\frac{\log\pi}{2}\right)^k\left\{\left(\frac{1}{2}\right)^k+\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{k-n}(it)^n}{n!}\prod_{j=0}^{n-1}(k-j)\right\} \quad (24)$$

$$B=\left(\frac{1}{2}\right)^k\left(\frac{\log\pi}{2}\right)^k+\left(\frac{\log\pi}{2}\right)^k\left\{\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{k-n}(it)^n}{n!}\prod_{j=0}^{k}(k-j)\right\} \quad (25)$$

To evaluate the value of B^2, we simply compute the square of (25) *such that;*

$$B^2=\left(\frac{1}{2}\right)^{2n-2}\left(\frac{\log\pi}{2}\right)^{2n-2}+2\left(\frac{1}{2}\right)^{2n-2}\left(\frac{\log\pi}{2}\right)^{2n-2}\left\{\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{k-n}(it)^n}{n!}\prod_{j=0}^{n-1}(k-j)\right\}$$
$$+\left(\frac{\log\pi}{2}\right)^{2n-2}\left\{\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{k-n}(it)^n}{n!}\prod_{j=0}^{n-1}(k-j)\right\}^2 \quad (26)$$

The above equation allows us to write (17) as follows:

$$\zeta(z)=\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\left(\frac{\log\pi}{4}\right)^k(2^3nC)\left[\frac{A}{(n-1)!}+\frac{E}{2^{2n}}t^{2n}\right]\right]$$
$$+\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\log\pi(2^2nC)\left\{\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{k-n}}{n!}\prod_{j=0}^{k}(k-j)\right\}\left[\frac{A}{(n-1)!}t^n+\frac{E}{2^{2n}}t^{3n}\right]\right]i^n$$
$$+\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\log\pi(2^5n^2)\left\{\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{k-n}}{n!}\prod_{j=0}^{k}(k-j)\right\}\left[At^{n+1}+\frac{E}{2^{2n}}t^{3n+1}\right]\right]i^{n+1}$$
$$+\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\left(\frac{\log\pi}{4}\right)^k(2^6n^2)\left[At+\frac{E}{2^{2n}}t^{2n+1}\right]\right]i \quad (27)$$

If the above series is truncated at L= even number then, (27) becomes;

$$\zeta(z)=\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\left(\frac{\log\pi}{4}\right)^k(2^3nC)\left[\frac{A}{(n-1)!}+\frac{E}{2^{2n}}t^{2n}\right]\right]$$
$$+\delta\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\log\pi(2^2nC)\left\{\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{k-n}}{n!}\prod_{j=0}^{k}(k-j)\right\}\left[\frac{A}{(n-1)!}t^n+\frac{E}{2^{2n}}t^{3n}\right]\right]$$
$$+\rho\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\log\pi(2^5n^2)\left\{\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{k-n}}{n!}\prod_{j=0}^{k}(k-j)\right\}\left[At^{n+1}+\frac{E}{2^{2n}}t^{3n+1}\right]\right]i$$
$$+\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\left(\frac{\log\pi}{4}\right)^k(2^6n^2)\left[At+\frac{E}{2^{2n}}t^{2n+1}\right]\right]i \quad (28)$$

where *δ and ρ could be either -1 or $+1$.*

On the other hand, if L is an odd number then the series in (27) becomes;

$$\zeta(z)=\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\left(\frac{\log\pi}{4}\right)^k(2^3nC)\left[\frac{A}{(n-1)!}+\frac{E}{2^{2n}}t^{2n}\right]\right]$$
$$+\rho\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\log\pi(2^5n^2)\left\{\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^{k-n}}{n!}\prod_{j=0}^{k}(k-j)\right\}\left[At^{n+1}+\frac{E}{2^{2n}}t^{3n+1}\right]\right]i$$
$$+\delta\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\log\pi(2^2nC)\left\{\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^{k-n}}{n!}\prod_{j=0}^{k}(k-j)\right\}\left[\frac{A}{(n-1)!}t^n+\frac{E}{2^{2n}}t^{3n}\right]\right]$$
$$+\sum_{n=1}^{L=\infty}\frac{1}{D}\left[\left(\frac{\log\pi}{4}\right)^k(2^6n^2)\left[At+\frac{E}{2^{2n}}t^{2n+1}\right]\right]i \quad (29)$$

δ and ρ remain as defined above.

On multiplying (17) by its conjugate, we obtain $\zeta(z)\overline{\zeta(z)}$ to be;

$$\sum_{n=1}^{\infty}\frac{B^2}{D^2}\left\{\left[\frac{2^3nAC}{(n-1)!}+\frac{1}{2^{2n}}(2^3n\,EC)t^{2n}\right]^2+\left[(2^6n^2A)t+\frac{1}{2^{2n}}(2^6n^2E)t^{2n+1}\right]^2\right\} \quad (30)$$

This can be neatly written as;

$$\zeta(z)\overline{\zeta(z)}=(\gamma^2(t)+\beta^2(t)) \quad (31),$$

where

$$\gamma(t)=\sum_{n=0}^{\infty}\frac{B}{D}\left[\frac{2^3nAC}{(n-1)!}+\frac{1}{2^{2n}}(2^3n\,EC)t^{2n}\right]\ and\ \ \beta(t)=\sum_{n=0}^{\infty}\frac{B}{D}\left[(2^6n^2A)t+\frac{1}{2^{2n}}(2^6n^2E)t^{2n+1}\right] \quad (32)$$

From the above, it is clear that (17) gives *$\gamma(t)$ as the state variable and $\beta(t)$ as the control variable.*

$$\zeta(z)\overline{\zeta(z)}=$$
$$\left[\sum_{n=0}^{L=\infty}\left[\frac{2^6n^2E^2C^2}{2^{4n}}\right]+(2^{12}n^4A^2)t^2+\left[\frac{2^7n^2AEC^2}{2^{2n}(n-1)!}\right]t^{2n}+\left[\frac{2^6n^2E^2C^2}{2^{4n}}\right]t^{4n}\right]\left\{\left(\frac{1}{2}\right)^{2n-2}\left(\frac{\log\pi}{2}\right)^{2n-2}\right\}$$
$$\left[(2^{12}n^4EA)t^{2n+2}+\left[\frac{2^{16}n^4E^2}{2^{4n}}\right]t^{4n+2}\right]\left\{\left(\frac{1}{2}\right)^{2n-2}\left(\frac{\log\pi}{2}\right)^{2n-2}\right\}+$$

[illegible]

$$+\sum_{n=0}^{L=\infty}\left[(2^{12}n^4EA)t^{2n+2}+\left[\frac{2^{16}n^4E^2}{2^{4n}}\right]t^{4n+2}\right]\left[\left(\frac{\log\pi}{2}\right)^{2n-2}\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{2k-2n}(it)^{2n}}{(n!)^2}\prod_{j=0}^{n-1}(k-j)^2\right]$$

[illegible]

$$+\sum_{n=0}^{L=\infty}\left[(2^{12}n^4EA)t^{2n+2}+\left[\frac{2^{16}n^4E^2}{2^{4n}}\right]t^{4n+2}\right]\left(\frac{1}{2}\right)^{2n-2}\left(\frac{\log\pi}{2}\right)^{2n-2}\left\{\sum_{n=1}^{L=\infty}\frac{\left(\frac{1}{2}\right)^{k-n}(it)^n}{n!}\prod_{j=0}^{n-1}(k-j)\right\}$$

Conclusion

If we choose to minimize the integral of (31), we come to obtain;

$$\min\int_a^b\zeta(z)\overline{\zeta(z)}dz=\min\int_a^b[\gamma^2(t)+\beta^2(t)]dt \quad (35)$$

Furthermore, (35) is a quadratic function for which its bilinear transformation is given as;

$$\min\int_a^b[\gamma^2(t)+\beta^2(t)]dt=\min\int_a^b[\gamma^T(t)P\gamma(t)+\beta^T(t)M\beta(t)]dt \quad (36)$$

On imposing some **constraints** on (36), it becomes an optimization problem of the form;

$$\text{mimin}\int_a^b[\gamma^T(t)P\gamma(t)+\beta^T(t)M\beta(t)]dt \quad (37)$$

Subject to the constraints; $\quad \gamma(t) = \frac{d\ \mathbf{R}\zeta(z)}{dt}$

$$0\le t\le T, \qquad \gamma(0)=\frac{1}{2}$$

The constrained problem (37) can be turned into unconstrained problem via the penalty method and the multiplier method (34) as;

$$(Z,AZ)_{\downarrow}H=\min\int_{\downarrow a}^{\uparrow b}[\gamma^{\uparrow}T(t)P\gamma(t)+\beta^{\uparrow}T(t)M\beta(t)+\mu\|\gamma(t)^{\uparrow}/-(d\,\mathbf{R}\zeta(z))/dt\ \|^{\uparrow}2+(\lambda,\gamma(t)^{\uparrow}/-(d\,\mathbf{R}\zeta(z))/dt\ )]dt \quad (38)$$

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