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Enestrom- Kakeya Theorem and Zero-free regions of polynomials

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ABSTRACT

In the literature, some extensions and generalizations of Enestrom-Kakeya theorem are available. In this paper we refine results by defining the zero –free regions of polynomials.

1. Introduction and statement of results

In the theory of distribution of zeros of polynomials, the Enestrom-Kakeya theorem [4] given below in theorem A is a well known result.

Theorem A. Let P(z) = $\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq .....\geq a_1 \geq a_0 > 0$$

Then all the zeros of P(z) lie in the disk $|z| \leq 1$.

Many attempts have been made to extend and generalize the Enestrom-Kakeya theorem . A. Joyal et al [3] extended the Enestrom-Kakeya theorem to the polynomials with general monotonic coefficients by proving that if

$$a_n \geq a_{n-1} \geq .....\geq a_1 \geq a_0$$

Then P(z) has all its zeros in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$$

Further Aziz and Zargar [1] generalized the result of A. Joyal et al [3] and the Enestrom-Kakeya theorem as given below in theorem B.

Theorem B. Suppose P(z) = $\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that

For some $\lambda \geq 1$,

$$\lambda a_n \geq a_{n-1} \geq .....\geq a_1 \geq a_0$$

then all the zeros of P(z) = $\sum_{i=0}^{n} a_i z^i$, lie in the disk

$$|z + (\lambda - 1)| \leq \frac{\lambda a_n - a_0 + |a_0|}{|a_n|}$$

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Theorem C. Suppose $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that either

$a_n \geq a_{n-2} \geq .....\geq a_2 \geq a_0 > 0$ and $a_{n-1} \geq a_{n-3} \geq .....\geq a_3 \geq a_1 > 0$ if n is even

Or $a_n \geq a_{n-2} \geq .....\geq a_3 \geq a_1 > 0$ and $a_{n-1} \geq a_{n-3} \geq .....\geq a_2 \geq a_0 > 0$ if n is odd , then all the zeros of P(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \leq 1 + \frac{a_{n-1}}{a_n}$$

But Govil and Rahman [2] proved that if , $P(z) = \sum_{i=0}^{n} a_i z^i$ is a complex polynomial of degree n with $|arg.a_i - \beta| \leq \alpha \leq \frac{\pi}{2}$, (i= 0,1,2,.......n) for some β real and $|a_n| \geq |a_{n-1}| \geq ....\geq |a_1| \geq |a_0|$, then P(z) has all its zeros in the disk

$$|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{i=0}^{n-1}|a_i|$$

Again Shah and Liman [5] generalized theorem B and the result of Govil and Rahman [2] and proved theorems D and E as stated below.

Theorem D. Suppose $P(z) = \sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n with Re(a_i)= α_i and Im(a_i)= β_i , i=0,1,2,........,n. If for some $\lambda \geq 1$

$\lambda\alpha_n \geq \alpha_{n-1} \geq .....\geq \alpha_1 \geq \alpha_0,\ \beta_n \geq \beta_{n-1} \geq .....\geq \beta_1 \geq \beta_0 > 0$

Then P(z) has all its zeros in the disk

$$\left|z + \frac{(\lambda-1)\alpha_n}{a_n}\right| \leq \frac{\lambda\alpha_n - \alpha_0 + |\alpha_0| + \beta_n}{|a_n|}$$

Theorem E. Suppose $P(z) = \sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n such that

$|arg.a_i - \beta| \leq \alpha \leq \frac{\pi}{2}$, (i= 0,1,2,.......n) for some β real and for some $\lambda \geq 1$

$\lambda|a_n| \geq |a_{n-2}| \geq ....\geq |a_2| \geq |a_0|$

Then P(z) has all its zeros in the disk

$$|z + (\lambda - 1)| \leq \frac{1}{|a_n|}\left\{(\lambda|a_n| - |a_0|)(\cos\alpha + \sin\alpha) + |a_0| + \frac{\sin\alpha\sum_{i=0}^{n-1}|a_i|}{2}\right\}$$

The main purpose of this paper is to refine some results mentioned above and define the zero -free regions of polynomials in theorems C, D and E .

2. Theorems And Proofs

Theorem 1. Suppose $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that either

$a_n \geq a_{n-2} \geq .....\geq a_2 \geq a_0$ and $a_{n-1} \geq a_{n-3} \geq .....\geq a_3 \geq a_1$ if n is even

Or, $a_n \geq a_{n-2} \geq .....\geq a_3 \geq a_1$ and $a_{n-1} \geq a_{n-3} \geq .....\geq a_2 \geq a_0$ if n is odd

Then P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| + a_n + |a_{n-1}| + a_{n-1} + |a_1| - a_1 - a_0}$$

Proof. To prove the theorem , we consider a polynomial F(z) defined by

F(z) = $(1 - z^2)$ P(z)

$= (1 - z^2)(a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_{n-1}z^{n-1} + a_n z^n)$

$= -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_1(n-2))z^{\uparrow}n + (a_{n-1} - a_1(n-3))z^{\uparrow}(n-1) + \ldots\ldots + (a_3 - a_1 1)z^{\uparrow}3 + (a_2 - a_0)z^2 + a_1 z + a_0$

$= g(z) + a_0$,

where $g(z) = -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_1(n-2))z^{\uparrow}n + (a_{n-1} - a_1(n-3))z^{\uparrow}(n-1) + \ldots\ldots + (a_3 - a_1 1)z^{\uparrow}3 + (a_2 - a_0)z^2 + a_1 z$

If $|z| < 1$ then

$|g(z)| \leq |a_1 n| + |a_1(n-1)| + (a_1 n - a_1(n-2)) + (a_{n-1} - a_1(n-3)) + \ldots\ldots + (a_3 - a_1 1) + (a_2 - a_0) + |a_1|$

since by hypothesis $a_n \geq a_{n-2} \geq \ldots\ldots \geq a_2 \geq a_0$ and $a_{n-1} \geq a_{n-3} \geq \ldots\ldots \geq a_3 \geq a_1$.

On simplification , we have

$|g(z)| \leq |a_n| + |a_{n-1}| + a_n + a_{n-1} - a_1 - a_0 + |a_1|$

Also we have, g(0) = 0, therefore by Schwarz lemma , it follows that $|g(z)| \leq M|z|$ for $|z| < 1$ where M = $|a_n| + |a_{n-1}| + a_n + a_{n-1} - a_1 - a_0 + |a_1|$

Again for $|z| < 1$, we have

$|F(z)| = |g(z) + a_0| = |a_0 + g(z)| \geq |a_0| - |g(z)|$

$\geq |a_0| - M|z|$

> 0 if $|a_0| > M|z|$

i.e. if $|z| < \dfrac{|a_0|}{M}$

Also we can show that M $\geq |a_0|$ as $|z| < 1$ and hence the desired result follows.

Theorem 2. Let P(z) = $\sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n with Re(a_i)= α_i and Im(a_i)= β_i , i=0,1,2,........,n . Let for some $\lambda \geq 1$,

$\lambda\alpha_n \geq \alpha_{n-1} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0$,

$\beta_n \geq \beta_{n-1} \geq \ldots\ldots \geq \beta_1 \geq \beta_0$, then P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| + (\lambda-1)|a_n| + \lambda\alpha_n - \alpha_o + \beta_n - \beta_0}$$

Proof. To prove the theorem, we consider a polynomial F(z) defined by

F(z) =(1 – z) P(z) = (1 – z) $\sum_{i=0}^{n} a_i z^i$

$= (1-z)(a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_{n-1}z^{n-1} + a_n z^n)$

On simplification , we have

F(z) = $-a_n z^{n+1} + (a_n - a_1(n-1))z^{\uparrow}n + \ldots\ldots\ldots + (a_1 - a_0)z + a_0$

$= g(z) + a_0$, where $g(z) = -a_n z^{n+1} + (a_n - a_1(n-1))z^{\uparrow}n + \ldots\ldots\ldots + (a_1 - a_0)z$

Using hypothesis, we can write g(z) as

$g(z) = -a_n z^{n+1} + (\alpha_n - \alpha_1(n-1))z^{\uparrow}n + (\alpha_1(n-1) - \alpha_1(n-2))z^{n-1} + \ldots\ldots\ldots + (\alpha_1 - \alpha_0)z +$ i $\{(\beta_1 n - \beta_1(n-1))z^{\uparrow}n + (\beta_1(n-1) - \beta_1(n-2))z^{n-1} + \ldots\ldots + (\beta_1 - -\beta_0)z\}$

$= -a_n z^{n+1} - (\lambda-1)\alpha_n z^n +$

$(\square\lambda\alpha\square_1 n - \alpha_1(n-1))z^{\uparrow}n + (\alpha_1(n-1) - \alpha_1(n-2))z^{\uparrow}(n-1) + \ldots\ldots$

$+(\alpha_1 - \alpha_0)z +$ i $\{(\beta_1 n - \beta_1(n-1))z^{\uparrow}n + (\beta_1(n-1) - \beta_1(n-2))z^{n-1} + \ldots\ldots + (\beta_1 - \beta_0)z\}$

Now if $|z| < 1$,then on simplification , we have

$|g(z)| \leq |a_n| + (\lambda-1)|a_n| + \lambda\alpha_n - \alpha_o + \beta_n - \beta_0$

From above , g(0) = 0, therefore by Schwarz lemma , it follows that

$|g(z)| \le M|z|$ for $|z| < 1$, where M= $|a_n| + (\lambda - 1)|a_n| + \lambda a_n - \alpha_o + \beta_n - \beta_0$

Again for $|z| < 1$,

$|F(z)| = |g(z) + a_0| \ge |a_0| - |g(z)| \ge |a_0| - M|z| > 0,$ if $|a_0| > M|z|$

i.e., if $|z| < \dfrac{|a_0|}{M}$

where M = $|a_n| + (\lambda - 1)|a_n| + \lambda a_n - \alpha_o + \beta_n - \beta_0$

Also we can show that M $\ge |a_0|$ as $|z| < 1$

Hence the desired result follows.

Theorem 3. Suppose P(z) = $\sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n such that

$|arg. a_i - \beta| \le \alpha \le \frac{\pi}{2}$, (i= 0,1,2,.......n) for some β real and for some $\lambda \ge 1$

$\lambda|a_n| \ge |a_{n-1}| \ge$ ....$\ge |a_1| \ge |a_0|$

Then P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{\lambda|a_n| + (\lambda|a_n| - |a_0|)\cos\alpha + (\lambda|a_n| + |a_0|)\sin\alpha + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|}$$

Proof. To prove the theorem , we consider a polynomial F(z) defined by

F(z) = (1 – z) P(z)

= (1 – z) ($a_0 + a_1 z + a_2 z^2$ +......+$a_{n-1}z^{n-1}$ + $a_n z^n$)

= $-a_n z^{n+1}$ +(a_n - $a_1(n-1))z^{\uparrow}n$ + ...+(a_1 - a_0)z + a_0

= $-a_n z^{n+1} - (\lambda - 1)a_n z^n$ + (□λa□↓n − $a_1(n-1))z^{\uparrow}n$ + ...+(a_1 - a_0)z + a_0

= g(z) + a_0 , where

g(z) = $-a_n z^{n+1} - (\lambda - 1)a_n z^n$ + (□λa□↓n − $a_1(n-1))z^{\uparrow}n$ + ...+(a_1 - a_0)z

It was shown in [2] that for two complex numbers b_0 , b_1 if

$|b_0| \ge |b_1|$ and $|arg. b_i - \beta| \le \alpha \le \frac{\pi}{2}$, (i= 0,1)

for some β then

$|b_0 - b_1| \le (|b_0| - |b_1|)\cos\alpha + (|b_0| + |b_1|)\sin\alpha$

Hence for $|z| < 1$,

$|g(z)| \le |a_n|$+(λ -1) $|a_n|$ + $|\lambda a_n - a_{n-1}|$ + $|a_{n-1} - a_{n-2}|$ + ...+ $|a_1 - a_0|$

$$\le \lambda|a_n| + (\lambda|a_n| - |a_0|)\cos\alpha + (\lambda|a_n| + |a_0|)\sin\alpha + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|$$

Again we have , g(0) = 0, therefore by Schwarz lemma we obtain

$|g(z)| \le M|z|$ for $|z| < 1$, where

$$M = \lambda|a_n| + (\lambda|a_n| - |a_0|)\cos\alpha + (\lambda|a_n| + |a_0|)\sin\alpha + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|$$

Therefore for $|z| < 1$, we have

$|F(z)| = |g(z) + a_0| \ge |a_0| - |g(z)| \ge |a_0| - M|z| > 0,$ if $|a_0| > M|z|$

i.e, if $|z| < \dfrac{|a_0|}{M}$, where

$$M = \lambda|a_n| + (\lambda|a_n| - |a_0|)\cos\alpha + (\lambda|a_n| + |a_0|)\sin\alpha + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|$$

Also we can show that M $\ge |a_0|$ as $|z| < 1$

Hence the desired result follows.

References

[1] A.Aziz and B.A. Zargar , Some extensions of Enestrom-Kakeya theorem,*Glasnik Matematicki* , 31 (1996), 239-244.

[2] N.K.Govil and Q.I.Rahman, On the Enestrom-Kakeya theorem, *Tohoku Math. J.* , 20 (1968), 126-136 .

[3] A. Joyal, G.Labelle and Q.I.Rahman, On the location of zeros of polynomials, , *Canad. Math. Bull*., 10 (1967), 55-63.

[4] M. Marden, Geometry of polynomials , Math.Surveys, No. 3*; Amer.Math*. *Society* , (R.I.: Providence) (1966).

[5] W.M.Shah and A. Liman , On Enestrom-Kakeya theorem and related analytic functions *, Proc. Indian Acad. Sci., (Math.Sci.)* 117 (2007), 359-370.