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Congruence lattices of uniform lattices

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ABSTRACT

In this Chapater we prove two important results. Let L be a lattice. A congruence of L is said to be uniform, if any two congruence classes of are of the same size. The lattice L is said to be uniform, if all congruences of L are uniform. We prove that every finite distributive lattice D can be represented as the congruence lattice of a finite uniform lattice.

1.1 Introduction

In this chapter we prove that every finite distributive lattice D can be represented as the congruence lattice of finite uniform lattice L. Infact we prove that "For any finite distributive lattice D, there exists a finite uniform lattice L such that the congruence lattice of L is isomorphic to D, and L satisfies the properties (P) and (Q) where

(P) Every join-irreducible congruence of L is of the form θ (0,p), for a suitable atom p of L.

(Q) If θ_1, θ_2 ,…………., $\theta_n \in$ J (ConL) are pairwise incomparable, then L contains atoms p_1, p_2,………., p_n that generate an ideal isomorphic to B_n and satisfy $\theta_i=\theta$ $(0,p_i)$, for all $i \leq n$.

To prove this result, we introduce a new lattice construction which is described in section 1.2. Then we find the congruences on this new lattice in section 1.3. In 1.4, we introduce a very simple kind of chopped lattices. In section 1.5, we prove that the ideal lattice of this chopped lattice is uniform. The proof of the theorem is presented in section 1.6.

Notation:

B_n will denote the Boolean algebra with 2^n elements. For a bounded lattice A with bounds 0 and 1, A^- will denote the lattice A – {0,1}

We start with the definition of uniform lattices.

Definition : 1.1.1

A congruence θ of a lattice L is uniform, if any two congruence classes A and B of θ are of the same size. That is, $|A| = |B|$.

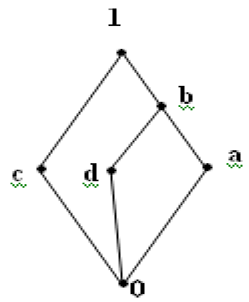
Definition : 1.1.2

A lattice L is said to be uniform, if all of its congruences are uniform.

Note : 1.1.3

Every lattice need not be a uniform lattice.

For example, the lattice N_6, given below is not uniform.



The lattice has exactly one non-trivial congruence θ and θ has exactly two congruence classes {0, a, b, d} and {e,l}.

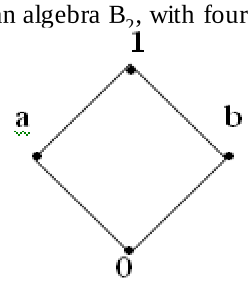
These two congruence classes are not of the same order.

Note : 1.1.4

There exists uniform lattices.

For example,

Consider the Boolean algebra B_2, with four elements.



Its congruence lattice is also B_2. It has 4 congruences. The null congruence ω, the all congruence i and two non-trivial congruences θ_1 and θ_2.

θ_1 has two congruence classes {{0, a,},{b, l}} and θ_2 has two congruence classes {{0,b},{a,1}} and both θ_1 and θ_2 are uniform congruences.

Hence B_2 is a uniform lattice.

1.2. A LATTICE CONSTRUCTION

Let A and B be lattices. Let us assume that A is bounded with bounds 0 and 1 with $0 \neq 1$. We introduce a new lattice construction N(A,B).

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If $u \in A x B$, then $u = (u_A, u_B)$ where $u_A \in A$ and $u_B \in B$. The binary relation $\leq_X$ will denote the partial ordering on AxB, and V_X and Λ_X the join and the meet in AxB respectively.

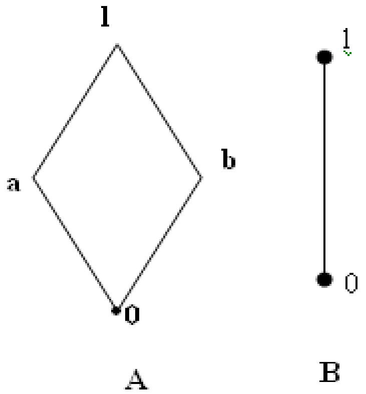
On the set AxB, we define a new binary relation denoted by $\leq_N$ as follows :

$\leq_N = \leq_X - \{(u,v) / u,v \in A^- x B \text{ and } u_B \neq v_B\}$.

We denote (AxB, $\leq_N$) by (N(A,B), $\leq_N$).

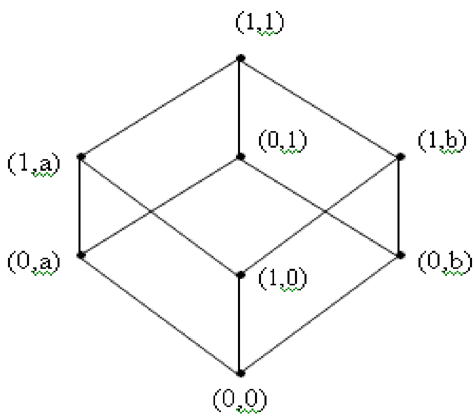
Example : 1.2.1

For example, consider A=B_2 and B=B_1



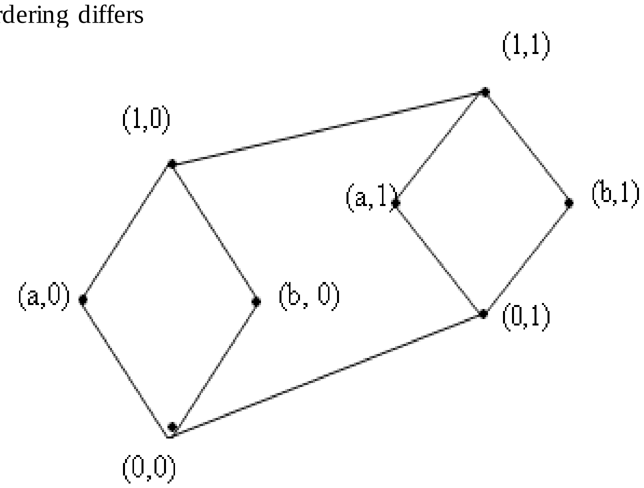
AxB= {(0,0), (0,1), (a,0), (a,1), (b,0), (b,1), (1,0), (1,1)}

AxB



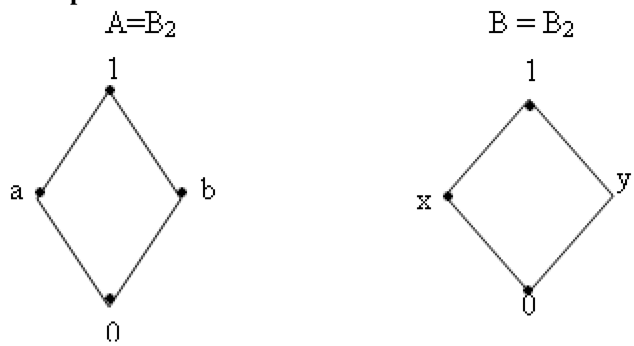
Then (AxB, $\leq_X$) is the lattice.

But N (A, B) has elements the same as AxB. But the partial ordering differs



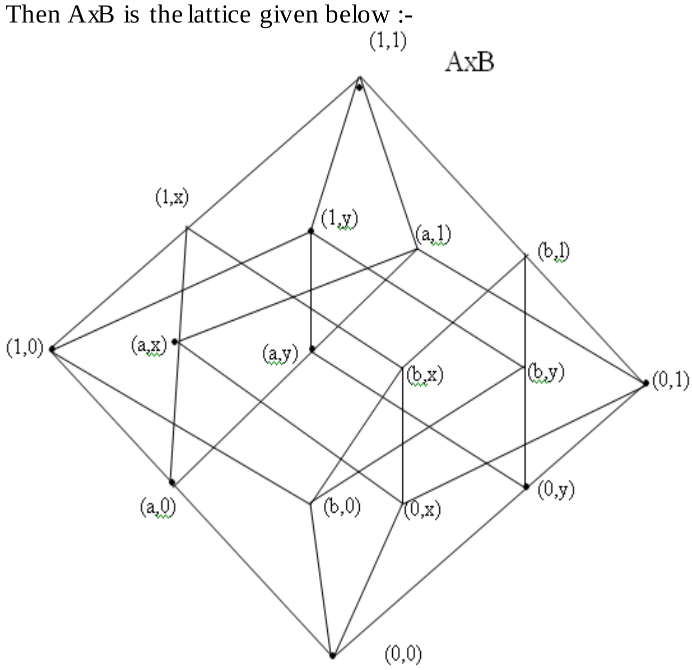
N (A,B)

Example: 1.2.2

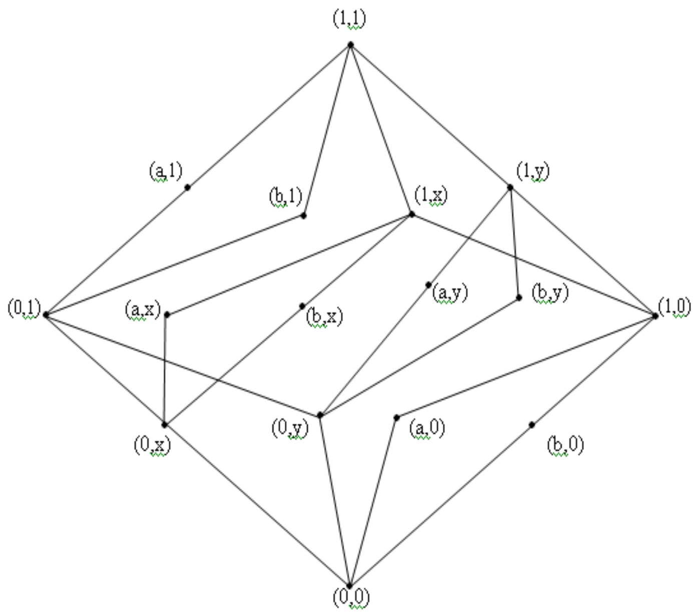


AxB = {(0,0), (0,x), (0,y), (0,1), (a,0), (a,x), (a,y), (a,1), (b,0), (b,x), (b,y), (b,1), (1,0), (1,x), (1,y), (1,1)}

Then AxB is the lattice given below :-

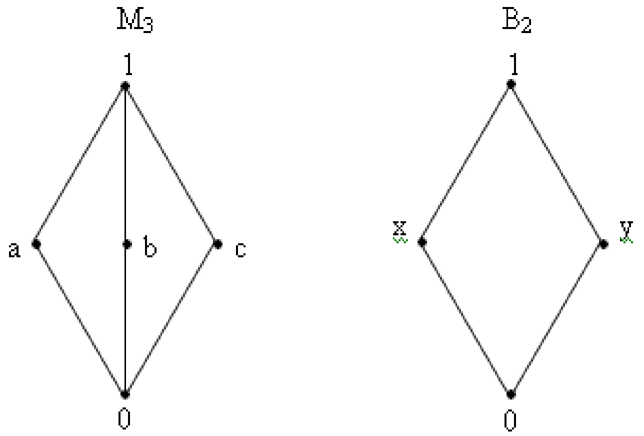


But N (B_2,B_2) is the lattice given below :



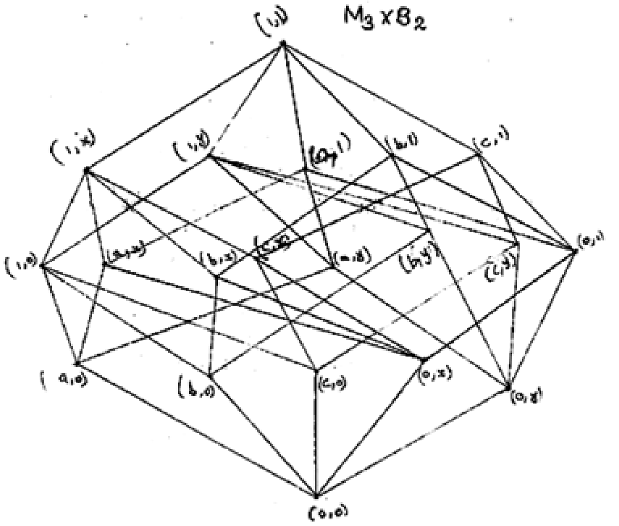
Example : 1.2.3

Consider the lattice $A=M_3$ and $B=B_2$

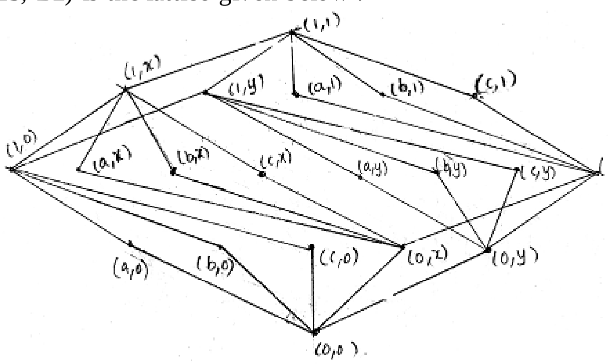


Then AxB = { (0,0) (0,x), (0,y), (0,1), (a,0), (a,x), (a,y), (a,1) (b,0), (b,x), (b,y), (b,1), (c,0), (c,x), (c,y), (c,1), (1,0), (1,x), (1,y), (1,1) }

Then the lattices $M_3 x B_2$ and $N(M_3, B_2)$ are as given below :



N (M3, B2) is the lattice given below :



Now we prove that N(A,B) is a lattice.

Lemma : 1.2.4

Let A and B be lattices. Let A be a bounded lattice with bounds 0 and 1 and $0\neq 1$. Then N(A,B) is a lattice. The meet and join in N(A,B) of $\leq_N$ -incomparable elements can be computed by the formulae.

$$u\Lambda_N v = \begin{cases} (0, u_B \Lambda v_B), & \text{if } u\Lambda_X v \in A^- x B \text{ and } u_B \neq v_B; \\ u\Lambda_X v, & \text{otherwise.} \end{cases}$$

$$uV_N v = \begin{cases} (1, u_B V v_B), & \text{if } uV_X v \in A^- x B \text{ and } u_B \neq v_B; \\ uV_X v, & \text{other wise.} \end{cases}$$

Proof:-

First we claim that (N(A,B), $\leq_N$) is a poset.

(i) $\leq_N$ is reflexive.

Let $a\in A$, $b\in B$.

Then $(a,b) \leq (a,b)$ in AxB.

Therefore, $((a,b), (a,b)) \in \leq_x$.

But $((a,b), (a,b)) \notin \{(u,v) / u,v \in A^- xB,\ u_B \neq v_B\}$ for b=b.

$\therefore$ $((a,b), (a,b)) \in \leq_N$.

$\therefore$ $\leq_N$ is reflexive.

(ii) $\leq_N$ is antisymmetric.

Let $(a,b) \leq_N (c,d)$ and $(c,d) \leq_N (a,b)$.

Then $(a,b) \leq_X (c,d)$ and $(c,d) \leq_X (a,b)$.

But $\leq_X$ is antisymmetric, hence (a,b) = (c,d) in AxB.

$\therefore$ a=c and b=d.

$\therefore$ (a,b) = (c,d) under $\leq_N$ for b=d.

$\therefore$ $\leq_N$ is antisymmetric.

(iii) $\leq_N$ is transitive.

Let $(a,b) \leq_N (c,d)$ and $(c,d) \leq_N (e,f)$

Then $(a,b) \leq_X (c,d)$ and $(c,d) \leq_X (e,f)$

But $\leq_X$ is a transitive relation.

Hence $(a,b) \leq_X (e,f)$.

$(a,b) \leq_N (c,d)$ implies $((a,b), (c,d)) \notin \{(u,v)/u,v \in A^- xB,\ u_B \neq v_B\}$

$\therefore$ b=d.

Similarly, $(c,d) \leq_N (e,f)$ implies d=f.

b=d, d=f implies b=f.

$(a,b) \leq_X (e,f)$ and b=f implies

$((a,b),(e,f)) \notin \{(u,v) / u,v \in A^- xB,\ u_B \neq v_B\}$.

$\therefore$ $(a,b) \leq_N (e,f)$.

$\therefore \leq_N$ is a transitive relation.

That is (N(A,B), $\leq_N$) is a poset.

To prove (N(A,B)$\leq_N$) is a lattice.

For that we have to prove $u\Lambda_N v$, $uV_N v$ exist for all elements $u,v \in N(A,B)$.

For that, it is enough if we prove that $u\Lambda_N v$, $uV_N v$ exist for $\leq_N$-incomparable elements $u, v \in N(A,B)$.

Because of duality principle, it is enough if we prove that $u\Lambda_N v$ exists for $\leq_N$-incomparable elements $u, v \in N(A,B)$.

Let $u, v \in N(A,B)$ and u, v be $\leq_N$ -incomparable.

Let t be a lower bound of u and v in N(A,B).

Case : 1

$u\Lambda_x v$ is not a lower bound of both u and v in N(A,B).

If $u\Lambda_x v$ is not a lower bound of both u and v in N(A,B), then either $u\Lambda_x v \Leftarrow_N u$ or $u\Lambda_x v \Leftarrow_N v$.

Suppose $u\Lambda_x v \Leftarrow_N u$,

then $u, u\Lambda_x v \in A^- xB$ and $u_B \neq (u\Lambda_x v)_B$.

But $(u\Lambda_x v)_B \neq u_B$ implies that $u_B \neq v_B$.

Since $t \leq_x u\Lambda_x v$, it follows that $t_B \leq (u\Lambda_x v)_B < u_B$ and so $t \notin A^- xB$.

For, if $t \in A^- xB$, $u \in A^- xB$ and $t_B \neq u_B$ implies that $t \leq_N u$.

Which is a contradiction to t is a lowerbound of u.

Since $t \notin A^- xB$, the first element of t must be 0 or 1.

If $t = (1, t_B)$, then it gives a contradiction to $t \leq_x u\Lambda_x v$.

$\therefore$ t must be equal to $(0, t_B)$.

That is, $t=(0,t_B)$.

Since $t_B \leq_x u_B$ and $t_B \leq_X v_B$, it follows that $t_B \leq u_B \Lambda v_B$.

$\therefore\ t \leq (0,u_B\Lambda_X v_B)$.

$\therefore\ u\Lambda_N v = (0,u_B\Lambda_X v_B)$.

Case : 2

$u\Lambda_X v$ is a lower bound of both u and v in N(A,B)

We claim that $u\Lambda_N v = u\Lambda_X v$

For that it is enough if we prove that $t \leq_N u\Lambda_X v$

Suppose $t \Leftarrow_N u\Lambda_X v$, then $t, u\Lambda_X v \in A^{-}xB$ and $t_B \neq (u\Lambda_X v)_B$

$\therefore\ t_B < (u\Lambda_X v)_B$.

$\therefore\ u\in A^{-}xB$ or $v\in A^{-}xB$

Suppose $u\in A^{-}xB$

$\therefore$The assumption of case 2, namely, $u\Lambda_X v \leq_N u$, implies that $(u\Lambda_X v)_B = u_B$,

$\therefore$t, $u\in A^{-}xB$ and $t_B \neq u_B$, contradicting that $t \leq_N u$.

Similarly, for $v\in A^{-}xB$

Thus, $t \Leftarrow_N u\Lambda_X v$ leads to a contradiction.

$\therefore\ t \leq_N u\Lambda_X v$.

Hence in case 2, $u\Lambda_N v = u\Lambda_X v$.

This verifies the meet formula.

Hence the lemma.

Notation :

$B_* = \{0\}$ x B, $B^* = \{1\}$ x B and for $b\in B$, A_b = A x {b}. We observe that B_* is an ideal of N(A,B) and B^* is a dual ideal of N(A,B).

1.3.CONGRUENCES ON N(A,B)

Definition : 1.3.1

Let K and L be lattices and let α be an embedding of K into L. Let θ be a congruence on L. We can define a congruence θ_1 on K via α. That is for a, $b\in K$ define θ_1 by $a \equiv b(\theta_1)$ if, and only if, $\alpha(a)\equiv \alpha(b)$ (θ).

We call θ_1 the restriction of θ transferred via the isomorphism α to K.

Remark : 1.3.2

Let A be a bounded lattice and B be a lattice. Then N(A,B) is a lattice. Define $\alpha : B \rightarrow N(A,B)$ by $\alpha(b)=(0,b)$ for all $b\in B$. Then α is an isomorphism of B into N(A,B) with image of α equal to B_*. Similarly, if we define $\beta : B \rightarrow N(A,B)$ by $\beta(b)=(1,b)$ for all $b\in B$, then β is an isomorphism of B into N(A,B) with image of β equal to B$*$. Define a map $\gamma_b : A \rightarrow N(A,B)$ by $\gamma_b(a) = (a,b)$ for all $a\in A$ and for a fixed $b\in B$. Then γ_b is an isomorphism of A into N(A,B) with the image of γ_b equal to A_b .

Remark : 1.3.3

Let Ψ be a congruence relation on N = N(A,B). Using the natural isomorphisam α of B into N(A,B), we define Φ_* as the restriction of Ψ to B_*. Using the natural isomorphism β of B into N(A,B), we define Φ^* as the restriction of Ψ to B^*. Using the natural isomorphism γ_b of A into N(A,B). We define θ_b as the restriction of ψ to A_b for $b\in B$.

Lemma : 1.3.4

$\Phi_* = \Phi^*$

Proof :-

Let $b_0 \equiv b_1$ (Φ_*).

Then $(0,b_0) \equiv (0, b_1)$ (ψ).

Joining both sides with $(1,b_0\Lambda b_1)$ we get,

$(0,b_0)$ V $(1,b_0\Lambda b_1) \equiv (0,b_1)$ V $(1,b_0\Lambda b_1)$ (ψ).

(ie) $(0V1,\ b_0 V (b_0\Lambda b_1)) \equiv (0V1,\ b_1 V (b_0\Lambda b_1))$ (ψ)

(ie) $(1,b_0) \equiv (1,b_1)$ (ψ)

$\therefore\ b_0 \equiv b_1$ (Φ^*).

Conversely, if $b_0 \equiv b_1(\Phi^*)$ then $(1,b_0) \equiv (1,b_1)(\Psi)$.

Taking meet on both sides with $(0, b_0 V b_1)$ we get,

$(1, b_0)\ \Lambda\ (0,b_0 V b_1) \equiv (1,b_1)\ \Lambda\ (0,b_0 V b_1)$ (Ψ)

(ie) $(1\Lambda 0,b_0\Lambda\ (b_0 V b_1)) \equiv (1\Lambda 0,\ b_1\ \Lambda\ (b_0 V b_1))$ (Ψ)

(ie) $(0,\ b_0) \equiv (0,b_1)$ (Ψ)

$\therefore\ b_0 \equiv b_1$ (Φ_*).

Hence $\Phi_* = \Phi^*$.

Note : 1.3.5

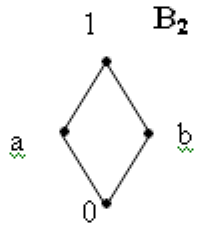
It is easy to see that $\Phi = \Phi_* = \Phi^* \in$ Con B and $\{\theta_b \mid b\in B\} \subseteq$ Con A. Further Φ and θ_b describe ψ.

Definition : 1.3.6

Let A be a bounded lattice. A congruence θ of A is said to separate 0 if $[0]\ \theta = \{0\}$. That is $x \equiv 0$ (θ) implies that x = 0. similarly, a congruence θ of A is said to separate 1 if $[1]\ \theta = \{1\}$. That is $x \equiv 1$ (θ) implies that x = 1. The lattice A is said to be non-separating, if 0 and 1 are not separated by any congruence $\theta \neq \omega$.

Example : 1.3.7

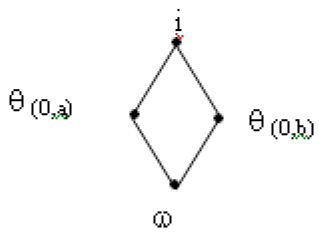
Consider the lattice B_2.



Then Con (B_2) = {ω, i, $\theta_{(0,a)}$, $\theta_{(0,b)}$}.

Con (B_2) is the lattice

Con (B$_2$)



Here ω is the only congruence separating 0 and 1.

Hence B_2 is a non-separating lattice.

Lemma : 1.3.8

Let A and B be lattices with $|A| > 2$ and $|B| > 1$. Let A be bounded with bounds 0 and 1. Let us further assume that A is non-separating. Let $\psi \neq \omega_N$ be a congruence of N(A,B). Define a map σ by $\sigma(\psi) = \Phi_*$,where Φ_* is the restriction of ψ to B_* via the natural isomorphism α. Then σ is a bijection between the non-ω_N congruences of N(A,B) and the congruences of B. Therefore, Con N(A,B) is isomorphic to Con B with a new zero added.

Proof : -

Let $\psi \neq \omega_N$ be congruence relation of N(A,B).

We start with the following statement.

Claim 1 :

There are elements $a_1 < a_2$ in A and an element $b_1\in B$ such that $(a_1,b_1) \equiv (a_2,b_2)$ (ψ)

Proof of Claim 1 :

Assume that $(u_1,v_1) \equiv (u_2,v_2)\ (\psi)$ with $(u_1,v_1) <_N (u_2,v_2)$

We distinguish two cases :

case (i) $u_1=u_2$

Then $(u_1,v_1) <_N (u_2,v_2)$ implies $v_1<v_2$ and either $u_1 = u_2 =0$ or $u_1 = u_2=1$.

That is either $(0,v_1) \equiv (0, v_2)\ (\psi)$ or $(1,v_1) \equiv (1, v_2)\ (\psi)$.

But $(0,v_1) \equiv (0,v_2)\ (\psi)$ implies $(1,v_1) \equiv (1,v_2)\ (\psi)$

and $(1,v_1) \equiv (1,v_2)\ (\psi)$ implies $(0,v_1) \equiv (0,v_2)\ (\psi)$

Hence we have both the congruences hold.

Since $|A| > 2$, we can choose $a \in A^-$.

Then $(a,v_1)\ V\ (0,v_1) = (aV0,\ v_1Vv_1) = (a,v_1)$

$(a,v_1) = (a,v_1)\ V\ (0,v_1) \equiv (a,v_1) V\ (0,v_2)$

$= (aV0,\ v_1Vv_2)$

$= (1,v_2)$ (since $v_1<v_2$ and by definition of $\leq_N$ in N(A,B), $aV0=1$)

$\therefore (a,v_1) \equiv (1,v_2)\ (\psi)$.

From this we get $(a,v_1)\ \Lambda\ (1,v_1) \equiv (1,v_2)\ \Lambda\ (1,v_1)\ (\psi)$.

That is $(a,v_1) \equiv (1,v_1)\ (\psi)$.

Hence the claim is true with $a_1 = a$, $a_2 = 1$ and $b_1 = v_1$.

Case (ii) $u_1< u_2$

Since we have assumed that $(u_1,v_1) <_N (u_2,v_2)$ it follows from the definition of $\leq_N$ that either $v_1 = v_2$ or $u_1 = 0$ or $u_2 = 1$.

If $v_1 = v_2$, then $(u_1,v_1) \equiv (u_2,v_1)\ (\psi)$ and so the claim is true with $a_1 = u_1$, $a_2 = u_2$ and $b_1 = v_1$.

If u = 0, then $(0, v_1) \equiv (u_2,v_2)\ (\psi)$

As $(u_1,v_1) <_N (u_2,v_2)$, (ie) $(0,v_1) <_N (u_2,v_2)$ we get $v_1 = v_2$

$\therefore (0,v_2) \equiv (u_2,v_2)\ (\psi)$.

Hence the claim is verified with $a_1 = 0$, $a_2 = u_2$ and $b_1 = v_2$.

If $u_2 = 1$, then $(u_1,v_1) \equiv (1,v_2)\ (\psi)$.

As $(u_1,v_1) <_N (1, v_2)$, it follows that $v_1 = v_2$.

$\therefore (u_1,v_1) \equiv (1,v_1)\ (\psi)$.

Hence the claim is true with $a_1 = u_1$, $a_2 = 1$ and $b_1 = v_1$.

Thus there are elements $a_1 < a_2$ in A and an element $b_1 \in B$ such that $(a_1,b_1) \equiv (a_2,b_1)\ (\psi)$.

Claim 2 :-

There is an element $b_2 \in B$ such that A_b is a single congruence class of ψ .

proof of claim 2 :-

By claim 1, there are $a_1<a_2$ in A and b_1 in B such that $(a_1,b_1) \equiv (a_2, b_1)\ (\psi)$.

Since A is non-separating, there exists $a_4 \in A$ with $a_4<1$ and $a_4 \equiv 1(\theta_{(0,a\ )})$.

As A_b is a sublattice of N(A,B), it follows that

$(0,b_1) \equiv (a_{3,}b_1)(\theta_{(a\ ,b\ ),(a\ ,b\ )})$

$\therefore (0,b_1) \equiv (a_3,b_1)\ (\psi)$.

So, for any $b_2 \in B$ with $b_1<b_2$, joining both sides with $(0,b_2)$ we obtain that $(0,b_1)\ V\ (0,b_2) \equiv (a_3,b_1)\ V\ (0,b_2)\ (\psi)$.

That is $(0,b_2) \equiv (1,b_2)\ (\psi)$.

(ie) A_b is in a single congruence ψ-class.

If b_1 is the unit element 1_B of B we cannot find a $b_2 \in B$ such that $b_1<b_2$.

Hence the proof is complete if b_1 is not the unit element of B.

If b_1 is the unit element of B, then we have $(0,\ 1_B) \equiv (a_3,1_B)\ (\psi)$.

Since A is non-separating, there exists $a_4 \in A$ with $a_4<1$ and $a_4 \equiv 1(\theta_{(0,a\ )})$.

Moreover A_1 is a sublattice of N(A,B).

So, it follows that $(a_4,1_B) \equiv (1,1_B)\ (\theta_{(0\ ,1\ B\ ),(a\ 3\ ,\ 1B\ )})$.

Therefore, $(a_4,1_B) \equiv (1,1_B)\ (\psi)$.

Now choose any $b_2 < 1_B$.

As $|B| >1$, such a b_2 exists.

Meeting bothsides with $(1,b_2)$, we obtain that

$(a_4,1_B)\ \Lambda\ (1,b_2) \equiv (1,1_B)\ \Lambda\ (1,b_2)\ (\psi)$.

That is $(0,b_2) \equiv (1,b_2)\ (\psi)$.

That is A_b is in a single congruence class of ψ.

Claim 3 :

A_b is in a single congruence class of ψ for each $b \in B$.

Proof of claim 3 :

Let $b \in B$

By claim 2, there is an element $b_2 \in B$ such that A_b is in a single congruence class of ψ.

(ie) $(1,b_2) \equiv (0,b_2)\ (\psi)$

$(1,b) = (\ (1,b_2)\ V\ (0,b\ V\ b_2)\ )\ \Lambda\ (1,b)$

$\equiv (\ (0,b_2)\ V\ (0,b\ V\ b_2)\ )\ \Lambda\ (1,b)$

$= (0,b)\ (\psi)$.

That is, $(1,b) \equiv (0,b)\ (\psi)$.

$\therefore A_b$ is in a single congruence class of ψ .

Proof of lemma :-

Let $\psi \in$ Con (N(A,B)) - $\{\omega_N\}$

Define σ : Con (N(A,B)) - $\{\omega_N\} \rightarrow$ Con (B) by

$\sigma(\psi) = \Phi_*$, where Φ_* is the restriction of ψ to B_*.

Claim : σ is one-one

Let $\psi_1, \psi_2 \in$ Con (N(A,B)) -$\{\omega_N\}$ be such that $\sigma(\psi_1) = \sigma(\psi_2)$.

That is $(\Phi_1)_* = (\Phi_2)_*$

Let $b_1 \equiv b_2\ (\Phi_1)_*$, then $(0,b_1) \equiv (0,b_2)\ (\psi_1)$.

$(\Phi_1)_* = (\Phi_2)_*$ and $b_1 \equiv b_2\ (\Phi_1)_*$ implies that $b_1 \equiv b_2 (\Phi_2)_*$

$\therefore (0,b_1) \equiv (0,b_2)\ (\psi_2)$.

Thus $(0, b_1) \equiv (0, b_2)\ (\psi_1)$ implies $(0,b_1) \equiv (0,b_2)\ (\psi_2)$.

Again $b_1 \equiv b_2\ (\Phi_1)_*$ implies $b_1 \equiv b_2\ (\Phi_2)_*$

$\therefore (0,b_1) \equiv (0,b_2)\ (\psi_1)$ implies $(1,b_1) \equiv (1,b_2)\ (\psi_1)$ and

$(0,b_2) \equiv (0,b_2)\ (\psi_2)$ implies $(1,b_1) \equiv (1,\ b_2)\ (\psi_2)$.

Thus $(1,b_1) \equiv (1,b_2)(\psi_1)$ implies $(1,b_1) \equiv (1,b_2)\ (\psi_2)$.

This implies $\psi_1=\psi_2$.

$\therefore \sigma$ is one-one.

Claim :- σ is onto

Let $\Phi \in$ Con (B)

Define a relation ψ on N(A,B) by

$(u_1,v_1) \equiv (u_2,v_2)\ (\psi)$ if, and only if, $v_1 \equiv v_2\ (\Phi)$.

Claim :- ψ is a congruence relation

(i) ψ is reflexive

Let $(u_1,v_1) \in$ N(A,B).

Then $v_1 \in B$.

Since Φ is reflexive, $v_1 \equiv v_2\ (\Phi)$.

By definition of ψ ,$(u_1,v_1) \equiv (u_1,v_1)(\psi)$.

$\therefore \psi$ is reflexive.

(ii) ψ is symmetric

Let (a,b) ,(c,d) $\in$ N (A,B) be such that $(a,b) \equiv (c,d)\ (\psi)$.

$(a,b) \equiv (c,d)\ (\psi) \Rightarrow b \equiv d\ (\Phi)$

$\Rightarrow d \equiv b\ (\Phi)$

$\Rightarrow (c,d) \equiv (a,b)\ (\psi)$

$\therefore \psi$ is symmetric.

(iii) ψ is transitive

Let (a,b), (c,d), (e,f)$\in$N(A,B) be such that

$(a,b) \equiv (c,d)(\psi)$ and $(c,d) \equiv (e,f)\ (\psi)$

Then $b \equiv d\ (\Phi)$ and $d \equiv f\ (\Phi)$.

$\therefore\ b \equiv f\ (\Phi)$.

This implies $(a,b) \equiv (e,f)\ (\psi)$.

$\therefore\ \psi$ is transitive.

$\therefore\ \psi$ is an equivalence relation.

Let $(u_1,v_1), (u_2,v_2), (u_3,v_3), (u_4,v_4) \in N(A,B)$ be Such that $(u_1,v_1) \equiv (u_2,v_2)\ (\psi)$ and $(u_3,v_3) \equiv (u_4,v_4)\ (\psi)$.

Then $v_1 \equiv v_2\ (\Phi)$ and $v_3 \equiv v_4\ (\Phi)$.

Since Φ is a congruence relation,

$v_1 V v_3 \equiv v_2 V v_4(\Phi)$ and $v_1 \Lambda v_3 \equiv v_2 \Lambda v_4\ (\Phi)$.

$\therefore\ (u_1 V u_3, v_1 V v_3) \equiv (u_2 V u_4, v_2 V v_4)\ (\psi)$ and

$(u_1 \Lambda u_3, v_1 \Lambda v_3) \equiv (u_2 \Lambda u_4, v_2 \Lambda v_4)\ (\psi)$.

$\therefore \psi$ is a congruence relation.

By definition of σ and ψ, we get $\sigma(\psi) = \Phi$

$\therefore \sigma$ is onto

$\therefore \sigma$ is a bijection from Con (N(A,B)) - $\{\omega_N\} \rightarrow$ Con (B).

Hence the lemma.

1.4 CHOPPED LATTICES

Definition : 1.4.1

Let M be a finite poset satisfying the following two conditions.

(i) Inf {a,b} exists in M, for any $a,b \in M$

(ii) Sup {a,b} exists for any $a,b \in M$ having a common upper bound in M.

In M, we define $a \Lambda b = \inf\{a,b\}$ and $aVb = \sup\{a,b\}$ whenever sup{a,b} exists in M.

Then M is a partial lattice called a chopped lattice.

Definition : 1.4.2

Let M be a finite chopped lattice. An equivalence relation θ of a chopped lattice M is a congruence relation if, and only if, $a \equiv b(\theta)$ and $c \equiv d\ (\theta)$ imply that $a\Lambda c \equiv b\Lambda d(\theta)$ and whenever aVc and bVd exist, $aVc \equiv bVd(\theta)$. The set Con M of all congruence relations of M partially ordered by set inclusion is again a lattice.

Definition : 1.4.3

Let M be a finite chopped lattice. A subset I of M is said to be an ideal of M if

(i) $i \in I$ and $a \in M$ imply $a\Lambda i \in I$

(ii) $a,b \in I$ implies $aVb \in I$ provided that aVb exists in M. The set IdM of all ideals of M partially ordered by set inclusion is a lattice.

Lemma : 1.4.4

Let M be a finite chopped lattice. Then for every congruence θ of M, there exists exactly one congruence $\overline{\theta}$ of IdM, such that for $a, b \in M$,

$(a] \equiv (b]\ (\overline{\theta})$ if, and only if, $a \equiv b\ (\theta)$

Proof :-

Since arbitrary meet exists in M, (m] is a finite lattice for every $m \in M$.

If {x, y} has an upperbound then xVy exists.

Let θ be a congruence relation on M.

For $X \subset M$, set $[X]\ \theta = \cup\ \{\ [x]\ \theta \mid x \in X\ \}$.

That is, $[X]\ \theta = \{y \mid x \equiv y(\theta)$ for some $x \in X\}$.

If, $I, J \in IdM$, define $I \equiv J(\overline{\theta})$ if, and only if, $[I]\theta = [J]\theta$.

Then $\overline{\theta}$ is an equivalence relation.

For,

(i) $[I]\theta = [I]\theta$ implies $I \equiv I(\overline{\theta})$.

$\therefore\ \overline{\theta}$ is reflexive.

(ii)Let $I \equiv J(\overline{\theta})$. Then $[I]\theta = [J]\theta$.

$\therefore$ $[J]\theta = [I]\theta$, which implies $J \equiv I\ (\overline{\theta})$.

$\therefore \overline{\theta}$ is symmetric.

(iii) Let $I \equiv J(\overline{\theta})$ and $J \equiv K(\overline{\theta})$.

Then $[I]\ \theta = [J]\theta$ and $[J]\ \theta = [K]\ \theta$.

Hence $[I]\ \theta = [K]\theta$.

$\therefore\ I \equiv K\ (\overline{\theta})$.

$\therefore\ \overline{\theta}$ is transitive.

So, $\overline{\theta}$ is an equivalence relation.

Let $I \equiv J(\overline{\theta})$, $N \in IdM$ and $x \in I \cap N$.

$I \equiv J(\overline{\theta}) \Rightarrow [I]\theta = [J]\theta$.

$\therefore x \in I$ implies $[x]\theta = [y]\theta$ for some $y \in J$.

$\therefore x \equiv y\ (\theta)$ for some $y \in J$.

$\therefore x\Lambda x \equiv x\Lambda y(\theta)$.

That is $x \equiv x\Lambda y(\theta)$.

$y \in J$, $x \in N$ implies $x\Lambda y \in J \cap N$.

$x\Lambda y \in J \cap N$, $x \equiv x\Lambda y(\theta)$ implies $x \in J \cap N$.

Thus $x \in I \cap N$ implies $x \in J \cap N$.

$\therefore [I \cap N]\theta \subseteq [J \cap N]\theta$.

Similarly, we can prove that $[J \cap N]\theta \subseteq [I \cap N]\theta$.

Hence $[I \cap N]\theta = [J \cap N]\theta$.

$\therefore\ I \cap N \equiv J \cap N(\overline{\theta})$.

Next we claim that $IVN \equiv JVN(\overline{\theta})$.

Let $A_o = I \cup N$.

Let $A_n = \{x \mid x \leq t_o V t_1,\ t_o, t_1 \in A_{n-1}\}$, for $0 < n < w$

Then $I \cup N = \cup\{A_n \mid n<w\}$.

We claim that $A_n \subseteq [JVN]\theta$.

We prove this result using induction on n.

When n = 0, $A_0 = I \cup N \subseteq [J]\theta \cup N \subseteq [JVN]\theta$

$\therefore A_0 \subseteq [JVN]\theta$.

Therefore the result is true when n = 0.

By induction assumption assume that $A_{n-1} \subseteq [JVN]\theta$.

Let $x \in A_n$. Then $x \leq t_0 V t_1$ for some $t_0, t_1 \in A_{n-1.}$

$t_0 \in A_{n-1}, A_{n-1} \subseteq [JVN]\theta$ implies

$t_0 \equiv u_0(\theta)$ for some $u_0 \in JVN$.

$t_1 \in A_{n-1}, A_{n-1} \subseteq [JVN]\theta$ implies

$t_1 \equiv u_1(\theta)$ for some $u_1 \in JVN$.

$\therefore t_0 \equiv t_0\Lambda u_o(\theta)$ and $t_1 \equiv t_1 \Lambda u_1(\theta)$.

$t_0 V t_1$ is an upper bound for $\{t_0\Lambda u_0, t_1\Lambda u_1\}$.

$\therefore (t_0 \Lambda u_0)\ V\ (t_1 \Lambda u_1)$ exists.

$\therefore t_0 V t_1 \equiv (t_0\Lambda u_0)\ V (t_1\Lambda u_1)(\theta)$.

$x = x\ \Lambda\ (t_0 V t_1)$

$\equiv x\ \Lambda\ [\ (t_0\Lambda u_0)\ V\ (t_1\Lambda u_1)\ ]\ (\theta)$.

Now $x\ \Lambda\ [\ (t_0\Lambda u_0)\ V\ (t_1\Lambda u_1)\ ] \in JVN$.

$\therefore x \in [\ JVN\ ](\theta)$.

$\therefore\ A_n \subseteq [\ JVN\ ](\theta)$.

Thus by induction each $A_n \subseteq [\ JVN\ ](\theta)$.

$\therefore \cup\ \{\ A_n\ /\ n < \omega\} \subseteq [\ JVN\ ](\theta)$.

$\therefore IVN \subseteq [\ JVN\ ](\theta)$.

Similarly $JVN \subseteq [\ IVN\ ]\ (\theta)$.

$\therefore IVN \equiv JVN(\overline{\theta})$.

$\therefore \overline{\theta}$ is a congruence relation on IdM.

Let $a \equiv b(\theta)$ and $x \in (a]$

$a \equiv b(\theta)$ implies $x\Lambda a \equiv x\Lambda b(\theta)$

That is $x \equiv x\Lambda b(\theta)$.

$\therefore (a] \subseteq (b](\theta)$.

Similarly, $(b] \subseteq (a](\theta)$

Hence $(a]\theta = (b]\theta$.

$\therefore (a] \equiv (b](\overline{\theta})$.

Thus $a \equiv b\ (\theta)$ implies $(a] \equiv (b]\ (\overline{\theta})$.

Conversely, let $(a] \equiv (b](\overline{\theta})$.

Then $a \equiv b_1(\theta)$ for some $b_1 \leq b$ and

$a_1 \equiv b(\theta)$ for some $a_1 \leq a$.

$\therefore a \vee a_1 \equiv b_1 \vee b(\theta)$.

That is $a \equiv b\ (\theta)$.

Thus $(a] \equiv (b](\overline{\theta})$ implies $a \equiv b(\theta)$.

Thus $\overline{\theta}$ has all the properties.

To prove uniqueness:-

Let Φ be a congruence relation of IdM satisfying $(a] \equiv (b](\Phi)$ if, and only if, $a \equiv b(\theta)$.

Let $I, J \in$ IdM, $I \equiv J(\Phi)$ and $x \in I$.

$I \equiv J(\Phi)$ implies $(x] \cap I \equiv (x] \cap J(\Phi)$.

But $x \in I$ implies $(x] \cap I = (x]$.

$(x] \cap J = (y]$ for some $y \in J$.

$\therefore (x] \cap I \equiv (x] \cap J(\Phi)$ implies $(x] \equiv (y](\Phi)$ for some $y \in J$.

$\therefore (x] \equiv (y](\Phi)$ implies $x \equiv y(\theta)$.

Thus given $x \in I$, there exists $y \in J$ such that $x \equiv y(\theta)$.

$\therefore x \in [J]\theta$.

That is $I \subseteq [J](\theta)$.

Similarly $J \subseteq [I](\theta)$.

Hence $[I](\theta) \equiv [J](\theta)$.

Therefore $I \equiv J(\overline{\theta})$.

Thus $I \equiv J(\Phi)$ implies $I \equiv J(\overline{\theta})$.

Conversely, let $I \equiv J(\overline{\theta})$.

Then $x \equiv y(\theta)$ for some $x \in I$ and $y \in J$.

Take all congruences of the form $x \equiv y(\theta)$, $x \in I$, $y \in J$.

By our assumption of Φ, $(x] \equiv (y](\Phi)$ and by our definition of θ, the join of all these congruences yields $I \equiv \overline{J(\theta)}$

Thus $\Phi = \overline{\theta}$.

Hence the lemma.

Definition : 1.4.5

Let C and D be finite lattices such that $J = C \cap D$ is an ideal in C and J is an ideal in D. Let m denote the generator of J. Then $M(C,D) = C \cup D$ is a finite chopped lattice with the natural partial ordering. We observe that if $a \vee b = c$ in M(C,D), then either $a,b,c \in C$ and $a \vee b = c$ in C or $a,b,c \in D$ and $a \vee b = c$ in D.

Lemma : 1.4.6

Let C and D be finite lattices such that $J = C \cap D$ is an ideal in C and an ideal in D. Let m denote the generator of J.

Let $\overline{M(C,D)} = \{(x,y) \in C \times D \ / \ x \wedge m = y \wedge m\}$, a subposet of CxD. Then $\overline{M(C,D)}$ is a finite lattice and Id $M(C,D) \cong \overline{M(C,D)}$.

Proof :-

Let I be an ideal of $M(C,\overline{D})$.

Then I can be written uniquely in the form $I_C \cup I_D$ where I_C is an ideal of C and I_D is an ideal of D satisfying $I_C \cap J = I_D \cap J$.

Let $I_C = (x]$ and $I_D = (y]$.

Then $I_C \cap J = I_D \cap J$ is the same as $x \wedge m = y \wedge m$.

Define a map Φ : Id $M(C,D) \rightarrow \overline{M(C,D)}$ by

$\Phi(I) = \Phi((x] \cup (y]) = (x, y)$.

We claim that Φ is an isomorphism.

(i) Φ is one-one

Let $\Phi(I) = \Phi(J)$ Where I and J are ideals of M(C,D).

Then $I = I_C \cup I_D$ where $I_C = (x]$ and $I_D = (y]$ and $J = J_C \cup J_D$ where $J_C = (a]$ and $J_D = (b]$.

$\Phi(I) = \Phi(J)$ implies $(x,y) = (a,b)$.

This implies $x = a$, $y = b$.

That is $(x] \cup (y] = (a] \cup (b]$.

That is $I = J$.

Hence Φ is one-one.

(ii) Φ is onto

Let $(x,y) \in C \times D$ be such that $x \wedge m = y \wedge m$.

Let $I = (x] \vee (y]$, Then $\Phi(I) = (x,y)$.

$\therefore$ Φ is onto.

(iii) Φ is a homomorphism

Let $I, J \in$ Id M(C,D).

Then $I = (x] \vee (y]$ and $J = (a] \vee (b]$ for some $x,a \in C$ and $y,b \in D$ such that $x \wedge m = y \wedge m$ and $a \wedge m = b \wedge m$.

$$
\begin{aligned}
\Phi(I \vee J) &= \Phi(((x] \vee (y]) \vee ((a] \vee (b])) \\
&= \Phi(((x] \vee (a]) \vee ((y] \vee (b])) \\
&= \Phi((x \vee a] \vee (y \vee b])) \\
&= (x \vee a, y \vee b)
\end{aligned}
$$

$$
\begin{aligned}
\Phi(I) \vee \Phi(J) &= \Phi((x] \vee (y]) \vee \Phi((a] \vee (b]) \\
&= (x,y) \vee (a,b) \\
&= (x \vee a, y \vee b)
\end{aligned}
$$

$\therefore \Phi(I \vee J) = \Phi(I) \vee \Phi(J)$.

$$
\begin{aligned}
\Phi(I \wedge J) &= \Phi(((x] \vee (y]) \wedge ((a] \vee (b])) \\
&= \Phi(((x] \wedge (a]) \vee ((y] \wedge (b])) \\
&= \Phi((x \wedge a] \vee (y \wedge b])) \\
&= (x \wedge a, y \wedge b)
\end{aligned}
$$

$$
\begin{aligned}
\Phi(I) \wedge \Phi(J) &= \Phi((x] \vee (y]) \wedge \Phi((a] \vee (b]) \\
&= (x,y) \wedge (a,b) \\
&= (x \wedge a, y \wedge b)
\end{aligned}
$$

$\therefore \Phi(I \wedge J) = \Phi(I) \wedge \Phi(J)$.

$\therefore \Phi$ is a homomorphism.

Hence Φ is an isomorphism.

$\therefore$ IdM(C,D) $\cong \overline{M(C,D)}$.

Lemma : 1.4.7

Let C and D be finite lattices such that $J = C \cap D$ is an ideal in C and an ideal in D. Let m be the generator of J. Let U be an ideal of C and let V be an ideal of D. Let us regard $U \cup V$ as a subset of IdM(C,D) by identifying an element with the principal ideal it generates. If $U \cap V = \{0\}$, then the sublattice generated by $U \cup V$ in IdM(C,D) is an ideal and it is isomorphic to UxV.

Proof :-

Let $U = (x]$ and $V = (y]$.

Then $x \in C$ and $y \in D$.

Define $\sigma : U \cup V \rightarrow$ IdM (C,D) by $\sigma(a) = \sigma((a \vee b)) = (a \vee b]$.

Then σ is an embedding of $U \cup V$ into IdM(C,D).

Suppose $U \cap V = \{0\}$

Let $\langle U \cup V \rangle$ be the sublattice generated by $U \cup V$.

Let $p,q \in \langle U \cup V \rangle$

Then $p \vee q \in \langle U \cup V \rangle$.

Let $x \in C \cup D$ and $p \in \langle U \cup V \rangle$

Then $x = x_1 \vee x_2$ and $p \leq t_1 \vee t_2$

$p \leq x \wedge p = (x_1 \vee x_2) \wedge (t_1 \vee t_2)$

$\leq (x_1 \wedge t_1) \vee (x_2 \wedge t_2) \in \langle U \cup V \rangle$

$\therefore p \in \langle U \cup V \rangle$.

Hence $\langle U \cup V \rangle$ is an ideal .

Let $a \in \langle U \cup V \rangle$, then $a \in ((x] \cup (y])$

Then $a \leq a_1 \vee a_2$ where $a_1 \in (x]$ and $a_2 \in (y]$.

$\therefore$ By identifying $a \rightarrow (a_1,a_2)$ we get

$\langle U \cup V \rangle \cong$ UXV.

Lemma : 1.4.8

Let C and D be finite lattices such that $J = C \cap D = (m]$ is an ideal in C and in D. Then Con IdM(C,D) $\cong \{(\theta, \Psi) \in$ Con C x Con D / $\theta\,|_J = \Psi\,|_J\}$.

Proof :-

Let Ω be a congruence of the chopped lattice M(C,D).

Let Ω_C and Ω_D be the restrictions of Ω to C and D respectively.

Then Ω_C is a congruence of C and Ω_D is a congruence of D Satisfying the condition Ω_C restricted to J equals Ω_D restricted to J.

$\therefore \sigma$: Con (M(C,D)) $\rightarrow \{(\theta,\psi) \in$ Con C x Con D / $\theta\,|_J = \psi\,|_J\}$

defined by $\sigma(\Omega) = (\Omega_C, \Omega_D)$ is a well defined map.

Conversely, let θ be a congruence on C and ψ be a congruence on D satisfying that θ restricted to J equals ψ restricted to J.

Define a congruence Ω on M(C,D) as follows :

(i) $x \equiv y(\Omega)$ if, and only if, $x \equiv y\ (\theta)$ for $x,y \in C$

(ii) $x \equiv y(\Omega)$if, and only if, $x \equiv y(\psi)$ for $x,\ y \in D$

(iii) If $x \in C$ and $y \in D$, $x \equiv y(\Omega)$ if, and only if,
$x \equiv x\Lambda y(\theta)$ and $y \equiv x\Lambda y\ (\psi)$ and symmetrically.

Then τ : $\{(\theta, \psi) \in$ Con C x ConD / $\theta\,|_J=\psi\,|_J\} \rightarrow$ Con(M(C,D))

defined by $\tau((\theta,\psi)) = \Omega$ is a well defined map.

$$(\tau \text{ B } \sigma)(\Omega) = \tau(\sigma(\Omega))$$
$$= \tau(\Omega_C, \Omega_D)$$
$$= \Omega$$
$$(\sigma \text{ B } \tau)(\Omega_C, \Omega_D) = \sigma(\tau(\Omega_C, \Omega_D))$$
$$= \sigma(\Omega)$$
$$= (\Omega_C, \Omega_D)$$

$\therefore \tau$ B σ = identity map and σ B τ = identity map.

$\therefore$ σ is an isomorphism.

Therefore Con M(C,D) $\cong \{(\theta,\psi) \in$ ConC x ConD / $\theta\,|_J=\psi\,|_J\}$.

But by lemma (2.4.4), Con M(C,D) $\cong$ Con(IdM(C,D)).

Hence Con(IdM(C,D)) $\cong \{(\theta,\psi) \in$ ConC x ConD / $\theta\,|_J=\psi\,|_J\}$.

Hence the lemma.

Lemma : 1.4.9

Let U be a finite lattice with an ideal V isomorphic to B_n. We identify V with the ideal $(B_n)_* = ((0,1)]$ of $N(B_2,B_n)$ to obtain the chopped lattice $K=M(U,N(B_2,B_n))$. Let m denote the generator of $V=(B_n)_*$. Then $IdK \cong M(U,N(B_2,B_n))$. Let $u \in U$. Then $\{y \in N(B_2,B_n)\,|\,(u,y) \in M(U,N(B_2,B_n))\}$ is isomorphic to B_2.

Proof :-

There are exactly four elements y of $N(B_2,B_n)$ satisfying that $u\Lambda m = y\Lambda m$, namely the elements of $(B_2)_{u\Lambda m}$.

They form a sublattice isomorphic to B_2.

Therefore $\{\ y \in N(B_2,B_n)\ /(u,y) \in M(U,N(B_2,B_n))\}$ is a four element set closed under co-ordinatewise meets and joins.

Hence the lemma.

1.5. CONGRUENCE CLASSES

Lemma: 1.5.1

Let U be a finite lattice with an ideal V isomorphic to B_n. Then $V \cong ((0,1)]$. Let us assume that U is uniform. Let K be a chopped lattice $M(U,N(B_2,B_n))$. Then $IdK \cong M\ (U,N(B_2,B_n))$. Then IdK is uniform.

Proof :-

A congruence Ω of IdK can be described by lemma (2.4.8) .

That is $\Omega \rightarrow (\theta, \psi)$ where θ is a congruence of U, ψ is a congruence of $N(B_2,B_n)$ and θ and ψ restrict to the same congruence of $V=(B_n)_*$.

The trivial congruences $\omega_{IdK} = (\omega_U, \omega_{N(B\ ,B\ )})$ and

$i_{IdK} = (i_U, i_{N(B\ ,B\ )})$ are obviously uniform.

We need to look at only two cases.

First case : Ω is represented by (θ, ω)

So $\theta\,|_V = \omega_V$. Let (x,y) be an element of IdK.

Then $[(x,y)](\theta,\omega) = \{(t,y) \in IdK\ |\ t \equiv x(\theta)\}$.

It $t \equiv x(\theta)$, then $t\Lambda m = x\Lambda m(\theta)$.

But $\theta\,|_V = \omega_V$ so $t\Lambda m = x\Lambda m$.

$\therefore [(x,y)\ ]\ (\theta,\omega) = \{\ (t,y)\ |\ t \equiv x(\theta)\ \}$ and so

$[(x,y)]\ (\theta,\omega) = [[x]\theta]$.

$\therefore$ Each congruence class of Ω is of the same size as a congruence class of θ.

So Ω is uniform.

Second case : Ω is represented by (θ, ψ) where $\psi \neq \omega$.

Let (x,y) be an element of IdK.

Then $[(x,y)]\ (\theta,\psi) = \{(\omega,z) \in IdK\,|\ x\ \equiv \omega(\theta)$ and $y \equiv z(\psi)\}$.

For a given ω, if (ω,t_1) and $(\omega,t_2) \in$ IdK, then $t_1 \equiv t_2(\psi)$ because $(B_2)_\omega$ is in a single congruence class of ψ by lemma 2.3.8 (Claim 3).

Therefore $\{t \in N(B_2,B_n)\,|\,(\omega,t) \in IdK\} = (B_2)_{\omega\Lambda m}$ by lemma 2.4.9.

Therefore $|\{t \in N(B_2,B_n)\,|\,(\omega,t)\ \in IdK\}| = |(B_2)_{\omega\Lambda m}| = 4$.

We conclude that

$[(x,y)]\ (\theta,\psi) = \{(\omega,z) \in IdK\,|\ x\ \equiv \omega(\theta)\ ,z \in (B_2)_{\omega\Lambda m}\}$

and so $|[(x,y)](\theta,\psi)\ | = 4\,|[x]\theta\,|$.

Therefore each congruence class of Ω is four times the size of a congruence class of θ .

Hence Ω is uniform.

Hence the lemma.

1.6. PROOF OF THE MAIN RESULT

Theorem : 1.6.1

For any finite distributive lattice D, there exists a finite uniform lattice L such that the congruence lattice of L is isomorphic to D and L satisfies the properties (P) and (Q) where

(P) : Every join-irreducible congruence of L is of the form $\theta(0,p)$, for a suitable atom p of L.

(Q) : If $\theta_1,\theta_2,\ldots,\theta_n \in$ J(ConL) are pairwise incomparable, then L contains atoms $p_1,p_2,\ldots,p_n$ that generate an ideal isomorphic to B_n and satisfy $\theta_i=\theta\ (0,p_i)$, for all $i \leq n$.

Proof :-

We prove the result using induction on n, where n is the number of join-irreducible elements.

Let D be a finite distributive lattice with n join-irreducible elements.

If n = 1, then $D \cong B_1$, so there is a lattice $L=B_1$ that satisfies the theorem1.6.1.

Let us assume that, for all finite distributive lattices with fewer than n join-irreducible elements, there exists a lattice L satisfying theorem 1.6.1 and properties (P) and (Q).

Assume that D has n join-irreducible elements.

Let q be a minimal element of J(D).

Let $q_1,q_2,\ldots,q_k (k \geq 0)$ be all upper bounds of q in J(D).

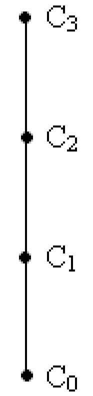
Let D_1 be a distributive lattice with $J(D_1)=J(D)-\{q\}$.

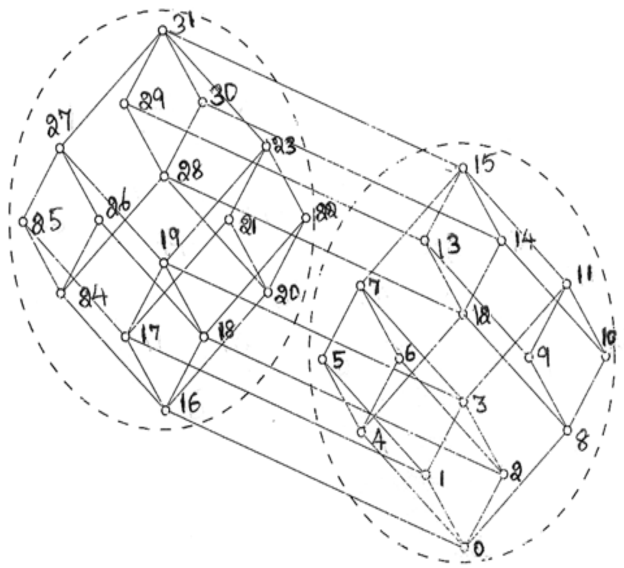
By induction assumption there exists a lattice L_1 satisfying Con $L_1 \cong D_1$ and (P) and (Q).
If k = 0, then $D \cong B_1 \times D_1$ and $L = B_1 \times L_1$, obviously satisfies all the requirements of the theorem and so the proof is over.
So, assume $k \geq 1$
The congruences of L_1 corresponding to the q_i's are pairwise incomparable and therefore can be written in the form $\theta(0,p_i)$ and the p_i's generate an ideal I_1 isomorphic to B_k.
The lattice $N(B_2,B_k)$ also contains an ideal $(B_k)_*$ isomorphic to B_k.
Identifying I_1 and $(B_k)_*$, We get the chopped lattice K and the lattice L=IdK.
By lemma1.5.1., IdK is uniform.
That is L is uniform.
Let θ be a join-irreducible congruence of L.
Then we can write θ as $\theta(a,b)$ where a is covered by b.
By lemma 1.4.6., it follows that we can assume that either a, b $\in L_1$, or a, b $\in N(B_2,B_k)$
In either case, there exists an atom q in L_1 or q in $N(B_2, B_k)$ so that
θ (a,b)= $\theta(0,q)$ in L_1 or $\theta(a,b)=\theta(0,q)$ in $N(B_2, B_k)$.
Obviously, q is an atom of Land $\theta(a,b)=\theta(0,q)$ in L verifying (P)for L.
Let θ_1, $\theta_2, \ldots, \theta_t$ be pairwise in-comparable join-irreducible congruences of L.
To verify condition (Q), we have to find atoms $p_1,p_2,\ldots,p_t$ of L satisfying $\theta_i = \theta(0,p_i)$ for all $i \leq t$ and such that $p_1,p_2,\ldots,p_t$ generate an ideal of L isomorphic to B_t.
Let p denote an atom in $N(B_2,B_k) - I_1$
Infact, there are two atoms but they generate the same congruence $\theta(0,p)$.
If $\theta(0,p)$ is not one of $\theta_1,\theta_2,\ldots,\theta_t$ then clearly we can find $p_1,p_2, \ldots, p_t$ in L_1 as required and $p_1,p_2, \ldots,p_t$ also serves in L.
If $\theta(0,p)$ is one of $\theta_1, \theta_2,\ldots,\theta_t$ say $\theta(0,p) = \theta_t$, then let $p_1,p_2,\ldots,p_{t-1}$ be the set of atoms establishing (Q) for $\theta_1,\theta_2,..,\theta_{t-1}$ in L_1 and therefore in L.
Then $p_1,p_2,\ldots,p_{t-1},p$ represent the congruences $\theta_1,\theta_2,\ldots,\theta_t$ and they generate an ideal isomorphic to B_t by lemma 1.4.7.
Therefore L satisfies (Q).
It is clear from this discussion that J(ConK) has exactly one more element than J(ConL_1), namely, $\theta(0,p)$.
This join-irreducible congruence relates to the join-irreducible congruences of ConL, exactly as q relates to the join-irreducible elements of D.
Therefore $D \cong$ ConL.
Hence the theorem.

Example : 1.6.2

The uniform construction for the four-element chain is
This lattice has four congruences.
C_o has 32 blocks.
C_0 is a null congruence
C_1 has 8 blocks.
C_1 = { {0,1,2,3}, {4,5,6,7},{8,9,10,11},{12,13,14,15}, {16,17,18,19},{20,21,22,23}, {24,25,26,27},{28,29,30,31} }.
C_2 has 2 blocks.
C_2 = { {0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}, {16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31} }.
C_3 has 1 block.
C_3 is all congruence.

The congruence lattice of this lattice is





$\therefore$ Every finite distributive lattice D can be represented as the congruence lattice of a finite uniform lattice L.

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