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Analysis of a discrete model of prey-predator interactions

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1. Introduction

Interactions of different species may take many forms such as competition, predation, parasitism and mutualism. One of the most important interactions is the predator-prey relationship. Many researchers studied predator - prey systems. Especially stability analyses of predator - prey systems are investigated. The Lotka-Volterra equations are a pair of first order differential equations used to describe the dynamics of interactions of two species. In 1926 Volterra came up with a model to describe the evolution of predator and prey fish populations in the Adriatic Sea. They were proposed independently by Alfred J.Lotka in 1925 [6, 8]. The equations are

x' = ax - bxy

y' = -cy + dxy

Where x is the number of prey, y is the number of predator. The parameters a, b, c and d are non-negative. The prey reproduce exponentially (a) in the absence of predator. The rate of predation (b) is proportional to the rate at which the predators and the prey meet. The parameter c is the death coefficient for predator in the absence of prey and d is the predator population growth coefficient from the interaction of the species.

2. Model Description

When populations have non overlapping generations, population models described by difference equations are more appropriate [3, 4]. Discrete models produce rich dynamical behavior [5, 7]. This paper considers the following system of difference equations which describes interactions between two species.

$$x(n+1) = (r+1)x(n) - rx(n)^{2} - cx(n)y(n)$$
(1)
$$y(n+1) = cx(n)y(n) + (1-d)y(n)$$

where 0 < d < 1. For the system (1), the equilibrium points are $E_0 = (0, 0); E_1 = (1, 0)$ and $E_2 = \left(\frac{d}{c}, \frac{r(c-d)}{c^2}\right)$. The equilibrium

point E_0 correspond to extinction of both species, E_1 corresponds to extinction of predator and E_2 corresponds to coexistence of both species. The equilibrium point E_2 is an interior equilibrium point and this is positive if c > d.

ABSTRACT

This paper investigates a 2-D discrete time predator - prey interaction model. The equilibrium points are obtained and stability of the equilibrium points is analyzed. The phase portraits are obtained for different sets of parameter values. Also a bifurcation diagram is presented. Numerical simulations are performed and they exhibit rich dynamics of the discrete model.

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3. Dynamic Behavior of the Model

In this section, we investigate the behavior of the system around the equilibrium points. Predator - Prey interactions have an oscillatory tendency [9]. However, the oscillations are often damped out in nature and stability is attained. More realistic mathematical models exhibit this behavior in the dynamics of predator - prey systems [1, 2]. The local stability analysis of the model can be carried out by computing the Jacobian corresponding to each equilibrium point. The Jacobian matrix J for the system (1) has the form

$$J(x,y) = \begin{pmatrix} (r+1) - 2rx - cy & -cx \\ cy & cx + 1 - d \end{pmatrix}$$

At E_0 , the Jacobian takes the form

$$J(E_0) = \begin{pmatrix} r+1 & 0\\ 0 & 1-d \end{pmatrix}$$

The Jacobian matrix is upper triangular. Hence the eigen values are $\lambda_1 = r + 1$ and $\lambda_2 = 1 - d$. Stability is ensured if $|\lambda_{1,2}| < 1$ which leads to the conclusion r < 0 which is not true.

Hence equilibrium with both species extinct is unstable. The Jacobian matrix for E_1 is evaluated as

$$J(E_1) = \begin{pmatrix} 1-r & -c \\ 0 & 1+c-d \end{pmatrix}$$

The eigen values are $\lambda_1 = 1 - r$ and $\lambda_2 = 1 + c - d$. The condition $|\lambda_{1,2}| < 1$ is satisfied when r > 0 and c < d. The equilibrium with only the prey present is locally asymptotically stable if r > 0 and c < d. Also Tr = 2 - r - d + c, Det = 1 + (1 - r)(c - d) - rCorresponding to the interior equilibrium point E_2 , the Jacobian is

$$J(E_2) = \begin{pmatrix} 1 - \frac{dr}{c} & -d \\ r(1 - \frac{d}{c}) & 1 \end{pmatrix}$$

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 $Tr = 2 - \frac{dr}{c}$ and $Det = 1 - \frac{dr}{c} + dr(1 - \frac{d}{c})$. The Calculations yield characteristic equation to the matrix $J(E_2)$ is

 $p(\lambda) = \lambda^2 - \lambda TrJ + Det J = 0 \quad (2)$

Using Jury conditions, the modulus of all roots of (2) is less than 1 (the equilibrium point is asymptotically stable) if p(1) = 1 - TrJ + DetJ > 0,

p(-1) = 1 + TrJ + DetJ > 0and

$$DetJ(E_2) < 1$$

We now find the conditions under which the equilibrium point E_2 is asymptotically stable. The inequality p(1) > 0 is equivalent to the condition

d < c(3)

The second inequality p(-1) > 0 holds if and only dr(2+d). Finally, DetJ(E2) <1 holds if and only if 4+dr

c < 1+*d*

(4)Combining the inequalities (3) and (4), we obtain

d < c < 1 + d

We can summarize the results as follows.

Result-1. The positive equilibrium point E_2 is asymptotically stable if d < c < l + d holds.

Result-2. The positive equilibrium point E_2 is unstable if c < dand c > 1 + d holds.

4. Numerical Simulations

In this section, we provide numerical simulations for the behavior of the system about the equilibrium points. We present the graphs of the solutions around the positive equilibrium points. In (1), the values r = 0.1; c = 0.8 and d = 0.2 are takentogether with the small initial values for prey and predator populations x = 0.1; y = 0.2.



As the prey population increases, their growth rate decreases as they approach carrying capacity because of the availability of limited resources.



Figure 1(b). Small Initial Values Phase Plane

When the prey-predator populations are small, the predator population declines due to lack of food. Then the predator population increases whereas the prey population is not able to recover quickly. When the limitation to resources is considered, a spiral emerges in phase plane. Periodical oscillation does not occur. The prey-predator populations eventually become stable and the two species can coexist in the same habitat at a stable equilibrium. Imposing limited growths on the prey population introduce damping effect to the system, see Figure-1(a,b).

The set of values r = 0.1; c = 0.05; d = 0.15 satisfy the conditions r > 0 and c < d. Therefore, the system is stable at E_1 and the predator goes extinct and prey population grows logistically as can be seen in Figure-2(a,b).



Choosing r = 0.5, c = 0.9 and d = 0.3, we obtain $E_2 = (0.33, -10.5)$ 0.37) and the eigen values are $\hat{\lambda}_{1,2} = 0.9150 \pm i0.3029$

and $|\lambda_{1,2}| = 0.964 < 1$. For the initial values x(0)=0.85, y(0)=0.3, the trajectory in the phase plane spirals in and moves towards the equilibrium point, see Figure-3(a,b). Hence the system is stable.





Figure 3(b). Behavior at E₂ Phase Plane

Bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value. In the bifurcation diagram (Figure-4) for prey population, we consider the values c = 0.009; d = 0.1 and $r \in (1.4, 2.9)$.



Figure 4. Bifurcation for Prey Population

This paper analyzed the discrete model of interactions between two species (1) with equilibrium points and obtained conditions under which the system attains stability. Numerical simulations are presented to study the dynamical behavior of the system. Finally, bifurcation in prey population is presented. **References**

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