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Bayesian inference for exponential distribution based on progressive type-II censored data with random scheme

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ABSTRACT
In this paper, we propose Bayes estimator of parameter of exponential distribution Under General Entropy Loss Function (GELF) for progressive type-II censored data with random scheme. The proposed estimator has been compared with corresponding Bayes estimator under Square Error Loss Function (SELF) and Maximum Likelihood Estimator (MLE) in terms of their risks based on simulated samples from exponential distribution.

1. Introduction

The simplest model that was thought for life time data was exponential distribution. It has constant hazard rate. Due to mathematical ease in using this distribution, it has widely been used for life testing and reliability analysis (See, Johnson and Kotz (1970)). In life testing and reliability, the time and cost involved in the experimentation, often investigators not to observe the complete failure times of all items put on test and partly observes data is called censored data. The procedure by which we get the part of sample data is called censoring scheme. There are various type of censoring Schemes discussed in the literature such as type-I censoring, type-II censoring and Hybrid censoring etc. Each censoring scheme has its own advantages. In type-II censored sample, n items are put on test and experiment is terminated after getting r failures which is pre decided $(i.e., 1 \le r \le n)$. For example, consider that a doctor investigate an experiment with n HIV- patients but after the death of first patient, some patient leave the experiment and go for treatment to other doctor/ hospital. Similarly, after the second death a few more leave and so on. A generalization of type -II censoring is progressive type-II censoring. In the censoring scheme, a few items following some probability law removed at each observed failure rate. Raqab et al (2001) have discussed the mechanism of this censoring scheme as follows: Under this general censoring scheme, n units are placed on a life-testing experiment and only $(m < n)$ are completely observed until failure. The censoring occurs progressively in m stages. These m stages offer failure times of the m completely observed units. At the time of the first failure (the first stage), r_1 of the $(n-1)$ surviving units are randomly withdrawn (censored) from the experiment, r_2 of the $n-2-r_1$ surviving units are withdrawn (censored) at the time of the second failure (the second stage), and so on. Finally, at the time of the m^{th} failure (the m^{th} stage), all the remaining $r_m = n-m-r_1 -\cdots r_{m-1}$ surviving units are withdrawn. We will refer to this as progressive type-II right censoring scheme. It is clear that this scheme includes the conventional type-II right censoring scheme $(r_1 = r_2 = r_{m-1} = 0, r_m = n-m)$ and the complete sampling scheme $(r_1 = r_2 = \cdots = r_{m-1} = 0, n = m)$. Progressive type censoring is time and cost effective. The added advantage of this scheme lies in the fact that it allows the removal of serving unit before the termination of test. Thus, it suites the need of industrial and clinical settings, particularly in those situations where the removal of units prior to failure is pre planned in order to save time and money associated with testing

Several authors have been considered this scheme for estimating the unknown parameters for different distributions, see, for example, Balakrishnan and Aggrawala (2000), Balakrishnan et al. (2004), Ng (2005), Mousa and Al-Sagheer (2006), and Balakrishnan (2007).

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In this paper, we propose Bayes estimator of parameter of Exponential distribution Under SELF and GELF for progressive type-II censored data with random removal. The proposed estimator has been compared with corresponding Bayes estimator under SELF and MLE in terms of their risks based on simulated samples from exponential distribution.

2. The Model

Let random variable X have an exponential distribution (ED) with parameter θ. The probability density function of X takes the following form

$$f(x,\theta)=\theta e^{\theta x}, \qquad \theta>0, x>0. \tag{1.1}$$

The survival function of X is

$$S(x)=1-e^{-\theta x}, \qquad \theta>0, x>0. \tag{1.2}$$

Let $(X_1,R_1),(X_2,R_2),(X_3,R_3)...(X_m,R_m)$, denote a Progressive type II censored sample, where $X_1<X_2<X_3,\cdots,X_m$. With pre-determined number of removals, say $R_1=r_1,R_2=r_2,R_3=r_3,\cdots,R_m=r_m$, the conditional likelihood function can be written as, Cohen (1963),

$$L(\theta;x\mid R=r)=c^*\prod_{i=1}^{m}f(x_i)[S(x_i)]^{r_i} \tag{1.3}$$

$$c^*=n(n-r_1-1)(n-r_1-r_2-2)(n-r_1-r_2-r_3-3)$$

Where $\cdots(n-r_1-r_2-r_3-3,...,r_m\text{- m+1}),$ for and $0\le r_i\le(n-m-r_1-r_2-r_3\cdots r_{i-1}),$ $i=1,2,3,\cdots,m-1.$ from (1.1), (1.2) and (1.3), we get

$$L(\theta;x\mid R=r)=c^*\theta^m e^{\sum_{i=1}^{m}-\theta(1+r_i)x_i} \tag{1.4}$$

The number R_i of units removed at the i^{th} failure, $i=1,2,3,\cdots,m-1,$ follows binomial distribution with parameters $(n-m-r_i)$ and p. Therefore,

$$P(R_1=r_1)=\frac{(n-m)!}{r_1!(n-m-r_1)!}p^{r_1}(1-p)^{n-m-r_1}, \tag{1.5}$$

And for $i=1,2,3,...,m-1,$

$$P(R_i=r_i\mid R_{i-1}=r_{i-1},\cdots R_1=r_1)=\frac{n-m-\sum_{l=1}^{l=i-1}r_l!}{r_i!\left(n-m-\sum_{l=1}^{i-1}r_l-r_i\right)!}p^{r_i}(1-p)^{n-m-\sum_{l=1}^{i-1}r_l-r_i} \tag{1.6}$$

Now, we further suppose that R_i is independent of X_i for all i. Then the full likelihood function takes the following form

$$L(\theta,\ p;x,\ r)=L(\theta,\ x\mid R=r)P(R=r), \tag{1.7}$$

$$P(R=r)=P(R_1=r_1)P(R_2=r_2\mid R_1=r_1)$$
$$P(R_3=r_3\mid R_2=r_2,R_1=r_1)\cdots P(R_{m-1}=r_{m-1}\mid R_{m-2}=r_{m-2},\cdots R_1=r_1), \tag{1.8}$$

From (1.5), (1.2) and (1.8), we get

$$P(R=r)=\frac{(n-m)!}{\left(n-m-\sum_{l=1}^{i-1}r_l-r_i\right)!\prod_{i=1}^{m-1}r_i!}p^{\sum_{i=1}^{m-1}r_i}(1-p)^{(m-1)(n-m)-\sum_{l=1}^{m-1}(m-i)r_i} \tag{1.9}$$

Now, using (1.1), (1.7) and (1.9), we can write the full likelihood function as in the following form

$$L(\theta,\ p;x,\ r)=AL_1(\theta)L_2(p) \tag{1.10}$$

Where $A=\dfrac{c^*(n-m)!}{\left(n-m-\sum_{l=1}^{i-1}r_l-r_i\right)!\prod_{i=1}^{m-1}r_i!}$, does not depend on the parameters θ and p

$$L_1(\theta)=\theta^m\exp\{-\theta\sum_{i=1}^{m}(1+r_i)x_i\}, \tag{1.11}$$

and

$$L_2(p)=p^{\sum_{i=1}^{m-1}r_i}(1-p)^{(m-1)(n-m)-\sum_{l=1}^{m-1}(m-i)r_i}. \tag{1.12}$$

3. Classical and Bayesian Estimation of Parameters

3.1 Maximum Likelihood Estimation

In this section, we have obtained the MLE of the parameters θ and p based on progressive type-II censored data with binomial removals. We observed that from (1.10), (1.11) and (1.12) that likelihood function is multiplication of three terms, namely, A, L_1 and L_2. Out of these, A does not dependent on the parameters θ and p; thus, it behaves as a constant for maximum likelihood

estimation. L_1 does not involve p and can be treated as function of θ only, where as L_2 involves p only. Therefore, the MLE's of θ can be derived by maximizing L_1 with respect to θ. Similarly, the MLE of p can be obtained by maximizing L_2. Taking log of both sides of (1.11), we have

$$L_1(\theta) = m\ln(\theta) - \theta\sum_{i=1}^{m}(1+r_i)x_i \qquad (1.13)$$

The first partial derivative of $L_1(\theta)$ with respect to θ is

$$\frac{\partial L_1(\theta)}{\partial \theta} = \frac{m}{\theta} - \sum_{i=1}^{m}(1+r_i)x_i \qquad (1.14)$$

Setting $\frac{\partial L_1(\theta)}{\partial \theta} = 0$, we get the likelihood equation for θ.

Solving the equation (1.14), we get the MLE of θ is given by

$$\hat{\theta}_M = \frac{m}{\sum_{i=1}^{m}(1+r_i)x_i} \qquad (1.15)$$

Similarly, since $L_2(p)$ does not involved θ, the maximum likelihood estimator of p can be derived by maximizing (1.12) directly. The log-likelihood function of $L_2(p)$ takes the following form

$$\ln p\sum_{i=1}^{m-1} r_i + \ln(1-p)[(m-1)(n-m) - \sum_{l=1}^{m-1}(m-i)r_i]. \qquad (1.16)$$

The first partial derivative of $L_2(p)$ with respect to p is

$$\frac{\partial L_2(p)}{\partial p} = \frac{\sum_{i=1}^{m-1} r_i}{p} - \frac{(m-1)(n-m) - \sum_{l=1}^{m-1}(m-i)r_i}{1-p} \qquad (1.17)$$

Setting $\frac{\partial L_2(p)}{\partial p} = 0$, we get the normal equation for p. By solving the above equation, we obtained the MLE of p as in the following form

$$\hat{p}_M = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{l=1}^{m-1}(m-i-1)r_i}. \qquad (1.18)$$

4. Bayes Procedure

In this section, we have obtained the Bayes estimator of the parameters θ and p based on progressively type-II censored data with binomial removals. In order to obtain the Bayes estimator, we need the prior distribution of θ and p respectively. Here we take conjugate family of prior for θ as gamma prior and prior for p as beta first kind. It is to be mentioned here that both prior are frequently used in literature because having the property of flexibility computational ease. Therefore, the prior for θ as:

$$g_1(\theta) = \frac{a^b e^{-a\theta}\theta^{b-1}}{\Gamma b};\ 0 < \theta < \infty,\ a > 0, b > 0. \qquad (1.19)$$

and prior for p as:

$$g_2(p) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1};\ 0 < p < 1, \alpha > 0, \beta > 0. \qquad (1.20)$$

Thus, the joint prior pdf for θ and p as

$$g(\theta, p) = g_1(\theta)g_2(p);\quad \theta > 0, 0 < p < 1. \qquad (1.21)$$

Let us consider the joint prior pdf for θ and p as defined in (1.21), so the conjunction of likelihood (1.10) and the joint prior $g(\theta, p)$, by Bayes theorem, the formula for the evaluation of joint posterior of θ and p is obtained for the given sample $x_1, x_2, x_3, ..., x_n$ as

$$\pi(\theta, p \mid x, r) = \frac{L(x \mid \theta, p)g(\theta, p)}{\int_0^{\infty}\int_0^{1} L(x \mid \theta, p)g(\theta, p)\partial p \partial \theta} \qquad (1.22)$$

Substituting $L(\theta, p)$ and $g(\theta, p)$ from (10) and (21) respectively in (22), the joint posterior pdf of (θ, p) became

$$\pi(\theta, p \mid x, r) = \frac{\theta^{(m+b-1)}}{j_0} e^{-\theta a^*} p^{\alpha^*-1}(1-p)^{\beta^*-1};\ \theta > 0, 0 < p < 1. \qquad (1.23)$$

Where

$$a^* = a + \sum_{i=1}^{m}(1+r_i)x_i, \alpha^* = \alpha + \sum_{i=1}^{m}(1+r_i)x_i,$$

$$\beta^* = \beta + (m-1)(n-m) - \sum_{i=1}^{m-1} r_i(m-i)$$

And

$$J_0 = \frac{\Gamma(m+b)B(\alpha^*, \beta^*)}{a^{*(m+b)}}$$

Therefore, the marginal posterior pdf's of θ and p are given by

$$\pi_1(\theta \mid x,r) = \frac{a^{*(m+b)}}{\Gamma(m+b)}\theta^{(m+b-1)}e^{-\theta a^*};\ \theta>0, a>0, b>0, \tag{1.24}$$

and

$$\pi_2(p \mid x,r) = \frac{1}{B(\alpha^*,\beta^*)}p^{\alpha^*-1}(1-p)^{\beta^*-1};\quad 0<p<1 \tag{1.25}$$

respectively. Note that the posterior distribution of θ is gamma with parameters $(m+b)$ and a^*, while the posterior distribution of p is beta first kind with parameters α^* and β^*.

Usually the Bayes estimators are obtained under SELF

$$l_1(\phi,\hat{\phi}) = \varepsilon_1(\phi-\hat{\phi})^2;\qquad \varepsilon_1>0. \tag{1.26}$$

Where $\hat{\phi}$ is the estimate of the parameter ϕ and the Bayes estimator $\hat{\phi}_S$ of ϕ comes out to be $E_\phi[\phi]$, where E_ϕ denotes the posterior expectation. This loss function is a symmetric loss function and can only be justified, if over estimation and under estimation of equal magnitude are of equal seriousness. But in real situation it may not exits. A number of asymmetric loss functions are also available in statistical literature Basu and Ebrahimi (1991) . In this problem we consider the General Entropy Loss Function (GELF), proposed by Calabria and Pulcini (1996), defined as follows:

$$l_2(\phi,\hat{\phi}) = \varepsilon_2\left(\left(\frac{\hat{\phi}}{\phi}\right)^{\delta} - \delta\ln\left(\frac{\hat{\phi}}{\phi}\right) - 1\right);\qquad \varepsilon_2>0. \tag{1.27}$$

The constant δ, involved in(27), is its shape parameter. It reflects departure from symmetry. When $\delta>0$, it considers over estimation (i.e., positive error) to be more serious than under estimation (i.e., negative error) and converse for $\delta<0$. The Bayes estimator $\hat{\phi}_G$ of ϕ under GELF is given by,

$$\hat{\phi}_G = \left[E_\phi\left(\phi^{-\delta}\right)\right]^{-\left(\frac{1}{\delta}\right)} \tag{1.28}$$

provided the posterior expectation exits. It may be noted here that for $\delta=-1$, the Bayes estimator under loss (26) coincides with the Bayes estimator under SELF l_1. Expressions for the Bayes estimators $\hat{\theta}_G$ and $\hat{p}_G$ for θ and p respectively under GELF can be given as

$$\hat{\theta}_G = \left[\int_0^\infty \theta^{-\delta}\pi_1(\theta \mid x,r)\partial\theta\right]^{\left(-\frac{1}{\delta}\right)}, \tag{1.29}$$

and

$$\hat{p}_G = \left[\int_0^\infty p^{-\delta}\pi_1(p \mid x,r)\partial p\right]^{\left(-\frac{1}{\delta}\right)}. \tag{1.30}$$

Substituting the posterior pdf's from (24) and (25) in (29) and (30) respectively and then simplifying, we get the Bayes estimators $\hat{\theta}_G$ and $\hat{p}_G$ of θ and p is obtained as follows

$$\hat{\theta}_G = \frac{1}{a^*}\left[\frac{(m+b-\delta-1)!}{(m-1)!}\right]^{\left(-\frac{1}{\delta}\right)} \tag{1.31}$$

and

$$\hat{p}_G = \left[\frac{(\alpha^*-\delta-1)!(\alpha^*-\beta^*-1)!}{(\alpha^*-1)!(\alpha^*+\beta^*-\delta-1)!}\right]^{\left(-\frac{1}{\delta}\right)} \tag{1.32}$$

Further, by putting $\delta=-1$ in (31) and (32) we respectively get Bayes estimators of θ and p as follows,

$$\hat{\theta}_S = \frac{(m+b)!}{a^*(m+b-1)!} \tag{1.33}$$

and

$$\hat{p}_S = \frac{\alpha^*}{(\alpha^*+\beta^*)} \tag{1.34}$$

5. Algorithm to Simulate Progressive Type-II Censored Sample with Binomial Removal

For the study of behavior of the estimators obtained in previous sections, we need to simulate progressive type-II censored samples with Binomial removals from specified ED. To get such a sample, we propose the use of following algorithm:

I. Specify the value of n.

II. Specify the value of m.

III. Specify the value of parameters θ and p.

IV. Generate a random sample (S_r) of size n from $E(\theta)$.

V. Generate random number r_i from $B\left(n-m-\sum_{l=0}^{i-1} r_l, p\right)$, at i^{th} stage for $i=1,2,3,...,m-1.(r_0=0)$

VI. Get ordered sample S_0 from S_r to choose the minimum which will be first observation in desired progressive type-II censored sample S_p.

VII. Drop the observation selected at VI from S_r to have a random sample S_r^* of size n^* (less 1 than that of S_r).

VIII. Generate r_i integers (at i^{th} stage) between 1 to n^* and observations corresponding to these numbers are dropped from S_r^* to have a random sample S_r^{**} of size

VIII. Generate r_i integers (at i^{th} stage) between 1 to n^* and observations corresponding to these numbers are dropped from S_r^* to have a random sample S_r^{**} of size $n^{**}=n^*-r_i$ and re-designate the random sample in hand as new random sample S_r.

IX. Repeat steps V to VIII $(m-1)$ times.

X. Set r_m according to the following relation.

$$r_m=\begin{cases} n-m-\sum_{l=1}^{m-1} r_l & \text{if } n-m-\sum_{l=1}^{m-1} r_l>0 \\ 0 & \text{otherwise} \end{cases}$$

and discard all the remaining r_m observations.

6. Simulation Studies

The estimators $\hat{\theta}_M$, $\hat{\theta}_G$ and $\hat{\theta}_S$ are the mle and corresponding the Bayes estimators under GELF and SELF of θ respectively. Similarly, the estimators $\hat{p}_M$, $\hat{p}_G$ and $\hat{p}_S$ are mles and corresponding the Bayes estimators under GELF and SELF of p respectively. We shall compare the estimators obtained under GELF with corresponding Bayes estimators under SELF and their mles. The comparisons are based on the simulated risks (average loss over sample space) under GELF and SELF both and the risks of the estimators are estimated on the basis of Monte-carlo simulation study of 2000 samples. It may be noted that the risks of the estimators will be the function of $n,m,p,\theta,a,b,\alpha,\beta$ and δ. In order to consider variation in the values of these parameters, we have obtained the simulated risks for $m=12[2]18$, when $n=20,\theta=2,\delta=\pm 1.5$ and $p=0.3$. The hyper parameter are chosen in such a way that the prior mean is same as the true value of the parameter with belief in considering as prior mean is strong or weak in the sense that the prior variance is small and large. Generating the progressive sample as mentioned in section 5, the simulated risks under SELF and GELF have been obtained for selected values of $n,m,\theta,a,b,p,\alpha,\beta$ and δ. After extensive study of numerical results conclusion are drawn regarding the behavior of proposed estimator which are summarized in various graphs.

7. Discussion of the Results

From figure (1) it is clear that the risks of all estimators of θ decreases and the risk of all estimators of p increases as m increases in GELF and SELF for the situation when over estimation is to be considered more serious than under estimation and vice-versa or equal importance is to be given for over and under estimation. When $\delta=-1.5$ i.e, under estimation is to be considered more serious than over estimation, the performance of the Bayes estimator under GELF is well in comparison to the Bayes estimator under SELF and corresponding MLE's in terms of their smaller risks. Such type of trend has noticed for both considered loss functions. When we considered the prior mean as the true value of the parameter and our belief in considering the this prior mean is strong in the sense that the prior variance is small, it is observed that Bayes estimators under GELF perform well in comparison to Bayes estimator under SELF and corresponding MLE's, for $\delta=-1.5$. But when we considered the prior mean is same as true value of the parameter and our belief in considering is weak in the sense that prior variance is large, it observed that performance of all the estimators of θ and p behave same as discussed above (see figure 3). But there is increment in the magnitude of risk of all estimators has noticed. Further when $\delta=1.5$ i.e, over estimation is to be considered more serious than under estimation, the Bayes estimators under SELF perform well in comparison to the Bayes estimators under GELF and their corresponding MLE's (see figures 2 and 4). It is also observed that Bayes estimator under GELF has smaller risk in comparison to the risk of MLE's in the both loss functions. For variation of the hyper parameters, performance of all estimators of θ and p followed same trend of $\delta=-1.5$.

8. Conclusions

It is expected that the estimator obtained under a particular loss function shall in general perform better than the estimators obtained under other loss functions. We have seen above that risk under GELF and SELF for the estimators $\hat{\theta}_G$ and $\hat{p}_G$ are always less than those of $\hat{\theta}_M$, $\hat{p}_M$, $\hat{\theta}_S$ and $\hat{p}_S$, when $\delta=-1.5$, the risks associated with

$\hat{\theta}_G$ and $\hat{p}_G$ are always smaller than the risk associated with other estimators. On other hand if $\delta = 1.5$ risk associated with $\hat{\theta}_S$ and $\hat{p}_S$ are noted to be smaller than other estimators.

1. $\hat{\theta}_G$ and $\hat{p}_G$ may be used as an estimator of θ and p when under estimation is to be considered more serious than over estimation.
2. $\hat{\theta}_S$ and $\hat{p}_S$ may be used as an estimator of θ and p when over estimation is to be considered more serious than under estimation.

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$n = 20,\ \delta = 1.5,\ p = .4,\ a = 5,\ b = 10,\ \alpha = 2,\ \beta = 6.$

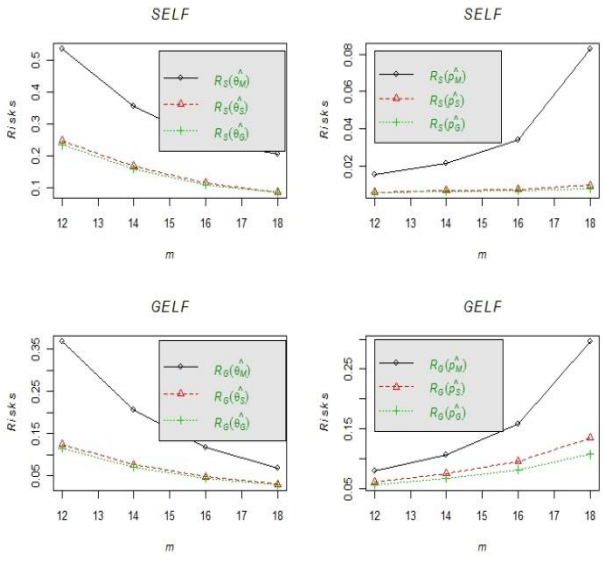

Figure 1: Risk of estimators of θ and p under SELF and GELF for fixed
$n = 20, \delta = -1.5, p = .4, a = 5, b = 10, \alpha = 2, \beta = 6.$

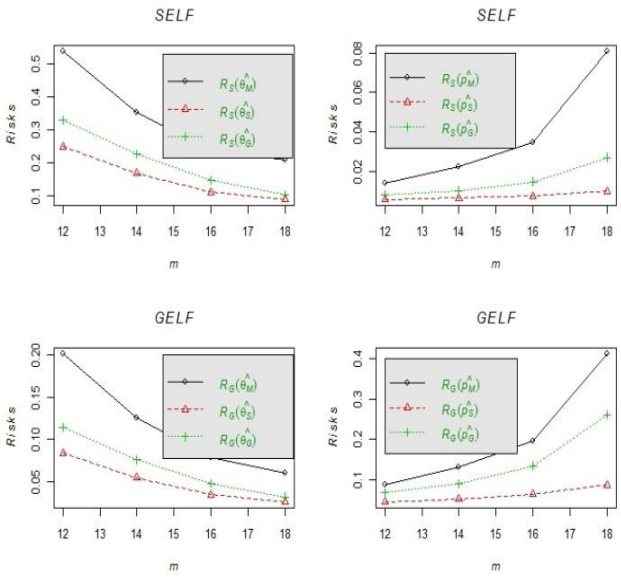

Figure 2: Risk of estimators of θ and p under SELF and GELF for fixed
$n = 20, \delta = 1.5, p = .4, a = 5, b = 10, \alpha = 2, \beta = 6.$

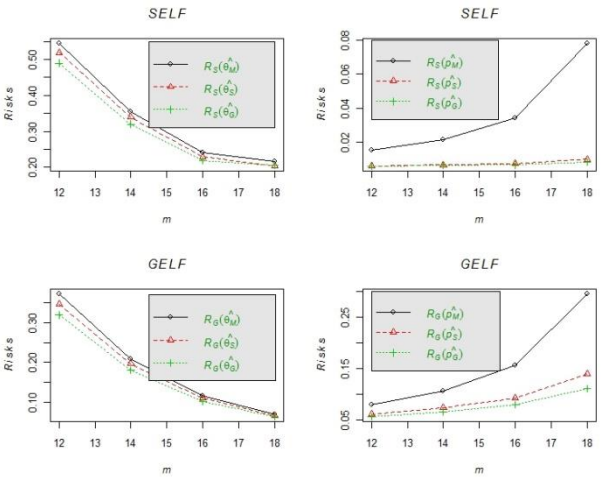

Figure 3: Risk of estimators of θ and p under SELF and GELF for fixed
$n = 20, \delta = -1.5, p = .4, a = .25, b = .50, \alpha = 2, \beta = 6.$

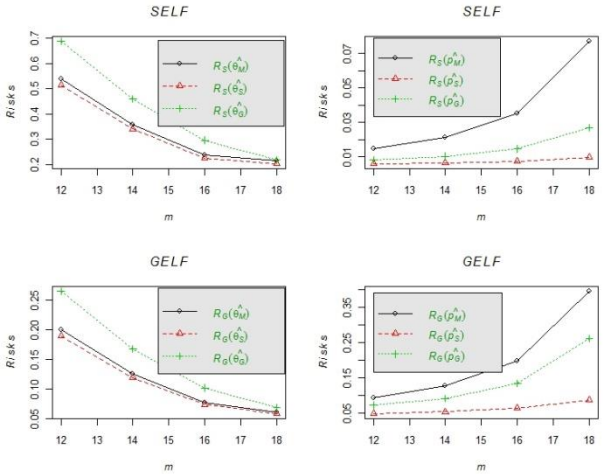

Figure 4: Risk of estimators of θ and p under SELF and GELF for fixed
$n = 20, \delta = 1.5, p = .4, a = .25, b = .5, \alpha = 2, \beta = 6.$