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Weighted drazin inverse of con-s-k-EP matrices

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ABSTRACT
The definition of the Drazin inverse of a square matrix with complex elements is extended to Con-s-k-EP matrices by showing that for any B and W, n by n respectively, there exists a unique matrix, X, such that $(BW)^k = (BW)^{k+1} XW$ for some positive integer k, $XWBWX = X$, and $BWX = XWB$. Various expressions satisfied by B,W,X and related matrices are developed.

1. Introduction

Let $C_{n\times n}$ be the space of nxn complex matrix of order n. Let C_n be the space of all complex n tuples. For $A \in C_{n\times n}$.Let $\bar{A}, A^T, A^*, A^s, \bar{A}^s, A^\dagger, R(A), N(A)$ and $P(A)$ denote the conjugate transpose, conjugate transpose, secondary transpose, conjugate secondary transpose, Moore-Penrose inverse range space, null space and rank of A respectively. A solution x of the equation AXA=A is called generalized inverse of A and is denoted by $\bar{A}$. if $A \in C_{n\times n}$ then the unique solution of the equations $AXA = A,\ XAX = X\,[AX]^* = AX,\,[XA]^* = XA$ [2] is called the Moore-Penrose inverse of A and is denoted by $A^\dagger$. A matrix A is called con-s-k-EP$_r$ if P(A)=r and $N(A) = N(A^T VK)$ (or) $R(A) = R(KVA^T)$. Throughout this paper let '$\mathcal{K}$' be the fixed product of disjoint transposition in $S_n = \{1,2,...n\}$ and k be the associated permutation matrix.

Let us define the function $k(x) = (x_{k(1)}, x_{k(2)}, ..., x_{k(n)})$. A matrix $A = (a_{ij}) \in C_{n\times n}$ is s-k-symmetric if $a_{ij} = a_{n-k(j)+1}, n-k(i)+1$ for i,j=1,2,…N. A matrix $A \in C_{n\times n}$ is said to be con-s-k-EP if it satisfies the condition $A_x = 0 <=> A^s k(x) = 0$ or equivalently $N(A) = N(A^T VK)$. In addition to that A is con-s-k-EP $<=> kvA$ is con-EP or AVK is con-EP and A is con-s-k-EP $<=> A^T$ is con-s-k-EP moreover A is said to be con-s-k-EP$_r$ if A is con-s-k-EP and of rank for further properties of con-s-k-EP matrixes one may refer[4].

Let A be any con-s-k-EP matrix with complex elements. Then the Moore-Penrose inverse of A is the unique matrix $X = A^\dagger$ such that

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$$AXA = A, XAX = X, (AX)^H = AX, (XA)^H = XA \qquad (1.1)$$

(where the superscript H denotes conjugate transpose). On the other hand, if A is squre, the Drazin inverse of A is the unique matrix $X = A_d$ such that

$$A^k = A^{k+1}X \quad \text{for some positive integer k,} \qquad (1.2)$$

$$X = X^2 A \qquad (1.3)$$

$$AX = XA \qquad (1.4)$$

It is the purpose of this paper to show that a Drazin inverse can be defined for Con-s-k-EP matrices in such a way that both $A^\dagger$ and A_d (when A is square) follow as special cases.

2. THE W- WEIGHTED DRAZIN INVERSE OF B CON-s-k-EP

Although the Drazin inverse was originally considered for elements in an associative ring [3] and Lemma 1 was established in that context [2], we use this result only for matrices and restate it accordingly have prove the result for con-s-k-EP matrix.

Lemma 1

For Con s-k-EP matrices B and W of order n by n

$$(BW)_d = KVB^TVK\left(KVW^TVKKVB^TVK\right)_d^2 KVW^TVK$$

$$= KVB^TVK\left(KVW^TVKKVB^TVK\right)_d^2 KVW^TVK$$

$$= KVB^TVK\left(KVW^TB^TVK\right)_d^2 KVW^TVK$$

$$= KVB^TVK\left(KVW^TB^TVK\right)_d\left(KVW^TB^TVK\right)_d KVW^TVK$$

$$= KVB^TVK\left(KVB_d^TW_d^TVK\right)\left(KVB_d^TW_d^TVK\right)KVW^TVK$$

$$= KVB^T\left(B_d^TW_d^T\right)\left(B_d^TW_d^T\right)W^TVK$$

$$= KVB^T\left(W^TB^T\right)_d^2 W^TVK$$

$$(BW)_d = KVB^T\left(W^TB^T\right)_d^2 W^TVK$$

The reader can also verify Lemma 1 by taking X equal to the right member of the equation and $A = KVB^TW^TVK$ and verifying that (1.2) - (1.4) are satisfied using (1.4) rewrite (1.3) as $A_d = A_d A A A_d$. The expression in Corollary1.1 now follow at once by induction.

Corollary 1.1

For every Positive integer P,

$$W(BW)_d^p = \left(KVW^TVKKVB^TVK\right)_d^p KVW^TVK$$

$$=\left(KVW^TB^TVK\right)_d^p KVW^TVK$$

$$=\left(KVW^TB^T\right)_d^p W^TVK$$

and $$B\left(WB\right)_d^p=\left(KVB^TW^TVK\right)_d^p KVB^TVK$$

$$B\left(WB\right)_d^p=\left(KVB^TW^T\right)_d^p B^TVK$$

our first result, Theorem 2 is established for an arbitrary positive integer P, as will be indicated following the proof however, the general case can always be reduced to P=2 by a simple transformation.

Theorem 2

For con-s-k-EP matrices B and W, and for every positive integer p, there is a unique X such that

$$\left(BW\right)_d\left(XW\right)=\left(KVB^TW^TVK\right)_d^p \qquad (2.1)$$

$$BWX = X\,KVW^TB^TVK \qquad (2.2)$$

$$BW\left(BW\right)_d X = X \qquad (2.3)$$

Also there is a unique X such that

$$XW = KVB^TW^TVK\left(KVB^TW^TVK\right)_d^p$$

$$XW = KVB^TW^T\left(B^TW^TVK\right)_d^p \qquad (2.4)$$

$$XW = KVW^TB^TVK\left(KVW^TB^TVK\right)_d^p$$

$$XW = KVW^TB^T\left(W^TB^TVK\right)_d^p \qquad (2.5)$$

$$XW\left(BW\right)^{p-1} X = X \qquad (2.6)$$

The unique matrix X which satisfies both sets of equation is

$$X = KVB^TVK\left(KVW^TB^TVK\right)_d^p$$

$$X = KVB^T\left(W^TB^TVK\right)_d^p \qquad (2.7)$$

Proof

Using (1.3) and corollary 1.1 it is easily seen that X in (2.7) satisfies (2.1) to (2.6).

To establish uniqueness, we show first that (2.1), (2.2) and (2.3) imply (2.4), (2.5) and (2.6) and then that (2.4), (2.5) and (2.6) imply (2.7).

Now, $$XW = KVB^TW^TVK\left(KVB^TW^TVK\right)_d$$

$$= KVB^TW^TVK\left(KVW_d^TB_d^TVK\right)$$

$$= KVB^TW^TVK\left(KVW_d^TB_d^TVK\right)$$

$$= KVB^TW^T\left(W_d^TB_d^TVK\right)$$

$$= KVB^TW^T\left(B^TW^TVK\right)_d$$

$$XW = KVB^TW^T\left(B^TW^TVK\right)_d^p$$

Now, $XW = KVB^TW^T\left(B^TW^TVK\right)_d$ $XW = KVB^TW^T\left(B^TW^TVK\right)_d^p$

by (2.3) and (2.1) thus (2.4) holds, and combines with (2.3) to give

$$XW\left(BW\right)^{p-1}X = KVB^TW^T\left(B^TW^T\right)_d^p\left(BWVK\right)^{p-1}X = KVB^TW^T\left(B^TW^TVK\right)_dX = X$$

Thus (2.6) also holds. Finally, using (2.2), (2.3), (2.4) and corollary 1.1, we have,

$$WX = KVW^TB^TW^T\left(B^TW^TVK\right)_dX = \left(KVW^TB^T\right)_dW^TB^TW^TVKX$$

$$= \left(KVW^TB^T\right)_dW^TVKXKVW^TB^TVK = \left(KVW^TB^T\right)_dW^TB^TW^T\left(B^TW^T\right)_d^pB^TVK$$

$$= \left(KVW^TB^T\right)\left(KVW^TB^T\right)_d^{P+1} = \left(KVW^TB^T\right)\left(KVW^TB^T\right)_d^P$$

That is (2.5). Hence (2.1),(2.2) and (2.3) imply (2.4), (2.5) and (2.6). If (2.4), (2.5) and (2.6) hold, then

$$X = XKVW^T\left(B^TW^TVK\right)^{p-1}X = KVB^TW^T\left(B^TW^T\right)_d^p\left(B^TW^TVK\right)^{p-1}X$$

$$= \left(KVB^TW^T\right)_dB^TW^TVKX = \left(KVB^TW^T\right)_dB^TW^TB^T\left(W^TB^TVK\right)_d^p$$

$$= KVB^TW^TB^T\left(W^TB^TVK\right)_d^{p+1} = KVB^T\left(W^TB^TVK\right)_d^p$$

Hence (2.7) holds.

Observe next that if $p \geq 1$, $q \geq -1$ and $r \geq 0$ are integers such that $q + 2r + 2 = p$, and if we let $\left(WB\right)^q = \left(KVW^TB^TVK\right)_d$ when $q = -1$, then $B\left(WB\right)_d^p = KVB^T\left(W^TB^T\right)^q\left[\left(\left(W^TB^T\right)^rW^T\right)\left(B^T\left(W^TB^TVK\right)^q\right)\right]_d^2$

Consequently Considerations fo X is Theorem 2 can always be reduced to the case p=2 if B is replaced by $B\left(WB\right)^2$ and W is replaced by $\left(WB\right)^rW$.

Corollary 2.1

The matrix $X = KVB^T\left(W^TB^T\right)_d^2VK$ is the unique solution to the equations.

$$\left(BW\right)^k = \left(KVB^TW^TVK\right)^{k+1}XKVW^TVK \text{. For some positive integer } l \quad (2.8)$$

$$X = XKVW^TB^TW^TVKX \qquad (2.9)$$

$$BWX = XKVW^TB^TVK \qquad (2.10)$$

Proof

That $X = KVB^T\left(W^T B^T\right)_d^2 VK$ is a solution is apparent by noting that the relation in (2.10) and (2.2) are identical (2.9) is (2.6) when p=2 and with $XW = \left(KVB^T W^T VK\right)_d$ by (2.4), (2.8) is (1.2) for $A = KVB^T W^T VK$.

To show uniqueness suppose both X_1 and X_2 are solutions to (2.8) for some positive integers l_1 and l_2 respectively (2.9) and (2.10) let $\hat{l} = \max\left(l_1, l_2\right)$. Then using respected applications of Equations (2.3), (2.9) and (2.10) we have

$$X_1 = X_1 KVW^T B^T W^T VKX_1 = KVB^T W^T VKX_1 KVW^T VKX_1 = \left(KVB^T W^T VK\right)^2 \left(X_1 KVW^T VK\right)^2 X_1$$

$$= .... = \left(KVB^T W^T VK\right)^{\hat{l}} = \left(X_1 KVW^T VK\right)^{\hat{l}} X_1 = \left(KVB^T W^T VK\right)^{\hat{l}+1} X_2 KVW^T VK\left(X_1 KVW^T VK\right)^{\hat{l}} X_1$$

$$= X_2\left(KVW^T B^T VK\right)^{\hat{l}+1} KVW^T VK\left(X_1 KVW^T VK\right)^{\hat{l}} X_1 = X_2 KVW^T B^T W^T VK\left(B^T W^T VK\right)^{\hat{l}}\left(X_1 KVW^T VK\right)^{\hat{l}}\left(X_1 KVW^T VK\right)^{\hat{l}} X_1$$

$$= X_2 KVW^T B^T W^T VKX_1$$

Continuing is a similar manner, $X_2 KVW^T B^T W^T X_1$ can be reduced to X_2. Thus $X = KVB^T\left(W^T B^T\right)_d^2 VK$ is unique.

It should be noted in corollary 2.1 that when B is square and W=I. Also there is a direct Correspondence between the relation in (2.8) and (1.2), (2.9) and (1.3) when written as X=XAX, and (2.10) and (1.4), in which the role of W in (2.8), (2.9) and (2.10) is to act as a "Sandwich" matrix so that products such as BWX and XWB can be defined. In view of the Correspondence between the defining equations for A_d and those in corollary 2.11, we define the Drazin inverse of a rectangular matrix in the following manner.

A DRAZIN INVERSE

Definition 2.11

For any matrices B and W$\in$ n x n respectively, the matrix B is con-s-k-EP matrix and W be the toeplitz matrix $X = B\left(WB\right)_d^2$ is called the W- weight Drazin inverse of B con-s-k-EP.

References

[1] Anna Lee, "Secondary symmetric, Secondary Skew symmetric, Secondary Orthogonal matrices" *Period. Math. Hungary* **7** (1976), 63-76.

[2] Ben Israel, A. and Greviue, T.N.E., "Generalized Inverses, Theory and Applications" *Wiley, New York*, (1974).

[3] Drazin, M.P., "Pseudo-inverses in associative rings and semi groups" *Amer. Math.Monthly*, **65**: 506-513 (1958).

[4] Krishnamoorthy, S., Gunasekaran, K. and Muthugobal, B.K.N., "Con-s-k-EP Matrices." *Journal of Mathematical Sciences and Engineering Applications*, Vol. 5, No.**1** (2011), 353-364.

[5] Penrose, R., "A generalized inverse for matrices" *Proc. Cambridge Philos. Soc*. **51**: 406-413 (1955).