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The Discrete Spectrum of a Class of Self-adjoint Differential Operators

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ABSTRACT
In this paper, we consider the 2nth-order symmetric differential expressions with real-valued coefficients. We obtain a necessary and sufficient condition for the discreteness of the spectrum of 2nth-order self-adjoint differential operators.

1. Introduction

The spectrum of differential operators is a fundamental problem in the theory of differential operators. For a self-adjoint operator T in Hilbert space, its redial spectrum is empty. So the research of its essential spectrum and its discrete spectrum is the main task of the researchers. If denote by $\sigma(T)$ **Error! Reference source not found.**, **Error! Reference source not found.** and $\sigma_e(T)$ **Error! Reference source not found.** the spectrum, the discrete spectrum and the essential spectrum, respectively, then $\sigma(T)=\sigma_d(T)\cup\sigma_e(T)$ **Error! Reference source not found.**

Since last century much effort has been devoted to study of spectral analysis of operators, particular attention has been paid to the situation in which the spectrum of differential operators is discrete (see [1-5]). Since Molchanov established the celebrated criterion on the discreteness of the spectrum in 1953 (see [6]), this result has been developed by many authors. In [7], Muller-pfeiffer investigated the differential operator which consists of Euler differential expression. Muller-pfeiffer obtained the distribution of the essential and the discrete spectrum of the constant coefficient Euler differential operator and got the conclusion which perturbation of coefficients does not change the essential spectrum. Based on [7], Zhong Wang and Jiong Sun considered the spectrum of Euler differential operator and gave some necessary and sufficient conditions on coefficients, to ensure that the spectrum is discrete (see [8, 9]).

In recent years, few researchers consider the discreteness of spectrum of differential operators. In fact, it is very important and fundamental part for differential operators. So in this paper we continue to consider the spectrum of the following differential operator which consists of 2nth-order differential expression

$$\tau u := \sum_{k=0}^{n}(a_k(x)u^{(k)}(x))^{(k)}, x\in(a,\infty), a\geq 0, \tag{1.1}$$

where **Error! Reference source not found.** are the real value functions. The method follows the general approach of ([7]). Here we improve the conditions in ([10]) and condition (i) of [10] are delete. In particular, we notice that not only the last coefficient can decide the discreteness of spectrum of differential operator when it tends to infinity according to a certain way and any one of the other coefficients can decide the discreteness of spectrum in the same way.

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2. Preliminaries

Let Hilbert space H denote the direct sum of closed subspaces **Error! Reference source not found.** and **Error! Reference source not found.**, i.e., **Error! Reference source not found.**. Define the operator **Error! Reference source not found.** on**Error! Reference source not found.**, and

$D(A_i) \subseteq H_i$, $R(A_i) \subseteq H_i$, $i = 1,2$.

If **Error! Reference source not found.** is a self-adjoint operator, then the direct sum operator **Error! Reference source not found.**, i.e.,

$Au = A_1 u_1 \oplus A_2 u_2, u = u_1 + u_2, u_i \in D(A_i), i = 1,2$

is also self-adjoint, and

$\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$ **Error! Reference source not found.**,

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If **Error! Reference source not found.** A is a self-adjoint realization of (1.1) in Hilbert space **Error! Reference source not found.** $L^2(a,\infty)$, then

$$L^2(a,\infty) = L^2(a,N) \cup L^2(N,\infty), \tag{2.1}$$

$A = A_1 \oplus A_2$ **Error! Reference source not found.** .

Error! Reference source not found.(2.2)

Here A_1 is a self-adjoint realization of (1.1) in Hilbert space **Error! Reference source not found.** and **Error! Reference source not found.** is a self-adjoint realization of (1.1) in Hilbert space **Error! Reference source not found.**. The operator **Error! Reference source not found.** A is self-adjoint, then $\sigma(A) = \sigma_d(A) \cup \sigma_e(A)$. Thus the spectrum of **Error! Reference source not found.** is discrete if and only if **Error! Reference source not found.** $\sigma_e(A) = \varnothing$. In addition, the operator **Error! Reference source not found.** is regular, $\sigma(A_1) = \sigma_d(A_1)$ **Error! Reference source not found.**. So the spectrum of **Error! Reference source not found.** is discrete if and only if $\sigma_e(A_2) = \varnothing$.

The operator **Error! No bookmark name given.** is said to be semebounded from below if **Error! Reference source not found.**is a densely symmetric linear operator and there exists **Error! Reference source not found.** such that $(Au,u) \ge c\|u\|^2, u \in D(A)$. If **Error! Reference source not found.** $c > 0$, then the operator A **Error! Reference source not found.** is positive definite. If choose $\gamma \in \mathrm{R}$ **Error! Reference source not found.**, then $A + \gamma E (D(A+\gamma E) = D(A)$ **Error! Reference source not found.** is also positive definite. Next define inner product and norm in **Error! Reference source not found.** as follows:

$[u,v] = ((A+\gamma E)u, v)$, $\|u\|_A = [u,u]^{1/2} (u,v) \in D(A)$.

The norm $\|u\|_A$ is called the energy norm of **Error! Reference source not found.**, complete space $H_A = (D(A), \|u\|_A)$ **Error! Reference source not found.** is a Hilbert space and **Error! Reference source not found.** is called energy space of the operator **Error! Reference source not found.**.

Lemma 2.1 [7] Let self-adjoint operator **Error! Reference source not found.** be semibounded from below. Then the spectrum of A is discrete if and only if a bounded set of energy space **Error! Reference source not found.** is precompact in Hilbert space **Error! Reference source not found.**.

We write space **Error! Reference source not found.** $C^{l}(x_1,x_2)$ for the set of all finite value functions on space $C^l(x_1,x_2)$ **Error! Reference source not found.** that satisfy

$$\|u\|_{C^{l}(x_1,x_2)}=\sum_{k=0}^{l}\sup_{x_1<x_2}|u^k(x)|.$$

write space $C_p^l(x_1,x_2)(1\le p<\infty)$ **Error! Reference source not found.** for the set of all finite value functions on space $C^l(x_1,x_2)$ **Error! Reference source not found.** that satisfy

$$\|u\|_{W_p^l(x_1,x_2)}=\left(\sum_{k=0}^{l}\int_{x_1}^{x_2}|u^{(k)}(x)|^p\,dx\right)^{\frac{1}{p}},$$

and Banach space **Error! Reference source not found.** is complete space **Error! Reference source not found.** with norm $\|\cdot\|_{W_p^l(x_1,x_2)}$ **Error! Reference source not found.**.

Lemma 2.2 [7] Let $l>k>0, 1<p<\infty$. Then the space **Error! Reference source not found.** is continuously embedded in space **Error! Reference source not found.**. And for arbitrary $\varepsilon>0$ **Error! Reference source not found.**, there exists constant C_ε, s.t.,

$$\|u\|_{C^k(x_1,x_2)}\le\varepsilon\|u^{(l)}\|+C_\varepsilon\|u\|_{L^p(x_1,x_2)},$$

(2.3)**Error! Reference source not found.**

where $u(x)\in W_p^l(x_1,x_2)$. If **Error! Reference source not found.** is finite, then the embedding from **Error! Reference source not found.** to **Error! Reference source not found.** is compact.

Lemma 2.3 [7] Differential operator

$$L_0u:=\sum_{k=0}^{n}(-1)^k a_k u^{(2k)}, D(L_0)=C_0^\infty(0,\infty)$$

is semibounded from below and self-adjoint. The essential spectrum of **Error! Reference source not found.** L_0 is a interval (Λ,∞) **Error! Reference source not found.**, where

$$\Lambda=\inf_{0<\xi<\infty}\sum_{k=0}^{n}b_k\xi^{2k}.$$

Lemma 2.4 [7] All self-adjoint extensions **Error! Reference source not found.** ofsymmetric operator A_0 with finite defect have the same essential spectrum and $\sigma_e(A)=\sigma_e(A_0)$.

Lemma 2.5 [11] Let **Error! No bookmark name given.** be the closure of the complex set $\{(A_0f,f): f\in D(A_0),\|f\|=1\}$. Then **Error! Reference source not found.** and $\pi(A)$ contains the complement of the set **Error! Reference source not found.** in complex plane. Here, the operator A_0 is the minimal operator generated by (1.1), the operator **Error! Reference source not found.** is the closed extension of **Error! Reference source not found.**, and**Error! Reference source not found.** $\pi(A)$ denotes the regular point set.

3. Main result and its proof

Consider the following differential expression

$$\tau u := \sum_{k=0}^{n} (-1)^k (a_k(x)u^{(k)}(x))^{(k)}, x \in (a,\infty), a \geq 0, \tag{3.1}$$

where $a_k(x), k = 1,2,\Lambda\ , n, x \in (a,\infty)$ **Error! Reference source not found.** are the real value functions. Here we assumed that the following condition holds:

(i) $a_k(x) \in W_2^k(a, X), X > a, k = 1,2,\Lambda\ , n$;**Error! Reference source not found.**

(ii) $a_n(x) \geq a > 0$ **Error! Reference source not found.**.

In the following we set

$$A_0 u := \tau u, D(A_0) = C_0^\infty(a,\infty). \tag{3.2}$$

Theorem 3.1 Suppose that the coefficients of symmetric differential expression (3.1) satisfy the conditions (i) and (ii) and

$$\sup_{a<x<\infty} \int_x^{x+1} | a_k(t) - b_k |dt < \infty, k = 0,1,2,\Lambda\ , n-1, k \neq p. \tag{3.3}$$

Here $(u(x))^- = \min\{u(x),0\}$ and **Error! Reference source not found.**. Then the spectrum of any self-adjoint extension **Error! Reference source not found.** of the operator A_0 is discrete if and only if

$$\lim_{x\to\infty} \int_x^{x+1} a_p(t)dt = \infty, p \in \{0,1,\Lambda\ , n\}. \tag{3.4}$$

Proof. Proof of sufficiency. Firstly, we consider the case for $0 \leq p < n$, i.e., **Error! Reference source not found.**.

For sufficiently large X, we introduce self-adjoint operators **Error! Reference source not found.** and **Error! Reference source not found.**. Here **Error! Reference source not found.**. A_1 and **Error! Reference source not found.** are self-adjoint extension of operators **Error! Reference source not found.** and **Error! Reference source not found.** respectively, and **Error! Reference source not found.**, where

$$A_{10}u = \tau u = \sum_{k=0}^{n} (-1)^k (a_k(x)u^{(k)}(x))^{(k)}, D(A_{10}) = C_0^\infty(a, X].$$

and

$$A_{20}u = \tau u = \sum_{k=0}^{n} (-1)^k (a_k(x)u^{(k)}(x))^{(k)}, D(A_{20}) = C_0^\infty[X,\infty).$$

According to direct sum divided theorem, $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$ **Error! Reference source not found.**. To study the discreteness of the operator **Error! Reference source not found.**, we can consider the discreteness of the operators **Error! Reference source not found.** and **Error! Reference source not found.**. The operator **Error! Reference source not found.** is regular, so we have $\sigma(A_1) = \sigma_d(A_1)$ **Error! Reference source not found.**. Thus if **Error! Reference source not found.**, then **Error! Reference source not found.**, i.e., the operator **Error! Reference source not found.** is discrete.

Since **Error! Reference source not found.**. To prove the discreteness of operator **Error! Reference source not found.**, in view of Lemma 2.4 and Lemma 2.5, we must define **Error! Reference source not found.**, that is, obtain the scope of the following equality:

$$(A_{20}u,u) = (A_2 u,u)_{[X,\infty)} = \sum_{k=0}^{n} \int_X^\infty a_k(x) | u^{(k)} |^2 dx.$$

By (3.4), for sufficiently large b_p and **Error! Reference source not found.**, there exists **Error! Reference source not found.**, s.t.,

$$\int_X^{X+1}(a_p(x)-b_p)dx > M$$ **.Error! Reference source not found.** (3.5)

For **Error! Reference source not found.** N of both (2.1) and (2.2), let $N = X$, we have

$$(A_{20}u,u)=\int_X^{\infty}a_n(x)|u^{(n)}|^2dx+\sum_{k=0}^{n-1}\int_X^{\infty}a_k(x)|u^{(k)}|^2dx$$

$$=\int_X^{\infty}a_n(x)|u^{(n)}|^2dx+\sum_{k=0}^{n-1}\int_X^{\infty}(a_k^+(x)+a_k^-(x))|u^{(k)}|^2dx.$$

From **Error! Reference source not found.** $p \neq n$, we see that

$$(A_{20}u,u)=\int_X^{\infty}a_n(x)|u^{(n)}|^2dx+\sum_{k\neq p,k=0}^{n-1}\int_X^{\infty}a_k(x)|u^{(k)}|^2dx+\int_X^{\infty}a_p(x)|u^{(p)}|^2\,dx. \quad (3.6)$$

We evaluate the following equality:

$$\frac{1}{2}\int_X^{\infty}a_n(x)|u^{(n)}|^2dx+\sum_{k\neq p,k=0}^{n-1}\int_X^{\infty}a_k^-(x)|u^{(k)}|^2dx$$

$$\geq \frac{1}{2}\int_X^{\infty}a|u^{(n)}|^2dx-\sum_{k\neq p,k=0}^{n-1}\int_X^{\infty}|a_k^-(x)||u^{(k)}|^2dx$$

$$=\frac{1}{2}\int_X^{\infty}a|u^{(n)}|^2dx-\sum_{k\neq p,k=0}^{n-1}\sum_{v=0}^{\infty}\int_{X+v}^{X+v+1}|a_k^-(x)||u^{(k)}|^2dx$$

$$=\frac{1}{2}\int_X^{\infty}a|u^{(n)}|^2dx-\sum_{k\neq p,k=0}^{n-1}\sum_{v=0}^{\infty}\int_0^1|a_k^-(X+v+t)|\left|\left(\frac{d}{dt}\right)^k u(X+v+t)\right|^2dt$$

$$\geq \frac{1}{2}\int_X^{\infty}a|u^{(n)}|^2dx-\sum_{k\neq p,k=0}^{n-1}\sum_{v=0}^{\infty}\max_{0<t<1}\left|\left(\frac{d}{dt}\right)^k u(X+v+t)\right|^2\int_{X+v}^{X+v+1}|a_k^-(x)|dx.$$

From (3,3), there exists constant **Error! Reference source not found.**,

$$=\int_X^{X+1}|a_k^-(t)dt \leq C$$

holds. Then by Lemma 2.2, for arbitrary small **Error! Reference source not found.**, we get

$$\sum_{k\neq p,k=0}^{n-1}\left(\max_{0<t<1}\left|\left(\frac{d}{dt}\right)^k u(X+v+t)\right|^2\right)$$

$$\leq \varepsilon\int_0^1\left|\left(\frac{d}{dt}\right)^n u(X+V+t)\right|^2dt+C_\varepsilon\int_0^1|u(X+v+t)|^2dt$$

$$=\varepsilon\int_{X+v}^{X+v+t}|u^{(n)}|^2dt+C_\varepsilon\int_{X+v}^{X+v+1}|u(x)|^2dt\cdot$$

Thus we can obtain

$$\frac{1}{2}\int_X^\infty a_n(x)|u^{(n)}|^2dx+\sum_{k\neq p,k=0}^{n-1}\int_X^\infty a_k^-(x)|u^{(k)}|^2dx$$

$$\geq\sum_{v=0}^{\infty}\left(\int_{X+v}^{X+v+1}\frac{1}{2}a|u^{(n)}|^2\ dx-C\sum_{k\neq p,k=0}^{n-1}\left(\max_{0<t<1}\left|\left(\frac{d}{dt}\right)^k u(X+v+t)\right|^2\right)\right)$$

$$\geq\sum_{v=0}^{\infty}\left(\int_{X+v}^{X+v+1}\frac{1}{2}a|u^{(n)}|^2\ dx-C\left(\varepsilon\int_{X+v}^{X+v+1}|u^{(n)}(x)|^2dt+C_\varepsilon\int_{X+v}^{X+v+1}|u(x)|^2dt\right)\right)$$

$$=\int_X^\infty\frac{1}{2}a|u^{(n)}|^2dx-C\varepsilon\int_X^\infty|u^{(n)}|^2dx-CC_\varepsilon\int_X^\infty|u(x)|^2dx\cdot$$

Since **Error! Reference source not found.** is arbitrary small, we choose $C_\varepsilon<\frac{1}{2}a$ and obtain

$$\frac{1}{2}\int_X^\infty a_n(x)|u^{(n)}|^2dx+\sum_{k\neq p,k=0}^{n-1}\int_X^\infty a_k^-(x)|u^{(k)}|^2dx$$

$$\geq -CC_\varepsilon\int_X^\infty|u(x)|^2dx$$

$$-C_1\|x\|^2_{(X,\infty)}.$$

where **Error! Reference source not found.**. Therefore, combining this and (3.6), we have

$$(A_{20}u,u)\geq\frac{1}{2}\int_X^\infty a|u^{(n)}|^2dx+\sum_{k\neq p,k=0}^{n-1}\int_X^\infty a_k^+(x)|u^{(k)}|^2dx+\int_X^\infty a_p(x)|u^{(p)}|^2\ dx-C_1\|x\|^2_{(X,\infty)}. \qquad (3.7)$$

By Lemma 2.3 and (3.5), for arbitrary large $b_p>0$, there exists **Error! Reference source not found.** X, such that

$$\frac{1}{2}\int_X^\infty a|u^{(n)}|^2dx+\sum_{k\neq p,k=0}^{n-1}\int_X^\infty a_k^+(x)|u^{(k)}|^2dx+\int_X^\infty a_p(x)|u^{(p)}|^2\ dx$$

$$\geq\int_X^\infty\left(\frac{a}{2}-\varepsilon\right)|u^{(n)}|^2dx+\sum_{k\neq p,k=0}^{n-1}\int_X^\infty(-\varepsilon)|u^{(k)}|^2dx+\int_X^\infty(b_p-\varepsilon)|u^{(p)}|^2\ dx.$$

Let

$$\Lambda=\inf_{0<\xi<\infty}\sum_{k=0}^{n}c_k\xi^{2k},$$

$$c_n=\frac{a}{2}-\varepsilon,\ c_k=-\varepsilon\ (0\leq k\leq n-1,k\neq p),\ c_p=b_p-\varepsilon,0<\varepsilon<\frac{\alpha}{2}.$$

So

$$\frac{1}{2}\int_X^\infty a|u^{(n)}|^2dx+\sum_{k\neq p,k=0}^{n-1}\int_X^\infty a_k^+(x)|u^{(k)}|^2dx+\int_X^\infty a_p(x)|u^{(p)}|^2\ dx\geq\Lambda\|u\|^2_{(X,\infty)}. \qquad (3.8)$$

Combining (3.7) and (3.8), we can conclude

$\sigma_e(A_{20})\cap(-\infty,b_p-C_1)=\varnothing$ **Error! Reference source not found.**.

Further, **Error! Reference source not found.** and is ε arbitrary small. Thus we see

Error! Reference source not found.,

where **Error! Reference source not found.** b_p is sufficiently large. Hence $\sigma_e(A)=\sigma_e(A_{20})=\varnothing$, i.e., **Error! Reference source not found.**. So, for $p\neq n$ **Error! Reference source not found.**, $\sigma(A)=\sigma_d(A)$ **Error! Reference source not found.** holds true.

Consider the case for **Error! Reference source not found.**. Its proof is similarly to the case for **Error! Reference source not found.**. From (3.5), for sufficiently large **Error! Reference source not found.** and b_n **Error! Reference source not found.**, there exists **Error! Reference source not found.** X, we have

$$\int_X^{X+1}(a_n(x)-b_n)dx>M$$ **Error! Reference source not found.**.

(3.9)

Next we can obtain

$$\sum_{k=0}^{n-1}\int_X^{\infty}a_k^-(x)\,|u^{(k)}|^2dx\geq -C\varepsilon\int_X^{\infty}|u^{(n)}|^2dx-CC_\varepsilon\int_X^{\infty}|u(x)|^2dx\,.$$

where**Error! Reference source not found.** ε is arbitrary small. Therefore, let **Error! Reference source not found.**, by (3.9), we have

$$(A_{20}u,u)=\int_X^{\infty}a_n(x)\,|u^{(n)}|^2dx-\sum_{k=0}^{n-1}\int_X^{\infty}a_k^-(x)\,|u^{(k)}|^2dx+\sum_{k=0}^{n-1}\int_X^{\infty}a_k^+(x)\,|u^{(k)}|^2dx$$

$$\geq\int_X^{\infty}(a_n(x)-C\varepsilon)\,|u^{(n)}|^2dx+\sum_{k=0}^{n-1}\int_X^{\infty}a_k^+(x)\,|u^{(k)}|^2dx-C_2\int_X^{\infty}|u(x)|^2dx$$

$$\geq\int_X^{\infty}(b_n-\varepsilon-C\varepsilon)\,|u^{(n)}|^2dx+\sum_{k=0}^{n-1}\int_X^{\infty}\varepsilon\,|u^{(k)}|^2dx-C_2\int_X^{\infty}|u(x)|^2dx$$

$$\geq\int_X^{\infty}(b_n-\varepsilon-C\varepsilon)\,|u^{(n)}|^2dx+\sum_{k=0}^{n-1}\int_X^{\infty}[-\varepsilon u^{(k)}|\square]^2dx-C_2\int_X^{\infty}|u(x)|^2dx$$

$$\geq(\Lambda-C_2)\,\|u\|^2_{(X,\infty)}\,,$$

where

$$\Lambda=\inf_{0<\xi<\infty}\sum_{k=0}^{n}c_k\xi^{2k}\,,$$

$$c_n=b_n-\varepsilon-C\varepsilon\,,\ c_k=-\varepsilon\,.$$

Therefore, we can conclude

$$\sigma_e(A_{20})\cap(-\infty,\Lambda-C_2)=\varnothing\,.$$

Further, **Error! Reference source not found.** and**Error! Reference source not found.** ε is arbitrary small. Thus we see

$$\sigma_e(A_{20})\cap(-\infty,b_n-C_2)=\varnothing\,,$$

i.e.,

$$\sigma_e(A)\cap(-\infty,b_n-C_2)=\varnothing$$

where **Error! Reference source not found.** is sufficiently large. Hence **Error! Reference source not found.**. So, if **Error! Reference source not found.** $p=n$, then **Error! Reference source not found.** holds true.

Proof of necessity. Assume that (3.4) does not holds true, then there exist a sequence $\{x_j\}_{j=1}^{\infty}$, $x_j \to \infty$ **Error! Reference source not found.**, and interval sequence $\omega_j = [x_j, x_j + 1]$, $j = 1,2,\Lambda$ **Error! Reference source not found.** such that

$$\sup_{j=1,2,\Lambda} \int_{x_j}^{x_j+1} a_p(x)dx < \infty . \tag{3.10}$$

where $\omega_i \cap \omega_j = \varnothing$, $i \neq j$. Next choose a function $u_1(x) \in C_0^{\infty}(0,\infty)$ with $\| u_1 \| = 1$ **Error! Reference source not found.**. Compact support of $u_1(x)$ **Error! Reference source not found.** contains in ω_1 **Error! Reference source not found.**. Next let

$u_j(x) = u_1(x - x_j + x_1)$, $j = 1,2,\Lambda$,

then $(u_i(x), u_j(x)) = \delta_{ij}$. Thus function set $\{u_j\}_{j=1,2,\Lambda}$ **Error! Reference source not found.** is not precompact. However,

$$(Au_j(x), u_j(x)) = \sum_{k=0}^{n} \int_0^{\infty} a_k(t) | u_j^{(k)} |^2 dt = \sum_{k=0}^{n} \int_{x_j}^{x_j+1} a_k(t) | u_j^{(k)} |^2 dt .$$

Therefore, from (3.3),

$$(Au_j(x), u_j(x)) = \sum_{k=0}^{n} \int_0^{\infty} a_k(t) | u_j^{(k)} |^2 dt = \sum_{k=0}^{n} \int_{x_j}^{x_j+1} a_k(t) | u_j^{(k)} |^2 dt$$

$$(Au_j(x), u_j(x)) \le \sum_{k=0}^{n} \max_{x_j < t < x_j+1} | u_j^{(k)}(t) |^2 \int_{x_j}^{x_j+1} a_k(t)dt$$

$$\le \sum_{k=0}^{n} \max_{x_1 < t < x_1+1} | u_j^{(k)}(t) |^2 \int_{x_j}^{x_j+1} a_k(t)dt$$

$\le C_0$ **Error! Reference source not found.**.

This means that $\{u_j(x)\}_{j=1}^{\infty}$ **Error! Reference source not found.** is bounded in energy space **Error! Reference source not found.**. So the spectrum of operator A is discrete. Thus function set $\{u_j(x)\}_{j=1}^{\infty}$ is not precompact. This is a contradiction. Hence (3.4) holds true.

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