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On Fuzzy Spaces

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ABSTRACT

In this paper, properties of fuzzy extremally disconnected spaces and fuzzy submaximal spaces are discussed.

1. Introduction

Extremally disconnected spaces play an important role in set-theoretical topology, in the study of Stone-Ĉech compactification and the Stone Space of any complete Boolean algebra, the theory of Boolean algebra, axiomatic set theory, functional analysis, C*-algebra and the study of Seq(ξ) space etc.

The concept of submaximality of general topological spaces was introduced by Hewitt [8] in 1943. He discovered a general way of constructing maximal topologies. In [1], Atlas et.al. proved that there can be no dense maximal subspace in a product of first countable spaces, while under Booth's lemma there exists a dense submaximal subspace in $[0, 1]^c$. It is established that under the axiom of constructability any submaximal Hausdorff space is σ-discrete. Any homogeneous submaximal space is strongly σ-discrete if there are no measurable cardinals. The first systematic study of submaximal spaces was undertaken in the paper by Arhangel'skiî and Collins [2]. They gave various necessary and sufficient conditions for a space to be submaximal and showed that every submaximal space is σ-discrete [2].

The notion of fuzzy extremally disconnected spaces was studied by Ghosh [6]. In this paper, properties of fuzzy extremally disconnected spaces and fuzzy submaximal spaces are discussed.

2. Preliminaries

In the present paper, X and Y are always fuzzy topological spaces. The class of fuzzy sets on a universal set X will be denoted by I^X and fuzzy sets on X will be denoted by Greek letters as μ, ρ, η, etc. A family τ of fuzzy sets in X is called a fuzzy topology for X if

(1) $0, 1 \in \tau$

(2) $\mu \wedge \rho \in \tau$, whenever $\mu, \rho \in \tau$ and

(3) $\vee\{\mu_\alpha : \alpha \in I\} \in \tau$, whenever each $\mu_\alpha \in \tau$ ($\alpha \in I$)

Moreover, the pair (X, τ) is called a fuzzy topological space. Every member of τ is called a fuzzy open set. The complement of a fuzzy open set is fuzzy closed.

Let μ be a fuzzy set in X. We denote the complement, the interior and the closure of a fuzzy set μ by μ^1 or $1 - \mu$, int(μ) and cl(μ), respectively. A fuzzy set in X is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α ($0 < \alpha \leq 1$) we denote this fuzzy point by x_α, where the point x is called its support [15]. For any fuzzy point $x_\in$ and any fuzzy set μ, we write $x_\in \in \mu$ if and only if $\in \leq \mu(x)$.

Definition 2.1. *A fuzzy set μ in a space X is called*

(1) *fuzzy β-open* [21] *if* $\mu \leq cl(int(cl(\mu)))$;

(2) *fuzzy semi-open* [3] *if* $\mu \leq cl(int(\mu))$;

(3) *fuzzy α-open* [4] *if* $\mu \leq int(cl(int(\mu)))$;

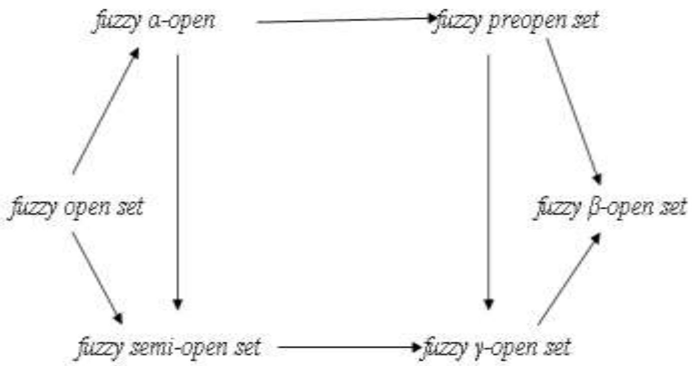
(4) *fuzzy preopen* [4] *if* $\mu \leq int(cl(\mu))$;

(5) *fuzzy γ-open* [7] *if* $\mu \leq int(cl(\mu)) \vee cl(int(\mu))$;

(6) *fuzzy regular open* [3] *if* $\mu = int(cl(\mu))$.

The complements of the above mentioned open sets are called their respective closed sets.

Remark 2.2. [13]



None of the above implications is reversible.

Remark 2.3. [4] *A fuzzy set μ in a space X is fuzzy α-open if and only if it is fuzzy semi-open and fuzzy preopen.*

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Definition 2.4. [6] *A space X is said to be fuzzy extremally disconnected if the closure of every fuzzy open set of X is fuzzy open in X.*

Definition 2.5. [16] *A subset ρ in a space X is said to be a fuzzy locally closed (briefly, a fuzzy LC) set if $\rho = \alpha \wedge \beta$, where α is a fuzzy open set and β is a fuzzy closed set.*

Remark 2.6. [14] *Every fuzzy regular open set is fuzzy open but not conversely.*

Theorem 2.7. [3]

(1) *The closure of a fuzzy open set is a fuzzy regular closed set.*

(2) *The interior of a fuzzy closed set is fuzzy regular open set.*

Lemma 2.8. [3] *For a fuzzy set λ of a fuzzy topological space (X, τ), we have*

(1) $1 - int(\lambda) = cl(1 - \lambda)$,

(2) $1 - cl(\lambda) = int(1 - \lambda)$.

Lemma 2.9. [9] *Let μ be a fuzzy set in a fuzzy topological space X. Then*

(1) $int(cl(int(cl(\mu)))) = int(cl(\mu))$ *and* $cl(int(cl(int(\mu)))) = cl(int(\mu))$.

(2) $[int(cl(\mu))]^1 = cl(int(\mu^1))$ *and* $[cl(int(\mu))]^1 = int(cl(\mu^1))$.

3. Some fuzzy spaces

In this section, we obtain some new results which are related to fuzzy extremally disconnected spaces and fuzzy submaximal spaces.

Theorem 3.1. [19] *For a space X, the following properties are equivalent.*

(1) *X is fuzzy extremally disconnected.*

(2) *$int(\mu)$ is fuzzy closed for every fuzzy closed subset μ of X.*

(3) *$cl(int(\mu)) \leq int(cl(\mu))$ for every subset μ of X.*

(4) *Every fuzzy semi-open set is fuzzy preopen.*

(5) *The closure of every fuzzy β-open subset of X is fuzzy open.*

(6) *Every fuzzy β-open set is fuzzy preopen.*

(7) *For every subset μ of X, μ is fuzzy α-open if and only if it is fuzzy semi-open.*

Theorem 3.2. [19] *For a subset μ of a space X, the following are equivalent.*

(1) *μ is a fuzzy LC set.*

(2) *$\mu = \lambda \wedge cl(\mu)$ for some fuzzy open set λ.*

Theorem 3.3. [19] *Let X be a fuzzy extremally disconnected space and $\mu \leq X$. Then the following properties are equivalent.*

(1) *μ is a fuzzy open set.*

(2) *μ is fuzzy α-open and a fuzzy LC set.*

(3) *μ is fuzzy preopen and a fuzzy LC set.*

(4) *μ is fuzzy semi-open and a fuzzy LC set.*

(5) *μ is fuzzy γ-open and a fuzzy LC set.*

Definition 3.4. *A subset μ of a space X is called fuzzy dense if $cl(\mu)=X$.*

Definition 3.5. *A subset μ of a space X is called fuzzy codense if μ^1 is fuzzy dense.*

Definition 3.6. *A fuzzy space X is called submaximal if every fuzzy dense subset of X is fuzzy open.*

Theorem 3.7. *For a space (X, τ), the following are equivalent.*

(1) *X is submaximal.*

(2) *Every fuzzy codense subset μ of X is fuzzy closed.*

Proof. (1) $\Rightarrow$ (2) Let μ be a fuzzy codense subset of X. Since μ^1 is fuzzy dense, by (1), μ^1 is fuzzy open. Thus, μ is fuzzy closed.

(2) $\Rightarrow$ (1) Let μ be a fuzzy dense subset of X. Since μ^1 is fuzzy codense, by (2), μ^1 is fuzzy closed. Thus, μ is fuzzy open. Hence, X is submaximal.

Theorem 3.8. *Let (X, τ) be a fuzzy topological space and $\mu \leq X$. Then the following are equivalent.*

(1) *μ is fuzzy LC set.*

(2) *$\mu = \alpha \wedge cl(\mu)$ for some fuzzy open set α.*

(3) *$cl(\mu) - \mu$ is fuzzy closed.*

(4) *$(cl(\mu))^1 \vee \mu$ is fuzzy open.*

(5) *$\mu \leq int[\mu \vee (cl(\mu))^1]$.*

Proof. (1) $\Rightarrow$ (2) If μ is fuzzy LC set, then there exist a fuzzy open set α and a fuzzy closed set β such that $\mu = \alpha \wedge \beta$. Clearly, $\mu \leq \alpha \wedge$ cl(μ). Since β is fuzzy closed, cl(μ) $\leq$ cl(β) = β and so $\alpha \wedge$ cl(μ) $\leq \alpha \wedge \beta = \mu$. Therefore $\mu = \alpha \wedge$ cl(μ).

(2) $\Rightarrow$ (3) Now $cl(\mu) - \mu = cl(\mu) \wedge (\mu^1) = cl(\mu) \wedge [\alpha \wedge cl(\mu)]^1 = cl(\mu) \wedge [\alpha^1 \vee (cl(\mu)^1)] = cl(\mu) \wedge \alpha^1$. Therefore cl($\mu$) $-$ μ is fuzzy closed.

(3) $\Rightarrow$ (4) Since $[cl(\mu) - \mu]^1 = [cl(\mu) \wedge \mu^1]^1 = (cl(\mu))^1 \vee \mu$, $(cl(\mu))^1 \vee \mu$ is fuzzy open.

(4) $\Rightarrow$ (5) Since $int[(cl(\mu))^1 \vee \mu] = [cl(\mu)]^1 \vee \mu$, $\mu \leq int[(cl(\mu))^1 \vee \mu]$.

(5)$\Rightarrow$(1)$(cl(\mu))^1 = int((cl(\mu))^1) \leq int[\mu \vee (cl(\mu))^1]$ which implies that $\mu \vee (cl(\mu))^1 \leq int[\mu \vee (cl(\mu))^1]$ and so $\mu \vee ((cl(\mu))^1)$ is fuzzy open. Since $\mu = (\mu \vee (cl(\mu))^1) \wedge cl(\mu)$, μ is fuzzy LC set.

Theorem 3.9. *For a space (X, τ), the following properties are equivalent.*

(1) *X is fuzzy extremally disconnected.*

(2) *Every fuzzy regular open subset of X is fuzzy closed in X.*

(3) *Every fuzzy regular closed subset of X is fuzzy open in X.*

Proof: (1) $\Rightarrow$ (2) Let X be a fuzzy extremally disconnected. Let μ be a fuzzy regular open subset of X. Then μ = int(cl(μ)). Since μ is fuzzy open set, cl(μ) is fuzzy open. Thus, μ=int(cl(μ))=cl(μ) and hence μ is fuzzy closed.

(2) $\Rightarrow$ (1) Suppose that every fuzzy regular open subset of X is fuzzy closed in X. Let $\mu \leq$X be a fuzzy open set. Since int(cl(μ)) is fuzzy regular open, it is fuzzy closed. Therefore cl(int(cl(μ)))=int(cl(μ)) and then cl(int(cl(μ)))=int(cl(μ)) for μ is fuzzy open. This implies cl(int(μ)) = int(cl(μ)) and then cl(μ) = int(cl(μ)). Thus cl(μ) is fuzzy open and hence X is fuzzy extremally disconnected.

(2) $\Leftrightarrow$(3) Proof is similar to (1) $\Leftrightarrow$ (2).

Theorem 3.10. *The following are equivalent for a space (X, τ),*

(1) *X is fuzzy extremally disconnected.*

(2) *The closure of every fuzzy semi-open subset of X is fuzzy open.*

(3) *The closure of every fuzzy preopen subset of X is fuzzy open.*

(4) *The closure of every fuzzy regular open subset of X is fuzzy open.*

Proof: **(**1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) Let μ be a fuzzy semi-open (fuzzy preopen) set. Then μ is fuzzy β-open and by Theorem 3.1, cl(μ) is fuzzy open in X.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) Let μ be any fuzzy regular open subset of X. Then μ is fuzzy semi-open and fuzzy preopen and hence by (2) and (3) respectively cl(μ) is fuzzy open in X.

(4) $\Rightarrow$ (1) Suppose that the closure of every regular open subset of X is fuzzy open.

Let $\mu \leq$X be a fuzzy open set. This implies that int(cl(μ)) is a fuzzy regular open set. Then cl(int(cl(μ))) is fuzzy open. Therefore cl(int(cl(μ))) = int(cl(int(cl(μ)))).

From this we have cl(int(cl(int(μ)))) = int(cl(int(cl(μ))) since μ is fuzzy open. Thus cl(int(μ)) = int(cl(μ)) which gives cl(μ) = int(cl(μ)). Thus, cl(μ) is fuzzy open and hence X is fuzzy extremally disconnected.

Definition 3.11. *A subset μ of a space (X, τ) is called*

(1) *fuzzy t-set* [17] *if $int(\mu)=int(cl(\mu))$.*

(2) *fuzzy semi-regular if* μ *is a fuzzy t-set and fuzzy semi-open.*
(3) *a fuzzy AB-set if* $\mu = \alpha \wedge \beta$ *where* $\alpha \in \tau$ *and* β *is fuzzy semi-regular.*

Definition 3.12. *A subset* μ *of a space* (X, τ) *is said to be*
(1) *fuzzy α*-set* [18] *if* $int(\mu)=int(cl(int(\mu)))$.
(2) *fuzzy A-set* [10] *if* $\mu=\alpha\wedge\beta$ *where* α *is fuzzy open and* β *is a fuzzy regular closed set.*
(3) *fuzzy B-set* [18] *if* $\mu=\alpha\wedge\beta$ *where* α *is fuzzy open and* β *is a fuzzy t-set.*
(4) *fuzzy C-set* [18] *if* $\mu=\alpha\wedge\beta$ *where* α *is fuzzy open and* β *is a fuzzy α*-set.*

Proposition 3.13. *Let* (X, τ) *be a fuzzy topological space and* $\mu\leq X$. *Then every fuzzy t-set is fuzzy α*-set.*

Proof: Let μ be a fuzzy t-set. Then $int(\mu)=int(cl(\mu))$ and $int(cl(int(\mu))) = int(cl(int(cl(\mu)))) = int(cl(\mu)) = in(\mu)$. Therefore μ is a fuzzy α*-set.

Proposition 3.14. *For a fuzzy subset* α *in a fuzzy topological spaces* (X, τ) *the following results are true.*
(1) *If* α *is a fuzzy A-set, then* α *is fuzzy LC set.*
(2) *If* α *is a fuzzy LC set, then* α *is fuzzy B-set.*
(3) *If* α *is a fuzzy B-set, then* α *is fuzzy C-set.*

Proof: (1) It follows from the fact that every fuzzy regular closed set is fuzzy closed.
(2) Since α is a fuzzy LC set, let $\alpha = \lambda \wedge \beta$ where λ is fuzzy open and β is fuzzy closed. Thus β is a fuzzy t-set and hence α is a fuzzy B-set. Then we have $\beta = cl(\beta)$ and hence $int(\beta)=int(cl(\beta))$.
(3) It follows from the fact that every fuzzy t-set is fuzzy α*-set.

Theorem 3.15. *For a space* (X, τ), *the following are equivalent.*
(1) *X is fuzzy submaximal.*
(2) $cl(\mu) - \mu$ *is fuzzy closed for every subset* μ *of X.*
(3) *Every subset of X is fuzzy LC set.*
(4) *Every subset of X is a fuzzy B-set.*
(5) *Every fuzzy dense subset of X is a fuzzy B-set.*

Proof: (1) $\Rightarrow$ (2) Suppose X is fuzzy submaximal. Let μ be a subset of X. Then
$cl(cl(\mu)-\mu)^1=cl(cl(\mu)\wedge\mu^1)^1=cl(\mu\vee(cl(\mu))^1)=X$ and so $(cl(\mu)-\mu)^1$ is fuzzy dense. By hypothesis,$(cl(\mu)-\mu)^1$ is fuzzy open and so $cl(\mu)-\mu$ is fuzzy closed.
(2) and (3) are equivalent by Theorem 3.8.
(3) $\Rightarrow$ (4) It follows from the fact that every fuzzy LC set is a fuzzy B-set by Proposition 3.14.
(4) $\Rightarrow$ (5) Obvious.
(5) $\Rightarrow$ (1) Let μ be a fuzzy dense subset of X. By (5), μ is a fuzzy B-set and so $\mu= \alpha \wedge \beta$ where α is a fuzzy open and $int(\beta)= int(cl(\beta))$. Since $\mu \leq \beta$, $cl(\mu) \leq cl(\beta)$
and so $X= cl(\beta)$. Therefore $X=int(cl(\beta))=int(\beta)$ which implies that $\beta=X$. Hence $\mu=\alpha \wedge \beta = \alpha \wedge X = \alpha$ and so μ is fuzzy open.

Theorem 3.16. *For a space* (X, τ), *the following are equivalent.*
(1) *X is fuzzy submaximal.*
(2) *Every subset of X is a fuzzy B-set.*
(3) *Every fuzzy* β*-open set is a fuzzy B-set.*
(4) *Every fuzzy dense subset of X is a fuzzy B-set.*

Proof:. (1) $\Rightarrow$ (2) It follows from Theorem 3.15.
(2) $\Rightarrow$ (3) Obvious.
(3) $\Rightarrow$ (4) It follows from the fact that every fuzzy dense subset of X is fuzzy β-open.
(4) $\Rightarrow$ (1) It follows from Theorem 3.15.

Theorem 3.17. *For a space* (X, τ), *the following properties are equivalent.*
(1) *X is fuzzy submaximal.*
(2) *Every subset of X is fuzzy LC set.*
(3) *Every subset of X is a union of a fuzzy open subset and a fuzzy closed subset of X.*
(4) *Every fuzzy dense subset of X is an intersection of a fuzzy closed subset and a fuzzy open subset of X.*

Proof:. (1) $\Rightarrow$ (2) It follows from Theorem 3.15.
(2) $\Leftrightarrow$ (3) Let $\mu\leq X$. By (2), we have $\mu^1=\alpha \wedge \beta$ where α is a fuzzy open and β is a fuzzy closed in X. This implies that $\mu=\alpha^1 \vee \beta^1$ where α^1 is fuzzy closed and β^1 is fuzzy open in X. The converse is similar.
(2) $\Rightarrow$ (4) Obvious.
(4) $\Rightarrow$ (1) Let $\mu \leq X$ be a fuzzy dense set. Then $\mu=\alpha \wedge \beta$ where α is fuzzy open and β is fuzzy closed. Since $\mu \leq \beta$ and μ is fuzzy dense, β is fuzzy dense set. Then $int(\beta)=int(cl(\beta))=int(X)=X$. Hence $\beta=X$ and $\mu=\alpha$ is fuzzy open. Thus, X is fuzzy submaximal.

Definition 3.18. *A space* (X, τ) *is called fuzzy normal* [20] *if for each pair of fuzzy closed sets* μ *and* λ *in X with* $\mu \, \bar{q} \, \lambda$, *there exist fuzzy open sets* ρ *and* η *in X such that* $\mu \leq \rho$ *and* $\lambda \leq \eta$, *with* $\rho \, \bar{q} \, \eta$.

Remark 3.19. *In a fuzzy space* (X, τ), *fuzzy normality and fuzzy externally disconnectedness are independent.*

Example 3.20. *Let X be any nonempty set. Define* $C_a: X\to[0,1]$ *such that* $C_a(x)=a$ $\forall x\in X$ and $a\in[0,1]$. *Then* (X, τ) *is fuzzy topological space with* $\tau =\{C_0, C_{4/10}, C_1\}$. (X, τ) *is fuzzy normal but not fuzzy extremally disconnected.*
Since the only fuzzy closed subset in X, other than C_0 *and* C_1 *is* $C_{6/10}$, *there is no pair of fuzzy closed, non quasi coincident sets. Hence* (X, τ) *is fuzzy normal. But the closure of the fuzzy open set* $C_{4/10}$ *is* $C_{6/10}$ *which is not fuzzy open in* (X, τ). *Thus* (X, τ) *is not fuzzy extemally disconnected.*

Example 3.21. *Let* $C_{6/10}$ *and* $C_{7/10}$ *be as defined in Example 3.20. Then* (X, τ) *is a fuzzy topological space with* $\tau = \{C_0, C_{6/10}, C_{7/10}$ $C_1\}$ *and* (X, τ) *is fuzzy extremally disconnected but not fuzzy normal.*
The fuzzy open sets in (X, τ) *other than* C_0 *and* C_1 *are* $C_{6/10}$ *and* $C_{7/10}$. *Also* $cl(C_{6/10}) = C_1 = cl(C_{7/10})$ *which is fuzzy open. Hence* (X, τ) *is fuzzy extremally disconnected.*
But $C_{4/10}$ *and* $C_{3/10}$ *are fuzzy closed in X with* $C_{4/10} \, \bar{q} \, C_{3/10}$ *and there are no fuzzy open sets U and W in X such that* $C_{4/10} \leq U$ *and* $C_{3/10} \leq W$ *with* $C_{4/10} \, \bar{q} \, C_{3/10}$. *Hence* (X, τ) *is not fuzzy normal.*

Proposition 3.22. *In a space* (X, τ), *the following properties hold:*
(1) *Every fuzzy open set is a fuzzy AB-set.*
(2) *Every fuzzy semi-regular set is a fuzzy AB-set.*
(3) *Every fuzzy semi-regular set is a fuzzy t-set.*
(4) *Every fuzzy AB-set is a fuzzy B-set.*

Proposition 3.23. *For a subset* μ *of a space* (X, τ), *the following properties are equivalent:*
(1) μ *is a fuzzy open set.*
(2) μ *is a fuzzy* α*-open set and a fuzzy AB-set.*
(3) μ *is a fuzzy preopen set and a fuzzy AB-set.*

Proof: (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) Obvious.
(3) $\Rightarrow$ (1) Since μ is a fuzzy preopen set, $\mu\leq int(cl(\mu))$. Since μ is a fuzzy AB-set,
$\mu=\alpha \wedge \beta$ where $\alpha \in \tau$ and β is fuzzy semi-regular set. Now $\mu\leq int(cl(\alpha \wedge \beta))$
$\leq int(cl(\alpha) \wedge cl(\beta))$
$=int(cl(\alpha)) \wedge int(cl(\beta))$
$=int(cl(\alpha)) \wedge int(\beta)$ (β is a fuzzy t-set)
Now, we have $\mu\leq \alpha$) $\mu=\alpha \wedge \mu$

$\leq \alpha \wedge [\text{int}(\text{cl}(\alpha)) \wedge \text{int}(\beta)]$
$\leq \alpha \wedge [X \wedge \text{int}(\beta)] = \alpha \wedge \text{int}(\beta)=\text{int}(\alpha) \wedge \text{int}(\beta)$ since α is fuzzy open
$=\text{int}(\alpha \wedge \beta)$
$=\text{int}(\mu)$
Then $\mu \in \tau$.

Lemma 3.24. *If μ is fuzzy preopen set in (X, τ), then $\mu=\lambda \wedge \rho$ for some $\lambda \in \tau$ and fuzzy dense $\rho \leq X$.*

Proof: If μ is fuzzy preopen set in (X,τ) then $\mu \leq \text{int}(\text{cl}(\mu))$. Taking $\lambda=\text{int}(\text{cl}(\mu))$, we have $\mu\leq \lambda$ and λ is fuzzy regular open in (X, τ) and hence fuzzy open in (X, τ). Since $\lambda=\text{int}(\text{cl}(\mu))$, we have $\lambda \leq \text{cl}(\mu)$ and $(\text{cl}(\mu))^1 \leq \lambda^1$. Let $\rho=(\lambda - \mu)^1=(\lambda \wedge \mu^1)^1= \lambda^1 \vee \mu$. We get $\text{cl}(\rho)=\text{cl}(\mu \vee \lambda^1) = \text{cl}(\mu) \vee \text{cl}(\lambda^1) \geq \lambda^1 \vee \text{cl}(\mu) \geq [\text{cl}(\mu)]^1 \vee \text{cl}(\mu) = X$. Then ρ is fuzzy dense in (X, τ) with $\mu=\lambda \wedge \rho$. The converse is not true.

Example 3.25. *Let* X={a, b}*and λ :* X$\rightarrow$[0,1] *be defined as $\lambda(a)=.6$, $\lambda(b)=.4$.*
Then (X, τ) is a fuzzy topological space with $\tau=\{0_x, \lambda, 1_x\}$; $\lambda=(.6, .4)$ is fuzzy open in (X, τ). Taking $\rho=(.4, .7)$, $cl(\rho) = 1$ and hence ρ is fuzzy dense in (X, τ). But $\mu=\lambda \wedge \rho=(.6, .4) \wedge (.4, .7)=(.4,.4)$ is not fuzzy preopen for $\mu=(.4, .4) \nleq 0_x =int(cl(.4,.4))$.

Theorem 3.26. *For a space (X, τ), the following properties are equivalent.*
(1) *X is fuzzy submaximal.*
(2) *Every fuzzy preopen set is fuzzy open.*
(3) *Every fuzzy preopen set is fuzzy semi-open and every fuzzy α-open set is fuzzy open.*

Proof. (1) $\Rightarrow$ (2) It follows from the previous lemma.
(2) $\Rightarrow$ (3) Suppose that every fuzzy preopen set is fuzzy open. Then every fuzzy preopen set is fuzzy semi-open.
Let $\mu \leq X$ be a fuzzy α-open set. Since every fuzzy α-open set is fuzzy preopen set, by (2), μ is fuzzy open.
(3) $\Rightarrow$ (1) Let μ be a fuzzy dense subset of X. Since $\text{cl}(\mu)= X$, μ is fuzzy preopen. By (3), μ is fuzzy semi-open. Thus μ is fuzzy preopen and fuzzy semi-open implies μ is fuzzy α-open. Again by (3), μ is fuzzy open and hence, X is fuzzy submaximal.

Theorem 3.27. *For a space (X, τ), the following properties are equivalent.*
(1) *X is fuzzy submaximal and fuzzy extremally disconnected.*
(2) *Any fuzzy β-open subset of X is fuzzy open.*

Proof: (1) $\Rightarrow$ (2) Since X is fuzzy extremally disconnected, by Theorem 3.1, every fuzzy β-open set is fuzzy preopen. Again by Theorem 3.26, every fuzzy preopen set is fuzzy open since X is fuzzy submaximal. Thus, every fuzzy β-open set is fuzzy open.
(2) $\Rightarrow$ (1) Let μ be any fuzzy β-open subset of X. By assumption μ is fuzzy open and hence fuzzy preopen. Then by Theorem 3.1 X is fuzzy extremally disconnected. Let μ be any fuzzy preopen subset of X. Then μ is fuzzy β-open and by assumption μ is fuzzy open. Hence by Theorem 3.26, X is fuzzy submaximal.

Corollary 3.28. *If a space (X, τ) is fuzzy submaximal and fuzzy extremally disconnected, the following are equivalent for a subset $\mu \leq X$.*
(1) *μ is fuzzy β-open.*
(2) *μ is fuzzy semi-open.*
(3) *μ is fuzzy preopen.*
(4) *μ is fuzzy α-open.*
(5) *μ is fuzzy open.*

Proof: It follows from the previous theorem.

Theorem 3.29. *If a space (X, τ) is fuzzy submaximal and fuzzy extremally disconnected, the following properties are equivalent for a subset $\mu \leq X$.*
(1) *μ is fuzzy semi-open.*
(2) *μ is fuzzy AB-set.*

Proof: (1) $\Rightarrow$ (2) Let μ be a fuzzy semi-open set of X. By assumption on X and Corollary 3.28 μ is fuzzy open. Hence by Proposition 3.22, μ is a fuzzy AB-set.
(2) $\Rightarrow$ (1) Let μ be a fuzzy AB-set. Then $\mu=\lambda \wedge \rho$ where $\lambda\in\tau$ and ρ is fuzzy semiregular and hence ρ is fuzzy semi-open X. By assumption on X, ρ is fuzzy open in X and $\mu=\lambda \wedge \rho$ is fuzzy open and therefore fuzzy semi-open in X.

Theorem 3.30. *For any fuzzy topological space, the following are equivalent.*
(1) *X is fuzzy extremally disconnected.*
(2) *for each fuzzy closed set λ, $int(\lambda)$ is fuzzy closed.*
(3) *for each fuzzy open set λ, we have $cl(\lambda) + cl(1 - cl(\lambda)) = 1$.*
(4) *for every pair of fuzzy open sets λ, μ in X with $cl(\lambda) + \mu = 1$, we have $cl(\lambda) + cl(\mu) = 1$.*

Proof: (1) $\Rightarrow$ (2) Let λ be any fuzzy closed. Then $1 - \lambda$ is fuzzy open. By (1), $\text{cl}(1- \lambda)$ is fuzzy open. Since $\text{cl}(1 - \lambda) = 1 - \text{int}(\lambda)$, $\text{int}(\lambda)$ is fuzzy closed.
(2) $\Rightarrow$ (3) Let λ be any fuzzy open. Then $1 - \lambda$ is fuzzy closed. By (2), $\text{int}(1 - \lambda)$
is fuzzy closed. Thus $\text{cl}(\text{int}(1 - \lambda)) = \text{int}(1 - \lambda)$. Now
$\text{cl}(\lambda) + \text{cl}(1 - \text{cl}(\lambda)) = \text{cl}(\lambda) + \text{cl}(\text{int}(1 - \lambda))$
$= \text{cl}(\lambda) + \text{int}(1 - \lambda)$
$= \text{cl}(\lambda) + 1 - \text{cl}(\lambda)$
$= 1$.
(3) $\Rightarrow$ (4) Assume for any open set λ, $\text{cl}(\lambda) + \text{cl}(1 - \text{cl}(\lambda)) = 1$. Suppose λ and μ
be any two fuzzy open sets such that $\text{cl}(\lambda) + \mu = 1$.
Then $\text{cl}(\lambda) + \mu = 1$
$\Rightarrow \text{cl}(\lambda) + \mu = \text{cl}(\lambda) + \text{cl}(1 - \text{cl}(\lambda))$
$\Rightarrow \mu = \text{cl}(1 - \text{cl}(\lambda))$
$\Rightarrow \text{cl}(\mu) = \text{cl}(1 - \text{cl}(\lambda))$
$\Rightarrow \text{cl}(\mu) = 1 - \text{cl}(\lambda)$
$\Rightarrow \text{cl}(\mu) + \text{cl}(\lambda) = 1$.
(4) $\Rightarrow$ (1) Let λ be any fuzzy open set and put $\text{cl}(\lambda) + \mu = 1$. That is, $\mu = 1 - \text{cl}(\lambda)$.
by (4), $\text{cl}(\mu) + \text{cl}(\lambda) = 1$.
$\text{cl}(\lambda) = 1 - \text{cl}(\mu)$.
$\text{cl}(\lambda)$ is fuzzy open in X.
That is, X is fuzzy extremally disconnected.

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