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Distributional generalized finite Hankel type transformation

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ABSTRACT

In this paper we have extended generalized finite Hankel transform in [2] with the use of Zemanian's technique related to the transformations arising from orthonormal series expansions (see [6]). Inversion formula, characterization theorem and application of differential operator in solving certain type of differential equations have been established.

1. Introduction : Inspired by work of authors in [2], We define generalized finite Hankel type transformation by the linear operator

$$H_{\alpha,\beta}[f(x)] = \int_0^a x^{1-r} f(x) J_{\alpha-\beta}(t_n x)\,dx = \bar{f}(t_n) \tag{1.1}$$

where $f(x)$ belongs to a certain class of functions for which the integral exists and $0 \le x \le a$. If $t_n\ (n = 1,2,3,\ldots\ldots)$ are the positive roots of the transcendental equations

$$J_{\alpha-\beta}(t_n a) = 0 \tag{1.2}$$

then the corresponding inversion formula is

$$f(x) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{\overline{f(n)}}{\left[J_{3\alpha+\beta}(t_n a)\right]^2} \left[x^r J_{\alpha-\beta}(t_n x)\right] \tag{1.3}$$

For $r = 0$ and $\alpha = \frac{1}{4} + \frac{\mu}{2}$, $\beta = \frac{1}{4} - \frac{\mu}{2}$ the relations reduce to the case studied in [4] where the Kernel $J_{\alpha-\beta}(t_n)$ is the Bessel functions of first kind and order $\alpha - \beta$.

Now we state and prove the result about orthogonality of $x^r J_{\alpha-\beta}(t_i x)$ as the following lemma:

Lemma 1.1: The general solution of differential equation

$$\left(D_x^2 + \frac{(1-2r)}{x} D_x\right) y + \left[t^2 + \frac{(r^2 - \alpha^2 - \beta^2 + 2\alpha\beta)}{x^2}\right] y = 0 \tag{1.4}$$

$(\alpha - \beta) \ge 0,\quad t > 0$, is given by the equation

$$y(x) = x^r \left[C_1 J_{\alpha-\beta}(tx) + C_2\, r_{\alpha-\beta}(tx)\right] \tag{1.5}$$

where t_i are the roots of $J_{\alpha-\beta}(t_i a) = 0$ (see [2]) ;

where $D_x \equiv \frac{d}{dx}$. Then

$$\int_0^a x^{1-2r} \left[x^r J_{\alpha-\beta}(t_i x)\right]\left[x^r J_{\alpha-\beta}(t_j x)\right] = 0 \quad if\ i \ne j. \tag{1.6}$$

$$= \frac{a^2}{2} J_{3\alpha+\beta}^2(t_i a), \qquad if\ i = j$$

Proof: Case I $i \ne j$: If t_i and t_j are unequal roots of $J_{\alpha-\beta}(ta) = 0$. Therefore

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$$J_{\alpha-\beta}(t_i x) = 0 \;\&\; J_{\alpha-\beta}(t_j x) = 0 \tag{1.7}$$

Let

$$u(x) = x^r J_{\alpha-\beta}(t_i x), \qquad v(x) = x^r J_{\alpha-\beta}(t_j x) \tag{1.8}$$

Thus $u(x)$ and $v(x)$ are the solutions of differential equation (1.4). For $u(x)$, (1.4) can be written as

$$[x^2 D_x^2 + x(1-2r) D_x + (t_i^2 x^2 - (\alpha^2 + \beta^2 - 2\alpha\beta) - r^2)]\, u = 0 \tag{1.9}$$

and for $v(x)$, (1.4) can be written as

$$\left[x^2 D_x^2 + x(1-2r) D_x + \left(t_j^2 x^2 - (\alpha^2 + \beta^2 - 2\alpha\beta - l^2)\right)\right] v = 0 . \tag{1.10}$$

Multiplying (1.9) by $v(x)$ and (1.10) by $u(x)$ and subtracting and then multiplying throughout by x^{-2r}, we obtain

$$x^{1-2r}(u''.v - uv'') + x^{-2r}(1-2r)(u'v - uv') + \left(t_i^2 - t_j^2\right) x^{1-2r} uv = 0 \tag{1.11}$$

where $'$ denotes differentiation w.r.t.x.

Now,

$$\frac{d}{dx}[x^{1-2r}(u'v - v'u)] = x^{1-2r}(u''v - v''u) + (1-2r)x^{-2r}(u'v - v'u). \tag{1.12}$$

Now we use (1.12) in (1.11) and integrating (1.11) from 0 to a to obtain

$$[x^{1-2r}(u'v - v'u)]_0^a = (t_i^2 - t_j^2)\int_0^a x^{1-2r} u(x)\, v(x)\, dx . \tag{1.13}$$

Making use of (1.7) and (1.8) in (1.13) we obtain

$$(t_i^2 - t_j^2)\int_0^a x^{1-2r}\left(x^r J_{\alpha-\beta}(t_i x)\right)\left(x^r J_{\alpha-\beta}(t_j)\right) dx = 0. \tag{1.14}$$

As $t_i \neq t_j$ the above equation gives

$$\int_0^a x^{1-2r}\left(x^r J_{\alpha-\beta}(t_i x)\right)\left(x^r J_{\alpha-\beta}(t_j x)\right) dx = 0 \text{ when } i \neq j. \tag{1.15}$$

This proves case I.

Case II: $i = j$.

Similarly we can prove that for $i = j$,

$$\int_0^a x^{1-2r}\left[x^r J_{\alpha-\beta}(t_i x)\right]^2 dx = \frac{a^2}{2} J_{3\alpha+\beta}^2(t_i a).$$

The polynomial $x^r J_{\alpha-\beta}(x)$ form an orthogonal set over $I = (0,a)$ on the real line with respect to weight function $w(x) = x^{1-2r}$.

Thus proof is completed.

2. Two spaces $J(I)$ and $J'(I)$:

Let $I = (0,a)$. We define for $x \in I$,

$$\Delta_{\alpha,\beta,r} = x^{2\beta-1-r} D\, x^{4\alpha-2r} D\, x^{2\beta-1-r} . \tag{2.1}$$

Now we compare with Δ_v where $v = \alpha - \beta - r$ Δ_v is linear operator for finite Hankel type transform for $v \geq -\frac{1}{2}$.

Therefore,

$$(\alpha-\beta) - r \geq \frac{1}{2} \Rightarrow (\alpha-\beta) \geq -\frac{1}{2} + r \Rightarrow (\alpha-\beta) \geq -\left(\frac{1}{2} - r\right). \tag{2.2}$$

For $r = 0$, $\alpha = \frac{1}{4} + \frac{\mu}{2}$, $\beta = \frac{1}{4} - \frac{\mu}{2}$, $\Delta_{\alpha,\beta,r}$ converts to linear operator for finite Hankel type transform.

Now, we define for a non-negative integer n

$$\psi_n(x) = \frac{\sqrt{2}\ x^{(1-2r)^{\frac{1}{2}}}\left(x^r J_{\alpha-\beta}\left(y_{\alpha,\beta,n}x\right)\right)}{a \qquad J_{3\alpha+\beta}\left(a\, y_{\alpha,\beta,n}\right)};\ n = 1,2,\dots. \tag{2.3}$$

from inversion formula for $f(x)$, where $J_{\alpha-\beta}$ is the $(\alpha-\beta)^{th}$ order Bessel type function of first kind and $y_{\alpha,\beta,n}$ denote all positive roots of $J_{\alpha-\beta}(ay) = 0$ with

$0 < y_{\alpha,\beta,1} < y_{\alpha,\beta,2} < y_{\alpha,\beta,3} < \cdots$ and $\lambda_n = -y^2_{\alpha,\beta,n}$. (2.4)

Then

$$\Delta_{\alpha,\beta,r}\,\psi_n = \lambda_n\,\psi_n\ , \qquad n = 0,1,2,\dots.. \tag{2.5}$$

One can easily note that $|\lambda_n| \to \infty$ as $n \to \infty$.

The sequence $\{\phi_n\}_{n=0}^{\infty}$ of smooth functions in $L_2(I)$ form a complete orthonormal system on it.

Definition 2.1:

We define $J(I)$ as the collection of all complex valued smooth function $\phi(x)$ defined on I such that

(i) For any non-negative integer k

$$\rho_k(\phi) = \rho_0\left(\Delta^k_{\alpha,\beta,r}\,\phi(x)\right) = \left[\int_0^a \left|\Delta^k_{\alpha,\beta,r}\,\phi(x)\right|^2\right]^{\frac{1}{2}} < \infty \tag{2.6}$$

(ii) For each pair of non-negative integers n, k

$$\left(\Delta^k_{\alpha,\beta,r}\,\phi\,,\psi_n\right) = \left(\phi, \Delta^k_{\alpha,\beta,r}\,\psi_n\right). \tag{2.7}$$

Note that every member of sequence $\{\psi_n\}_{n=0}^{\infty}$ of eigen functions is a member of $J(I)$.

The operator $\Delta_{\alpha,\beta,r}$ is continuous linear mapping of $J(I)$ into itself. Continuity is established from the fact that

$\left(\Delta_{\alpha,\beta,r}\,\phi_v\,,\psi_n\right) = \left(\phi_v\,,\Delta_{\alpha,\beta,r}\,\psi_n\right) = \lambda_n\,(\phi_v,\psi_n) \to 0$

as $v \to \infty$ whenever $\{\phi_n\}_{n=0}^{\infty}$ converges to the zero function in $J(I)$. $J(I)$ is a linear space under addition and multiplication by a complex number. ρ_0 is a norm and $\{\rho_k\}_{k=0}^{\infty}$ is a separating collection of seminorms, hence it is a countable multinorm on $J(I)$. We equip with J (I) the topology generated by $\{\rho_k\}_{k=0}^{\infty}$. Thus J (I) is a countably multinormed space. Every Cauchy sequence in J (I) converges in it hence J (I) is complete and therefore it is a Frechet space. Also J (I) is a testing function space.

Lemma 2.2: Let $\phi \in J(I)$ then for $0 < x < a$, ϕ can have series expansion

$$\phi(x) = \sum_{n=0}^{\infty} (\phi(x), \quad \psi_n(x))\ \psi_n(x)\ . \tag{2.8}$$

This converges in J (I).

Lemma 2.3: $\Delta_{\alpha,\beta,r}$ is self adjoining differential operator.

That is

$$\left(\Delta_{\alpha,\beta,r}\,\phi_1,\phi_2\right) = \left(\phi_1,\Delta_{\alpha,\beta,r}\,\phi_2\right). \tag{2.9}$$

Lemma 2.4: For b_n to be complex numbers, the series $\sum_{n=0}^{\infty} b_n\,\psi_n$ converges in J (I) if and only if the series $\sum_{n=0}^{\infty} |\lambda_n|^{2k}\,|b_n|^2$ converges for every non-negative integer k.

Proof : By using (2.7) we have

$$\int_0^a \left|\Delta^k_{\alpha,\beta,r} \sum_{n=p}^{q} b_n\,\psi_n\right|^2 dx = \int_0^a \left|\sum_{n=p}^{q} b_n\,\Delta^k_{\alpha,\beta,r}\,\psi_n\right|^2 dx$$

$$= \int_0^a \left| \sum_{n=p}^{q} b_n \lambda_n^k \psi_n \right|^2 dx$$

$$= \int_0^a \sum_{n=p}^{q} \sum_{m=p}^{q} b_n \bar{b}_m \lambda_n^k \lambda_m^k \psi_n \bar{\psi}_m \, dx.$$

But due to orthonormality we have

$$\int_0^a \left| \Delta_{\alpha,\beta,r}^k \sum_{n=p}^{q} b_n \psi_n \right|^2 dx = \sum_{n=p}^{q} |\lambda_n|^2 |b_n|^2 .$$

Now we define the dual space $J'(I)$ as the collection of all linear continuous functional on $J(I)$. Since $J(I)$ is testing function space, $J'(I)$ is space of generalized functions. As usual, the number that $f \in J'(I)$ assigns to any $\phi \in J(I)$ is denoted by $\langle f, \phi \rangle$. We define

$$(f, \phi) = \langle f, \bar{\phi} \rangle, \qquad \phi \in J(I). \tag{2.10}$$

For any complex number b

$$(bf, \phi) = b\,(f, \phi) = (f, \bar{b}\,\phi). \tag{2.11}$$

Definitely $J'(I)$ is linear space. As $J(I)$ is complete, $J'(I)$ is also complete by the Theorem 1.8.3 of Zemanian [5]. We define the generalized differential operator $\overline{\Delta}'_{\alpha,\beta,l}$ on $J'(I)$ through

$$\left(f, \Delta_{\alpha,\beta,r}\phi\right) = \langle f, \overline{\Delta_{\alpha,\beta,r}\phi} \rangle = \langle \overline{\Delta_{\alpha,\beta,r}}' f, \bar{\phi} \rangle = \langle \overline{\Delta_{\alpha,\beta,r}}' f, \bar{\phi} \rangle.$$

Since $\Delta_{\alpha,\beta,r}$ is self adjoint $\overline{\Delta_{\alpha,\beta,r}}' = \Delta_{\alpha,\beta,r}$.

Thus $(\Delta f, \phi) = (f, \Delta\phi), \qquad f \in J'(I), \qquad \phi \in J(I).$ (2.12)

It can be easily proved that $\Delta_{\alpha,\beta,l} : J'(I) \to J'(I)$ is continuous linear mapping by making use of the fact $\Delta_{\alpha,\beta,l}$ is a continuous linear mapping of $J(I)$ into itself.

Now we state some properties of $J(I)$ and $J'(I)$ which will be useful in the sequel.

(i) $J(I) \subset L_2(I)$ when we identify that each function in $J(I)$ with corresponding equivalence class in $L_2(I)$. Convergence in $J(I)$ implies the convergence in $L_2(I)$.

(ii) $D(I) \subset J(I)$. Convergence in $D(I)$ implies the convergence in $J(I)$. The topology of $D(I)$ is stronger than that induced on it by $J(I)$. The restriction of any member $f \in J'$(I) to $D(I)$ is a member of $D'(I)$. Moreover convergence in $J'(I)$ implies the convergence in $D'(I)$. Hence in the sense of Zemanian [5], the members of $J'(I)$ are distributions.

(iii) $J(I) \subset \varepsilon(I)$. Furthermore if $\{\phi_v\}_{v=1}^{\infty}$ converges in $J(I)$ to the limit say ϕ then $\{\phi_v\}_{v=1}^{\infty}$ also converges in $\varepsilon(I)$ to the same limit ϕ.

(iv) Since $D(I) \subset J(I) \subset \varepsilon(I)$, and $D(I)$ is dense in $\varepsilon(I)$, $J(I)$ is also dense in $\varepsilon(I)$. The topology of $J(I)$ is stronger than the topology induced on $J(I)$ by $\varepsilon(I)$. Hence $\varepsilon'(I)$ is subspace of $J'(I)$.

(v) We make $L_2(I)$ as a subspace of $J'(I)$ by defining the number that $f \in L_2(I)$ assigns to any $\phi \in J(I)$ as

$$(f, \phi) = \int_0^a f(x)\, \overline{\phi(x)}\, dx . \tag{2.13}$$

Now since $J(I)$ is subspace of $L_2(I)$, it is clear that $J(I)$ is imbedded in $J'(I)$.

Also f is linear and continuous on $J(I)$.

(vi) If $f(x) = \Delta_{\alpha,\beta,r}^k g(x)$ for some $g \in L_2(I)$ and some k then $f \in J'(I)$. Indeed, $\Delta_{\alpha,\beta,l}$ is linear and continuous mapping of $J'(I)$ into itself and $L_2(I) \subset J'(I)$ implies that $f \in J'(I)$.

(vii) For each $f \in J'(I)$ there exists a positive constant C and a non-negative integer r such that for every $\phi \in J(I)$

$|(f, \phi)| \leq C\, \rho_r(\phi).$

where $\rho_r = \max\{r_1, r_2, \ldots, r_r\}$ *and* C, r depend on f but not on ϕ.

3. Orthogonal series expansion of a generalized Function in $J'(I)$:

In this section we provide fundamental theorem to represent an orthonormal series expansion of any $f \in J'(I)$ with respect to ψ_n which in turn yields an inversion formula for the generalized integral transformation.

Theorem 3.1: Every $f \in J'(I)$ has a series expansion

$$f = \sum_{n=0}^{\infty} (f, \psi_n)\, \psi_n \tag{3.1}$$

which converges in $J'(I)$.

Proof: By Lemma 2.2, for any $f \in J(I)$ we have

$$(f, \phi) = \left(f, \quad \sum_{n=0}^{\infty} (\phi, \quad \psi_n)\, \psi_n\right) = \sum_{n=0}^{\infty} \overline{(\phi, \psi_n)}\, (f, \psi_n)$$

$$= \sum_{n=0}^{\infty} (f, \psi_n)\, (\psi_n, \phi)\,. \tag{3.2}$$

Now the right hand side converges for every $\phi \in J(I)$.

Thus

$$(f, \phi) = \left(\sum_{n=0}^{\infty} (f, \psi_n)\, \psi_n, \qquad \phi\right)$$

proves our assertion.

The orthonormal series expansion (3.1) gives inversion formula for a distributional generalized finite Hankel transform $\mathcal{H}$ defined by

$$\mathcal{H} f = F(n) = (f, \psi_n) = f \in J'(I), n = 0,1,2, \ldots . \tag{3.3}$$

In this way $\mathcal{H}$ is a mapping of $J'(I)$ into the space of complex valued functions $F(n)$ defined on n and

$$\mathcal{H}^{-1} F(n) = f = \sum_{n=0}^{\infty} F(n)\, \psi_n. \tag{3.4}$$

$\mathcal{H}$ is a continuous linear mapping.

Thus proof is completed.

Theorem 3.2 (Uniqueness): Let $f, g \in J'(I)$ and $\mathcal{H}\, f = F(n)$, $\mathcal{H}\, g = G(n)$ satisfy $F(n) = G(n)$ for every n, then $f = g$ in the sense of equality in $J'(I)$.

4. Characterization of Distributional Generalized finite Hankel type transforms:

In this section we give characterization of the functions $F(n)$ which are generalized finite Hankel type transforms of distributions in $J'(I)$ as the following theorem.

Theorem 4.1: For b_n to be complex numbers, the series $\sum_{n=0}^{\infty} b_n\, \psi_n$ (4.1)

converges in $J'(I)$ if and only if there exists a non-negative integer q such that $\sum_{\lambda_n \neq 0} |\lambda_n|^{-2q}\, |b_n|^2$ converges. Moreover if f denotes the sum (4.1) in $J'(I)$ then $b_n = (f, \psi_n)$.

Proof: Necessary condition: We assume that the series (4.1) converges in $J'(I)$ say to f then since $\psi_n \in J(I)$

$$(f, \psi_m) = \left(\sum_{n=0}^{\infty} b_n\, \psi_n\,, \qquad \psi_m\right) = \sum_{n=0}^{\infty} b_n\, (\psi_n, \qquad \psi_m) = b_m = F(m)$$

by orthonormality of ψ_n This proves the last statement of the theorem. Now we denote by P the statement

" For every $\phi = \sum a_n\, \psi_n \in J(I)$, the series $\sum \bar{a}_n\, b_n$ converges" . (P)

Here we select a_n such that $|\bar{a}_n\, b_n| = |a_n\, b_n|$.

Firstly we prove that the sequence $(F(n)\,\lambda_n^{-q})_{n=1}^{\infty}$ is bounded for some q, say q_0. If not then the sequence is unbounded for every $q = 1,2,3, \ldots.$ Hence there is increasing sequence $\{n_q\}$ of positive integers such that

$$\left|F(n_q)\,\lambda_{n_q}^{-1}\right| \geq 1, \qquad q = 1,2,3 \ldots.$$

Now for every $q = 1,2,3 \ldots,$ we get

$$a_n = \begin{cases} \left|a\,\lambda_{n_q}^{q}\right|^{-1} & if\; n = n_q \\ 0 & if\; n = n_q. \end{cases}$$

The for any fixed non-negative integer k,

$$\sum_{n=1}^{\infty} \left|\lambda_n^k\, a_n\right|^2 = \sum_{q=1}^{\infty} \left[\left|\lambda_{n_q}^k\right| \left|\lambda_{n_q}^{q} q\right|^{-1}\right]^2 = \sum_{q=1}^{\infty} q^{-1} \left|\lambda_{n_q}\right|^{2k-2q}.$$

Since $\left|\lambda_{n_q}\right|^{2k-2q}$ is bounded for sufficiently large q, the series $\sum_{q=1}^{\infty} q^{-2} \left|\lambda_{n_q}\right|^{2k-2q}$ converges. Hence $\sum_{n=1}^{\infty} \left|\lambda_n^k\, a_n\right|^2$ converges for every non-negative integer k. But

$$\sum_{n=1}^{p} |\lambda_n^k\, a_n|^2 = \int_0^a \left|\Delta_{\alpha,\beta,r}^k \sum_{n=1}^{p} a_n\, \psi_n\right|^2 dx. \tag{4.2}$$

Thus the series $\sum_{n=1}^{p} a_n\, \psi_n$ converges in $J(I)$, say to $\phi \in J(I)$.

$$\sum_{n=1}^{\infty} |a_n\, b_n| \geq \sum_{q=1}^{\infty} \left|a_{n_q}\, \lambda_{n_q}^{q}\right| = \sum_{q=1}^{\infty} q^{-1} = \infty.$$

This contradicts the statement (P). Thus $\{F(n)\,\lambda_n^{-q_0}\}$ is bounded for some positive q_0. Now from the fact that $|\lambda_n| \to \infty$ as $n \to \infty$ we can say that $\left|\lambda_n^{-q}\, F(n)\right| \to as\; n \to \infty$ for each $q > q_0$. Next we prove that the series in (4.2) converges for some $q > q_0$. Let the series in (4.2) diverges for every $q > q_0$. Then there will be increasing sequence

$\{m_q\}$ of positive integers such that

$$1 \leq \sum_{n=m_{q-1}}^{m_{q-1}} |\lambda_n^{-q} F(n)|^2 < 2, \qquad q = q_0 + 1, \qquad q_0 + 2, \qquad q_0 + 3, \ldots.$$

Thus we select

$$|a_n| = \left|f(n)\,\lambda_n^{-2q}\, q^{-1}\right| when\; m_q - 1 < n < m_q, \qquad q > q_0.$$

Then for every non-negative integer k ,

$$\sum_{m_{q-1}}^{m_{q-1}} \left|\lambda_n^k\, a_n\right|^2 = \sum_{m_{q-1}}^{m_{q-1}} |\lambda_n|^{2k-2q} \left|\lambda_n^{-q}\, F(n)\right|^2 q^{-2} < 2q^{-2},$$

for sufficiently large q.

Hence the series

$$\sum_{n=1}^{\infty} \left|\lambda_n^k\, a_n\right|^2$$

converges for each k. Then again by (4.2) the series $\sum_{n=1}^{\infty} a_n\, \psi_n$ converges in $J(I)$, say to ϕ. On the other hand $\sum_{n=1}^{\infty} |a_n\, F(n)|$ diverges because

$$\sum_{n=m_{q-1}}^{m_{q-1}} |a_n\, F(n)| = \sum_{n=m_{q-1}}^{m_{q-1}} |a_n\, F(n)| = \sum_{n=m_{q-1}}^{m_{q-1}} \left|[F(n)]^2\, \lambda_n^{-2q}\, q^{-1}\right| \geq q^{-1}.$$

This again contradicts the statement (P). Hence the series (4.2) converges for every $q > q_0$.

Sufficient condition: Assume that the series (4.2) converges for some positive q.

Let $\phi \in J(I)$. Then for every $\phi \in J(I)$

$$\sum_{\lambda_n \neq 0} |(b_n \psi_n, \phi)| \leq \left[\sum_{\lambda_n \neq 0} |b_n (\psi_n, \phi)| = \sum_{\lambda_n \neq 0} |\lambda_n^{-q} b_n| \, |\lambda_n^q (\phi, \psi_n)|\right].$$

Now for sum of real numbers we use Schwarz inequality to obtain

$$\sum_{\lambda_n \neq 0} |(b_n \psi_n, \phi)| \leq \left[\sum_{\lambda_n \neq 0} |\lambda_n^{-q} b_n|^2 . \sum |\lambda_n^q (\phi, \psi_n)|^2\right]^{\frac{1}{2}}$$

by assumption the first series on the right side converges.

Now, since $\phi \in J(I)$, from Lemma 2.2, the series expansion $\phi = \sum_{n=1}^{\infty} (\phi, \psi_n) \psi_n$ converges in $J(I)$. Then using (4.2) for $k = q$ we confirm that the second series on the right side also converges. Thus the series $\sum_{\lambda_n \neq 0} (b_n \psi_n, \phi) = \left(\sum_{\lambda_n \neq 0} b_n \psi_n, \phi\right)$ converges which further implies that the series (4.1) converges in $J'(I)$. This completes the proof.

Theorem 4.2: f is a member of $J'(I)$ if and only if there exists some non-negative integer m and a $g \in L_2(I)$ such that

$$f = \Delta_{\alpha,\beta,r}^m g + \sum_{\lambda_n = 0} b_n \psi_n, \tag{4.3}$$

where b_n are complex numbers.

5. Application of differential operator $\Delta_{\alpha,\beta,r}$:

For a non-negative integer k , by using the series expansion (3.1) of $f \in J'(I)$ we have

$$\Delta_{\alpha,\beta,r}^k f = \sum_{n=0}^{\infty} (f, \psi_n) \Delta_{\alpha,\beta,r}^k \psi_n = \sum_{n=0}^{\infty} (f, \psi_n) \lambda_n^k \psi_n \tag{5.1}$$

from the differential equation

$$P\left(\Delta_{\alpha,\beta,r}^k\right) f = g, \tag{5.2}$$

where P is a polynomial, $g \in J'(I)$ is known and $f \in J'(I)$ is unknown. Since $f \in J'(I)$

$$P\left(\Delta_{\alpha,\beta,r}^k\right) f = P\left(\Delta_{\alpha,\beta,r}^k\right) \sum_{n=0}^{\infty} (f, \psi_n) \psi_n = \sum_{n=0}^{\infty} (f, \psi_n) P\left(\Delta_{\alpha,\beta,r}^k\right) \psi_n$$

$$= \sum_{n=0}^{\infty} (f, \psi_n) P(\lambda_n) \psi_n .$$

Applying $\mathcal{H}$ to (5.2) we get

$P(\lambda_n)(f, \psi_n) = (g, \psi_n)$.

Case I: $P(\lambda_n) \neq 0$, then

$$(f, \psi_n) = [P(\lambda_n)]^{-1} (g, \psi_n).$$

Now by applying $\mathcal{H}^{-1}$ to both the side we get

$$f = \sum_{n=0}^{\infty} (g, \psi_n) [P(\lambda_n)]^{-1} \psi_n . \tag{5.3}$$

Now by characterization Theorem 4.1 and uniqueness Theorem 3.2, the solution in (5.3) exists and is unique.

Case II: $P(\lambda_n) = 0$ for some λ_n .

Let $P\left(\lambda_{n_k}\right) = 0$ for $k = 1,2,3, \dots \dots, m$. Then the solution will be

$$f = \sum_{P(\lambda_n) \neq 0} (g, \psi_n) [P(\lambda_n)]^{-1} \psi_n \tag{5.4}$$

which is not unique in $J'(I)$. We may add to (5.4) any complementary solution

$$f_c = \sum_{s=1}^{m} a_s \, \psi_n \quad ,$$

where a_s are arbitrary numbers.

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