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Advances in Pure Mathematics

Elixir Adv. Pure Math. 59 (2013) 15494-15506

ISSN: 2229-712X

A Class of Multivalent β - Uniformly Starlike Functions Associated with a Convolution

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ARTICLE INFO

Article history:

Received: 19 April 2013;

Received in revised form:

5 June 2013;

Accepted: 5 June 2013;

Keywords

Analytic functions, Convolution,

β - uniformly starlike and convex.

ABSTRACT

In this paper, a class of p -valent β - uniformly starlike analytic functions associated with a convolution is introduced and certain properties of functions belonging to this class such as a necessary and sufficient coefficient inequality, growth and distortion properties, neighborhood properties are obtained. With the help of coefficient inequality, extreme points for functions belonging to this class are derived.

1. Introduction and Preliminaries

Let $S(p)$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k},\ a_{p+k}\in\mathbb{C},\ \ p,k\in\mathbb{N}=\{1,2,3,\ldots\}, \tag{1}$$

which are analytic and p - valent in the unit disk $\Delta=\{z\in\mathbb{C};;|z|<1\}$.

Let $g(z)\in S(p)$ be of the form:

$$g(z)=z^p+\sum_{k=1}^{\infty}b_{p+k}z^{p+k},\ \ b_{p+k}=0. \tag{2}$$

Convolution (Hadamard product), $f*g$ of f and g is defined as usual by

$$(f*g)(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}b_{p+k}z^{p+k}=(g*f)(z). \tag{3}$$

This convolution generalizes several convolution operators such as:

The convolution in (3) reduces to the operator $W_{q,s}^{p}([\alpha_1,A_1])f(z)$ involving a Wright's generalized hypergeometric function (see [18])

$$ {}_q\Psi_s[z]\equiv(..(\alpha_1,A_1),(\alpha_2,A_2),......(\alpha_q,A_q)(\beta_1,B_1),(\beta_2,B_2),......(\beta_s,B_s)z).$$

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The convolution operator $W_{q,s}^{p}([\alpha_1, A_1])f(z)$, for which

$$b_{p+k} = \frac{\prod_{i=1}^{q} \frac{\Gamma(\alpha_i + A_i k)}{\Gamma(\alpha_i)}}{\prod_{i=1}^{s} \frac{\Gamma(\beta_i + B_i k)}{\Gamma(\beta_i)} k!},$$

is studied by Aouf and Dziok [2], [3], Dziok and Raina [8], and Dziok et al. [9] and Sharma [38] in their respective work and taking $A_i = 1, i = 1,2,...q,\ B_i = 1, i = 1,2...s,$ for $q \le s+1$, it reduces to Dziok Srivastava operator [10] which involve a generalized hypergeometric function ${}_qF_s[z]$:

$${}_qH_s^p([\alpha_1])f(z) = z^p\ {}_q\mathrm{F}_s[z] * f(z)$$

where $${}_q\mathrm{F}_s[z] = {}_qF_S(\alpha_1, \alpha_2, \ldots \alpha_q;\ \beta_1, \beta_2, \ldots \beta_s; z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{q} (\alpha_i)_k}{\prod_{i=1}^{s} (\beta_i)_k k!} z^k, z \in \Delta$$

the symbol $(\alpha)_k$ is the Pochhammer symbol defined by $(\alpha)_k = \dfrac{\Gamma(\alpha + k)}{\Gamma(\alpha)}, k \in \mathbb{N}_0.$

The operator ${}_qH_s^p([\alpha_1])f(z)$ includes Hohlov operator [15] which involve Gaussian hypergeometric function ${}_2F_1$:

$${}_2H_1^p([\alpha_1])f(z) = z^p\ {}_2F_1(\alpha_1, \alpha_2;; \beta_1;; z) * f(z),$$

as well as Carlson and Shaffer operator [7] involving incomplete beta function:

$$L_p(\alpha_1, \beta_1)f(z) = z^p\ {}_2F_1(\alpha_1, 1;; \beta_1;; z) * f(z)$$

which again reduces to Ruschweyh derivative operator [31] (also see [8], [9]) :

$$D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)$$

if $\alpha_1 = n + p > 0, \beta_1 = 1$ and $D^0 f(z) \equiv f(z).$

Further, the convolution reduces to the Salagean operator [33] if

$$b_{p+k} = \left(\frac{p+k}{p}\right)^n, n = 0,1,2\ldots$$

and to a generalized Salagean operator [1] if

$$b_{p+k} = \left(\frac{p+\delta k}{p}\right)^n, \delta > 0, n = 0,1,2\ldots$$

Further, the convolution reduces to an integral operator involving generalized fractional integral operator, if

$$b_{p+k} = \frac{(p+1)_k (p-\mu+\nu+1)_k}{(p-\mu+1)_k (p+\lambda+\nu+1)_k}$$

$(0 \le \lambda < 1, \rho > \max\{0, \mu-\nu\}-1)$. Again, this convolution reduces to the derivative operator involving generalized fractional derivative operator, if

$$b_{p+k} = \frac{(p+1)_k (p-\mu+\nu+1)_k}{(p-\mu+1)_k (p-\lambda+\nu+1)_k}.$$

The generalized fractional calculus operators are studied in [5], [25], [39] .

A function $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, which is analytic and convex in Δ is said to be in class P if

$$\operatorname{Re}\{p(z)\} > 0, \quad p(0) = 1.$$

and $p(z)$ is said to be in $P(\alpha, \beta)$ if $\operatorname{Re}(p(z) - \alpha) > \beta |p(z) - 1|, 0 = \alpha < 1, \beta = 0.$

Note that $P(\alpha, 0) = P(\alpha)$.

Goodman ([12], [13]), Ronning ([28], [29]) introduced and studied the following subclasses:

A function $f(z)$ of the form (1) is said to be in the class $S_p(\alpha, \beta)$ of uniformly β - starlike functions if it satisfies the condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta \left|\frac{zf'(z)}{f(z)} - 1\right| \ (z \in \Delta), \text{ where } -1 \le \alpha < 1 \text{ and } \beta \ge 0.$$

A function $f(z)$ of the form (1) is said to be in the class $UCV(\alpha, \beta)$ of uniformly β - convex functions if it satisfies the condition:

$$\operatorname{Re}\left\{1 + \frac{zf'(z)}{f(z)} - \alpha\right\} > \beta \left|\frac{zf'(z)}{f(z)}\right| \ (z \in \Delta), \text{ where } -1 \le \alpha < 1 \text{ and } \beta \ge 0.$$

It follows that $f(z) \in UCV(\alpha, \beta) \Leftrightarrow zf'(z) \in S_p(\alpha, \beta)$.

Let $T(p)$ denote a subclass of $S(p)$ consisting of functions which are analytic p - valent, can be expressed in the form:

$$f(z)=z^p-\sum_{k=1}^{\infty}a_{p+k}z^{p+k}, \quad a_{p+k}=0. \tag{4}$$

Associated with the convolution in this chapter, a class $\beta-S_g(p,m,\alpha)$ of functions $f(z)\in T(p)$ is considered whose members satisfy the condition:

$$\mathrm{Re}\left\{\frac{z(f*g)^{m+1}(z)}{(f*g)^m(z)}-\alpha(p-m)\right\}>\beta\left|\frac{z(f*g)^{m+1}(z)}{(f*g)^m(z)}-(p-m)\right| \tag{5}$$

$(g(z)\in S(p)$ be of the form (2), $\beta\geq 0, -1\leq\alpha<1,\ p>m,\ m\in \mathbb{N}_0,\ z\in\Delta)$,

where $(f*g)^r(z)$ denotes the r^{th} derivative of $(f*g)$ and is given by:

$$(f*g)^r(z)=\frac{p!}{(p-r)!}z^{p-r}+\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-r)!}a_{p+k}b_{p+k}z^{p+k-r} \quad r\in \mathbb{N}_0=\{0,1,2,\ldots\}. \tag{6}$$

This class $\beta-S_g(p,m,\alpha)$ generalizes several classes studied earlier in [5], [14], [17], [34], [36] and [37] etc.

In particular taking $g(z)=\frac{z^p}{1-z}$ (or $b_{p+k}=1$) with $m=0$ and 1 respectively the class $\beta-S_g(p,m,\alpha)$ reduces to p - valent β-uniformly starlike and convex classes respectively of order 0 which are studied in [13], [14], [21], [24] and [29]. Also taking $g(z)=\frac{z}{1-z}$ and $g(z)=\frac{z}{(1-z)^2}$ respectivelywith $m=0$ the class $\beta-S_g(1,m,\alpha)$ reduces to univalent β-uniformly starlike and convex classes respectively of order α which are studied by Shams, Kulkarni and Jahangiri [35]. In addition to that if $\beta=0$ this class reduces to the p - valent starlike and convex classes respectively of order α (see [16]).

Also for $m=0$ and for $g(z)=z^p\ {}_qF_s(\alpha_1,\alpha_2,\ldots\alpha_q;\ \beta_1,\beta_2,\ldots\beta_s;\ z)$ the class $\beta-S_g(p,m,\alpha)$ reduces to the class studied by Marouf [20]. Further on taking $m=0$, $p=1$ and $g(z)=z\ {}_2F_1(\alpha_1,1;;\beta_1;;z)$ the class reduces to the special case of the class studied in [36]. Again for $p=1$, $b_{1+k}=(1+k)^n$, $n\in\mathbb{N}_0$ and $m=0$ the class $\beta-S_g(p,m,\alpha)$ reduces to the class studied by Kuang et al. [19].

Also, note that:

(i) If $g(z)=\varphi(z)$, the class $\beta-S_g(1,0,\alpha)$ studied by Raina and Bansal [27].

(ii) If $g(z)=\frac{z}{(1-z)^2}$, the class $1-S_g(1,0,\alpha)$ reduces to the class studied by Bharati et al. [6].

(iii) If $g(z)=z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k$, $(c\neq 0,-1,-2,.....)$ the class $\beta-S_g(1,0,\alpha)$ coincides with the class studied by Murugusundaramoorthy and Magesh [22], [23].

(iv) If $g(z)=z+\sum_{k=2}^{\infty}k^n z^k$, the class $\beta-S_g(1,0,\alpha)$ studied by Rosy and Murugusundaramoorthy [30].

(v) If $g(z)=z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n z^k$, the class $\beta-S_g(1,0,\alpha)$ reduces to the class studied by Aouf and Mostafa [5].

(vi) If $g(z)=z+\sum_{k=2}^{\infty}\binom{k+\lambda-1}{\lambda}z^k$, the class $\beta-S_g(1,0,\alpha)$ coincides with the class studied by Ruscheweyh [31].

Following earlier works of Ruscheweyh [32], Frasin and Darus [11] and Prajapat et al. [26], consider the (q,δ)-neighborhood of functions $f(z)\in T(p)$ of the form (4) for $q,\delta\geq 0$:

$$N_\delta^q(f)=\left\{h:h\in T(p), h(z)=z^p-\sum_{k=1}^{\infty}c_{p+k}z^{p+k} \text{ and } \sum_{k=1}^{\infty}(p+k)^{q+1}\left|a_{p+k}-c_{p+k}\right|\leq\delta\right\}. \quad (7)$$

It follows from the definition (7) that for the identity function $e(z)=z^p$

$$N_\delta^q(e)=\left\{h:h\in T(p), h(z)=z^p-\sum_{k=1}^{\infty}c_{p+k}z^{p+k} \text{ and } \sum_{k=1}^{\infty}(p+\mathrm{k})^{q+1}\left|c_{p+k}\right|\leq\delta\right\}. \quad (8)$$

It is observed that $N_\delta^0(f)=N_\delta(f)$ the δ-neighborhood defined by Ruscheweyh [32].

2 Coefficient Inequality

In this section, a necessary and sufficient coefficient condition for a functiom $f\in T(p)$ to be in $\beta-S_g(p,m,\alpha)$ is established.

Theorem 2.1 *Let $f(z)\in T(p)$ be of the form (4). Then $f\in\beta-S_g(p,m,\alpha)$, for $g(z)\in S(p)$ of the form (2), $\beta\geq 0, -1\leq\alpha<1, p>m, m\in\mathbb{N}_0$, if and only if*

$$\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-m)!}\left[k(1+\beta)+(1-\alpha)(p-m)\right]a_{p+k}b_{p+k}\leq\frac{(1-\alpha)p!}{(p-m-1)!}. \quad (9)$$

The result is sharp.

Proof. Let (9) holds.

Set $$p(z)=\frac{1}{(1-\alpha)(p-m)}\left\{\frac{z(f*g)^{m+1}(z)}{(f*g)^{m}(z)}-\alpha(p-m)\right\}.$$

To show $f\in\beta-S_g(p,m,\alpha)$, it is necessary to show that

$$\operatorname{Re}p(z)>\beta|p(z)-1|. \tag{10}$$

It is easily verified that (10) holds if and only if

$$\operatorname{Re}\{p(z)(1+\beta e^{i\theta})-\beta e^{i\theta}\}>0 \qquad for\ -\pi\le\theta<\pi$$

which is equivalent to

$$|p(z)(1+\beta e^{i\theta})-\beta e^{i\theta}-1|<|p(z)(1+\beta e^{i\theta})-\beta e^{i\theta}+1|. \tag{11}$$

Using series expansion of $(f*g)^{m+1}$ and $(f*g)^{m}$ from (6)

$$E:=|p(z)(1+\beta e^{i\theta})-\beta e^{i\theta}+1|$$

Thus

$$E>\frac{|z^{p-m}|\left\{\frac{2(1-\alpha)p!}{(p-m-1)!}-\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-m)!}[k(1+\beta)+2(1-\alpha)(p-m)]a_{p+k}b_{p+k}\right\}}{(1-\alpha)(p-m)|(f*g)^{m}(z)|}.$$

Similarly

$$F:=|p(z)(1+\beta e^{i\theta})-\beta e^{i\theta}-1|=|(1+\beta e^{i\theta})(p(z)-1)|=\frac{|z^{p-m}|\left|(1+\beta e^{i\theta})\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-m)!}(-k)a_{p+k}b_{p+k}z^{k}\right|}{(1-\alpha)(p-m)|(f*g)^{m}(z)|}$$

and hence $$F<\frac{|z^{p-m}|}{(1-\alpha)(p-m)|(f*g)^{m}(z)|}\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-m)!}ka_{p+k}b_{p+k}.$$

Thus

$$E-F>\frac{|z^{p-m}|\left[\frac{2(1-\alpha)p!}{(p-m-1)!}-\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-m)!}[2k(1+\beta)+2(1-\alpha)(p-m)]a_{p+k}b_{p+k}\right]}{(1-\alpha)(p-m)|(f*g)^{m}(z)|}\ge 0$$

if (9) holds. This proves that $f\in\beta-S_g(p,m,\alpha)$.

Conversely, suppose $f \in \beta - S_g(p,m,\alpha)$. Now using the fact that

$$\mathrm{Re}\{w-\alpha(p-m)\} > \beta|w-(p-m)|$$

if and only if $$\mathrm{Re}\{w-\alpha(p-m)-\beta[(p-m)-w]e^{i\theta}\} > 0, \quad -\pi \le \theta < \pi.$$

Let $w := \dfrac{z(f*g)^{m+1}(z)}{(f*g)^m(z)}$.

Thus from the hypothesis $\mathrm{Re}\left\{\dfrac{z(f*g)^{m+1}(z)}{(f*g)^m(z)}(1+\beta e^{i\theta})-(p-m)(\alpha+\beta e^{i\theta})\right\} > 0$

or $\mathrm{Re}\left\{\dfrac{(1-\alpha)p!}{(p-m-1)!}z^{p-m} - \sum_{k=1}^{\infty}\dfrac{(p+k)!}{(p+k-m)!}\left[k(1+\beta e^{i\theta})+(1-\alpha)(p-m)\right]a_{p+k}b_{p+k}z^{p+k-m} > 0\right\}$.

The above inequality holds for all z in Δ. Letting $z \to 1^-$ for $-\pi \le \theta < \pi$

$$\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-m)!}\left[k(1+\beta Re(e^{i\theta}))+(1-\alpha)(p-m)\right]a_{p+k}b_{p+k} < \frac{p!(1-\alpha)}{(p-m-1)!}$$

hence, $$\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-m)!}\left[k(1+\beta)+(1-\alpha)(p-m)\right]a_{p+k}b_{p+k} \le \frac{p!(1-\alpha)}{(p-m-1)!}$$

which is the required inequality (9).

Finally, sharpness follows for the extremal function:

$$f_{p+k}(z) = z^p - \frac{(1-\alpha)p!(p+k-m)!z^{p+k}}{(p+k)!(p-m-1)!\left[k(1+\beta)+(1-\alpha)(p-m)\right]b_{p+k}}, k \ge 1, b_{p+k} > 0. \tag{12}$$

Corollary 2.1 *If* $f \in \beta - S_g(p,m,\alpha)$, *then for* $g(z) \in S(p)$ *of the form (2),*

$$a_{p+k} \le \frac{(1-\alpha)p!(p+k-m)!}{(p+k)!(p-m-1)!\left[k(1+\beta)+(1-\alpha)(p-m)\right]b_{p+k}}, k \ge 1, b_{p+k} > 0. \tag{13}$$

The equality in (13) is attained for the function f_{p+k} given by (12).

By Theorem 2.1 following result is obtained, provided b_{p+k} has its positive lower bound.

Corollary 2.2 *If* $f \in \beta - S_g(p,m,\alpha)$, *then for* $g(z) \in S(p)$ *of the form (2) and* $\zeta := \sum_{k=1}^{\infty} b_{p+k}$,

$$\sum_{k=1}^{\infty} a_{p+k} \le \frac{(1-\alpha)(p-m)(p-m+1)}{(p+1)[(1+\beta)+(1-\alpha)(p-m)]\zeta}. \tag{14}$$

Again by Theorem 2.1 and using (14) following result is obtained.

Corollary 2.3 *If* $f \in \beta - S_g(p,m,\alpha)$, *then for* $g(z) \in S(p)$ *of the form (2) and* $\zeta := \min_{k\geq 1} b_{p+k}$,

$$\sum_{k=1}^{\infty}(p+k)a_{p+k} \leq \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)]\zeta}. \tag{15}$$

Remark 1 *Taking* $b_{p+k} = (1+k)^n, k \geq 1, n \in \mathbb{N}_0, m = 0$, *in Theorem 2.1 the result of Aouf and Mostafa [5] follows. Taking* $b_{p+k} = \binom{\lambda + k}{k}, \alpha = 0, m = 0$, *Theorem 2.1, yields a result which is the special case of the result obtained by Sharma and Singh in [37]. Taking* $p = 1$, $m = 0$, *in Theorem 2.1 yields a result which is again a special case of the result obtained by Aouf et al.[4].*

Moreover, when $b_{p+k} = \frac{(a_1)_k (a_2)_k \ldots (a_q)_k}{(b_1)_k (b_2)_k \ldots (b_s)_k}\frac{1}{k!}$, $q = s+1$, $b_i \neq 0,-1,-2,\ldots (i = 1,2,\ldots s)$ *and* $m = 0$, *Theorem 2.1 corresponds to the result obtained by Goyal and Bhagtani [14].*

Remark 2 *If* b_{p+k} *is non-decreasing, we replace* ζ *by* b_{p+1} *in Corollaries 2.2 and 2.3.*

3 Growth and Distortion Bounds

In this section, growth and distortion bounds of functions belonging to the class $\beta - S_g(p,m,\alpha)$ by using results of Corollaries 2.2 and 2.3 are derived.

Theorem 2 *Let* $f(z) \in T(p)$ *of the form (4) be in the class* $\beta - S_g(p,m,\alpha)$, *then for* $g(z) \in S(p)$ *of the form (2) and* $\zeta := \min_{k\geq 1} b_{p+k}$,

$$|z^p| - \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta}|z^{p+1}| \leq |f(z)| \leq |z^p| + \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta}|z^{p+1}| \tag{16}$$

and

$$|pz^{p-1}| - \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)]\zeta}|z^p| \leq |f'(z)| \leq |pz^{p-1}| + \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)]\zeta}|z^p|. \tag{17}$$

The bounds are sharp and extremal function is given by

$$f(z) = z^p - \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta}z^{p+1}. \tag{18}$$

Proof. Taking absolute value of $f(z)$ from (4) and using Corollary 2.2,

$$|f(z)| \le |z^p| + \sum_{k=1}^{\infty} a_{p+k} |z^{p+k}| \le |z^p| + \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta} |z^{p+1}|$$

and
$$|f(z)| \ge |z^p| - \sum_{k=1}^{\infty} a_{p+k} |z^{p+k}| \ge |z^p| - \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta} |z^{p+1}|$$

which prove assertion (16).

Again, from (4)
$$f'(z) = pz^{p-1} - \sum_{k=1}^{\infty} (p+k) a_{p+k} z^{p+k-1}$$

and using Corollary 2.3

$$|f'(z)| \le |pz^{p-1}| + \sum_{k=1}^{\infty} (p+k) a_{p+k} |z^{p+k-1}| \le |pz^{p-1}| + \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)]\zeta} |z^p|$$

and
$$|f'(z)| \ge |pz^{p-1}| - \sum_{k=1}^{\infty} (p+k) a_{p+k} |z^{p+k-1}| \ge |pz^{p-1}| - \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)]\zeta} |z^p|$$

which prove assertion (17).The bounds in (16) and (17) are sharp and extremal function is given by (18).

Corollary 3.1 ***(Covering Result)*** *If* $f \in \beta - S_g(p,m,\alpha)$, *then for* $g(z) \in S(p)$ *of the form (2) and* $\zeta = \min_{k \ge 1} b_{p+k}$, *the disk* $|z| < 1$ *is mapped by* f *onto a domain that contains the disk*

$$|f(z)| < \frac{(1+\beta)(p+1)\zeta + (1-\alpha)(p-m)[(\zeta-1)(p+1)+m]}{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta}.$$

The result is sharp with extremal function given by (18).

4 Neighborhood Properties

In this section, the neighborhood properties for the functions belonging to the class $\beta - S_g(p,m,\alpha)$ are determined.

Theorem 4.1 *Let* $f(z) \in T(p)$ *of the form (4) be in the class* $\beta - S_g(p,m,\alpha)$, *then for* $g(z) \in S(p)$ *of the form (2) and* $\zeta = \min_{k \ge 1} b_{p+k}$,
$$\beta - S_g(p,m,\alpha) \subset N_\delta(e),$$

where
$$\delta = \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)]\zeta}. \qquad (19)$$

Proof. Let $f(z) \in \beta - S_g(p,m,\alpha)$. Then from Corollary 2.3,

$$\sum_{k=1}^{\infty} (p+k) a_{p+k} \le \frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)]\zeta} = \delta, \ k \ge 1,$$

which directly proves that $f(z)\in N_{\delta}(e)$, where δ is given in (*19*). Hence the result.

4.1 Definition of Class $\beta-\mathbf{S}_{\mathbf{g}}^{(\gamma)}(\mathbf{p},\mathbf{m},\alpha)$

A function $f(z)\in T(p)$ is said to be in the class $\beta-S_{g}^{(\gamma)}(p,m,\alpha)$ if there exists a function $h(z)\in\beta-S_{g}(p,m,\alpha)$ such that $\left|\frac{f(z)}{h(z)}-1\right|\le p-\gamma.$

Theorem 4.2 *If* $h(z)\in\beta-S_{g}(p,m,\alpha)$ *with* $g(z)\in S(p)$ *of the form (2) and* $\zeta=\min_{k\ge1} b_{p+k}$, *then*

$$N_{\delta}^{q}(h)\subset\beta-S_{g}^{(\gamma)}(p,m,\alpha),$$

Where $$\gamma=p-\frac{\delta}{(p+1)^{q+1}}\times\frac{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta}{\{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta-(1-\alpha)(p-m)(p-m+1)\}}. \tag{20}$$

Proof. Let $f(z)\in T(p)$ of the form *(4)* be in $N_{\delta}^{q}(h)$, then from (*7*),

$$\sum_{k=1}^{\infty}\left|a_{p+k}-c_{p+k}\right|\le\frac{\delta}{(p+1)^{q+1}}. \tag{21}$$

Since $h(z)\in\beta-S_{g}(p,m,\alpha)$, then from Corollary 2.2,

$$\sum_{k=1}^{\infty}c_{p+k}\le\frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta}. \tag{22}$$

So that

$$\left|\frac{f(z)}{h(z)}-1\right|\le\frac{\sum_{k=1}^{\infty}\left|a_{p+k}-c_{p+k}\right|}{1-\sum_{k=1}^{\infty}c_{p+k}}\le\frac{\delta}{(p+1)^{q+1}\left[1-\frac{(1-\alpha)(p-m)(p-m+1)}{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta}\right]}$$

$$=\frac{\delta[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta}{(p+1)^{q+1}\{[(1+\beta)+(1-\alpha)(p-m)](p+1)\zeta-(1-\alpha)(p-m)(p-m+1)\}}=p-\gamma,$$

provided that γ is given by (*20*). Thus $f(z)\in\beta-S_{g}^{(\gamma)}(p,m,\alpha)$. This proves Theorem 4.2.

5 Extreme Points

Theorem 5.1 *Let* $f_{p}(z)=z^{p}$ *and*

$$f_{p+k}(z)=z^{p}-\frac{(1-\alpha)p!(p+k-m)!}{(p-m-1)![k(1+\beta)+(1-\alpha)(p-m)](p+k)!b_{p+k}}z^{p+k},\ k\ge1.$$

Then $f(z)\in\beta-S_g(p,m,\alpha)$ with $g(z)\in S(p)$ of the form *(2)* if and only if it can be expressed as:

$$f(z)=\lambda_p f_p(z)+\sum_{k=1}^{\infty}\lambda_{p+k} f_{p+k}(z),\ z\in\Delta,\text{ where } \lambda_{p+k}\geq 0 \text{ and } \lambda_p=1-\sum_{k=1}^{\infty}\lambda_{p+k}.$$

Proof. Let

$$f(z)=\lambda_p f_p(z)+\sum_{k=1}^{\infty}\lambda_{p+k} f_{p+k}(z)=\left[1-\sum_{k=1}^{\infty}\lambda_{p+k}\right]z^p+\sum_{k=1}^{\infty}\lambda_{p+k}\left\{z^p-\right.$$

$$\left.\frac{(1-\alpha)p!(p+k-m)!}{(p-m-1)![k(1+\beta)+(1-\alpha)(p-m)](p+k)!b_{p+k}}z^{p+k}\right\}$$

$$=z^p-\sum_{k=1}^{\infty}\frac{(1-\alpha)p!(p+k-m)!}{(p-m-1)![k(1+\beta)+(1-\alpha)(p-m)](p+k)!b_{p+k}}\lambda_{p+k}z^{p+k}.$$

Then from Corollary 2.2, $f(z)\in\beta-S_g(p,m,\alpha)$.

Conversely, let $f(z)\in\beta-S_g(p,m,\alpha)$, and setting

$$\lambda_{p+k}=\frac{(p-m-1)![k(1+\beta)+(1-\alpha)(p-m)](p+k)!b_{p+k}}{(1-\alpha)p!(p+k-m)!}a_{p+k},\ \ k=1,2,3....\text{and } \lambda_p=1-\sum_{k=1}^{\infty}\lambda_{p+k},$$

therefore $f(z)=z^p-\sum_{k=1}^{\infty}a_{p+k}z^{p+k}=z^p-\sum_{k=1}^{\infty}\lambda_{p+k}z^p+\sum_{k=1}^{\infty}\lambda_{p+k}z^p$

$$-\sum_{k=1}^{\infty}\lambda_{p+k}\frac{(1-\alpha)p!(p+k-m)!}{(p-m-1)![k(1+\beta)+(1-\alpha)(p-m)](p+k)!b_{p+k}}z^{p+k}$$

or

$$f(z)=\lambda_p f_p(z)+\sum_{k=1}^{\infty}\lambda_{p+k} f_{p+k}(z).$$

This completes the proof.

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