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Creeping flow past a porous sphere with solid pore embedded in porous medium

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ABSTRACT

The solution of the problem of slow flow of an incompressible viscous fluid past a porous sphere with solid pore embedded in another porous medium is investigated. The Brinkman equation for the flow inside and outside the porous sphere in their stream function formulation is used. Explicit expressions are investigated for stream function, shear stress and normal stress for inside and outside region. The drag force experienced by a porous sphere with solid pore embedded in a porous medium is also evaluated.

Introduction:

The problem of flow of fluids through porous medium is very important in ground water recharge, acquifers and oil technology. There has been considerable interest in the use of beds of porous particles for biological applications such as perfusion chromatography for purifying proteins and other biomolecules and cell or enzyme immobilization. Several researchers have been considered the flow past a body embedded in a porous medium using Darcy’s model. However, the darcy law appears to be inadequate for the flows with high porosity .to model such flows Brinkman[1] has suggested a modification to Darcy’s law.

Creeping flow past a solid sphere with a porous shell has been solved by Masliyah and Neale[2].In this paper they found the settling rates of a solid sphere with attached threads experimentally. Flow of a Newtonian fluid past an impervious sphere embedded in porous medium is studied by Bhupen Barman[3].They found that viscous sublayer increases with the increase of the permeability of the porous medium.Pop and Ingham[4] Investigate the flow past a sphere in a Porous medium. This paper shows that there is no flow separation for the flow. Flow past an axissymmetric body embedded in a saturated porous medium was investigated by Srinivasa Charya and Murthy[5]. In this paper they derived a general formula for the drag on the body .Deo and Gupta[6] studied the Drag on a Porous sphere embedded in another porous medium.In this paper they found the drag force experienced by a porous sphere embedded in another porous medium.

The present article deals the problem of slow flow of an incompressible viscous fluid past a porous sphere with solid pore embedded in another porous medium is investigated. The Brinkman equation for the flow inside and outside the porous sphere in their stream function formulation is used. Explicit expressions are investigated for stream function, shear stress and normal stress for inside and outside region. The drag force experienced by a porous sphere with solid pore embedded in a porous medium is also evaluated.

Mathematical formulation of the problem:

The configuration of a porous sphere enclosing a solid sphere in an unbounded porous medium is shown in the figure I. Let us consider the flow of an incompressible viscous fluid with a uniform velocity U directed in the positive z direction in a porous medium of permeability k_2 in which a porous sphere enclosing a solid sphere of different permeability k_1 is embedded. The inside and outside region of the sphere are fully saturated with the viscous fluid. We shall denote $i = 1$ in

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an entity for region I and $i = 2$ for outside region of porous sphere respectively. The governing Brinkman equation for both the region can be expressed as

$$\hat{\mu}_i \nabla^2 v^{(i)} - \frac{\mu_i}{k_i} v^{(i)} = \nabla p^{(i)}$$

$$\nabla^2 v^{(i)} - \sigma_i^2 v^{(i)} = \frac{1}{\hat{\mu}_i} \nabla p^{(i)}, \qquad i = 1,2 \qquad (1)$$

Here,

$$\sigma_i^2 = \frac{\mu_i a^2}{\hat{\mu}_i k_i} = \frac{\beta a^2}{k_i}, \qquad \beta = \frac{\mu_i}{\hat{\mu}_i}$$

where μ_i *and* $\hat{\mu}_i$ denotes the viscosities and effective viscosity of fluid and porous media respectively and k_i is the permeability in both regions. Since σ_i are dimensionless quantities related inversely with the permeability, we named σ_i as the dimensionless permeability parameter. In addition, the equation of continuity for incompressible fluids must be satisfied in both the regions:

$$div\, v^{(i)} = 0, \qquad i = 1,2 \qquad (2)$$

These equations of continuity for axisymmetric, incompressible viscous fluid in spherical polar coordinates (r, θ, φ) for both the regions can also be written as

$$\frac{\partial}{\partial r}\left[r^2 v_r^{(i)}\right] + \frac{r}{sin\theta}\frac{\partial}{\partial \theta}\left[v_\theta^{(i)} sin\theta\right] = 0$$

Where $v_r^{(i)}$ and $v_\theta^{(i)}$ are component of velocities in the direction of r and θ , respectively.

The stream function formulation of the above equation in spherical polar coordinates (r, θ, φ) reduces to the following fourth order partial differential equation

$$E^2\left(E^2\Psi^{(i)}\right) - \sigma^2\left(E^2\Psi^{(i)}\right) = 0 \qquad (3)$$

Where the operator

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\left(1-\zeta^2\right)}{r^2}\frac{\partial^2}{\partial \zeta^2}, \zeta = cos\theta$$

Furthermore, the non-vanishing velocity components $\left(v_r^{(i)}, v_\theta^{(i)}, 0\right), i = 1,2$ and tangential and normal stresses respectively are given by

$$v_r^{(i)} = -\frac{1}{r^2 sin\theta}\frac{\partial \Psi^{(i)}}{\partial \theta}; v_\theta^{(i)} = \frac{1}{r sin\theta}\frac{\partial \Psi^{(i)}}{\partial r}; \qquad (4)$$

$$T_{r\zeta}^{(i)}(r,\theta) = \frac{\mu}{r sin\theta}\left[\frac{\partial^2\Psi^{(i)}}{\partial r^2} - \frac{2}{r}\frac{\partial \Psi^{(i)}}{\partial r} - \frac{(1-\zeta^2)}{r^2}\frac{\partial^2\Psi^{(i)}}{\partial \zeta^2}\right], \qquad (5)$$

$$T_{rr}^{(i)}(r,\zeta) = -p^{(i)} - \frac{2\mu}{r^2}\left[\frac{2}{r}\frac{\partial \Psi^{(i)}}{\partial \zeta} - \frac{\partial^2\Psi^{(i)}}{\partial r \partial \zeta}\right] \qquad (6)$$

Also, the pressure may be obtained in both regions by integrating the following relations respectively:

$$\frac{\partial p^{(i)}}{\partial r} = -\frac{\mu}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left\{(E^2 - \sigma^2)\Psi^{(i)}\right\}; \qquad \frac{\partial p^{(i)}}{\partial \theta} = \frac{\mu}{sin\theta}\frac{\partial}{\partial r}\left\{(E^2 - \sigma^2)\Psi^{(i)}\right\} \qquad (7)$$

A particular regular solution on the symmetery axis z of Brinkman's equation that satisfies the requirement for the spherical case can be found as (Zlantanovski, 1999)

$$\Psi^{(i)}(r,\zeta) = \left[A_i r^2 + B_i \frac{1}{r} + C_i\sqrt{r}K_{\frac{3}{2}}(\sigma_i r) + D_i\sqrt{r}I_{\frac{3}{2}}(\sigma_i r)\right]G_2(\zeta), \qquad i = 1,2 \qquad (8)$$

Here $I_{\frac{3}{2}}(\sigma_i r)$ and $K_{\frac{3}{2}}(\sigma_i r)$ are the modified Bessel function of the first and second kind, respectively, as defined in Abramowitz and Stegun (1970),and A_i, B_i, C_i and D_i are constants of integration. Also $G_2(\zeta) = \frac{1-\zeta^2}{2}$, being the Gegenbauer function of first kind.

The solution given by equation (8) provides the flow for both the region with proper choice of constants. For the flow in region I the regular solution will be

$$\Psi^{(1)}(r,\zeta) = \left[A_1 r^2 + B_1 \frac{1}{r} + C_1\sqrt{r}K_{\frac{3}{2}}(\sigma_1 r) + D_1\sqrt{r}I_{\frac{3}{2}}(\sigma_1 r)\right]G_2(\zeta) \tag{9}$$

For the flow in the region II,the constant D_2 will be zero. Therefore the regular solution outside will be

$$\Psi^{(2)}(r,\zeta) = \left[A_2 r^2 + B_2 \frac{1}{r} + C_2\sqrt{r}K_{\frac{3}{2}}(\sigma_2 r)\right]G_2(\zeta) \tag{10}$$

Boundary Conditions:

In order to close the boundary-value problem for equation (1),it is necessary to specify boundary-conditions. the boundary condition that are physically realistic and mathematically consistent for this proposed problem can be taken as given subsequently.

On the surface of solid sphere:

No slip boundary condition i.e $v_r^{(1)} = 0$ and $v_\theta^{(1)} = 0$ (11)

On the surface of porous sphere:

The continuity of velocity component across the porous sphere implies that we may take

$$v_r^{(1)} = v_r^{(2)}, \quad v_\theta^{(1)} = v_\theta^{(2)} \tag{12}$$

Also, we assume that tangential and normal components of stresses are continuous across the surface, so that we take

$$T_{r\zeta}^{(1)}(r,\zeta) = T_{r\zeta}^{(2)}(r,\zeta), \qquad T_{rr}^{(1)}(r,\zeta) = T_{rr}^{(2)}(r,\zeta) \tag{13}$$

Since $\Psi^{(2)}(r,\zeta) = -\frac{1}{2}Ur^2\sin^2\theta = -Ur^2G_2(\zeta) \quad as\ r\to\infty$ (14)

for a uniform stream flowing with velocity U in the direction of the positive z axis.

Applying the preceding boundary condition (11)-(14), we get the following equations, respectively.

$$A_1 + B_1 + C_1K_{\frac{3}{2}}(\sigma_1) + D_1I_{\frac{3}{2}}(\sigma_1) = 0 \tag{15}$$

$$2A_1 - B_1 - C_1\left\{\sigma_1K_{\frac{1}{2}}(\sigma_1) + K_{\frac{3}{2}}(\sigma_1)\right\} + D_1\left\{\sigma_1I_{\frac{1}{2}}(\sigma_1) - I_{\frac{3}{2}}(\sigma_1)\right\} = 0 \tag{16}$$

$$A_1 + B_1 + C_1K_{\frac{3}{2}}(\sigma_1) + D_1I_{\frac{3}{2}}(\sigma_1) - A_2 - B_2 - C_2K_{\frac{3}{2}}(\sigma_2) = 0 \tag{17}$$

$$2A_1 - B_1 - C_1\left\{\sigma_1K_{\frac{1}{2}}(\sigma_1) + K_{\frac{3}{2}}(\sigma_1)\right\} + D_1\left\{\sigma_1I_{\frac{1}{2}}(\sigma_1) - I_{\frac{3}{2}}(\sigma_1)\right\} - 2A_2 + B_2 + C_2\left\{\sigma_2K_{\frac{1}{2}}(\sigma_2) + K_{\frac{3}{2}}(\sigma_2)\right\} = 0 \tag{18}$$

$$B_1(\sigma_1^2+12) - 2A_1\sigma_1^2 + 2C_1\{6K_{3/2}(\sigma_1) + 2\sigma_1K_{1/2}(\sigma_1)\} + 2D_1\{6I_{3/2}(\sigma_1) - 2\sigma_1I_{1/2}(\sigma_1)\} - \gamma^2[B_2(\sigma_1^2+12) - 2A_2\sigma_1 \tag{19}$$

$$6B_1 + C_1\{K_{3/2}(\sigma_1)(\sigma_1^2+6) + 2\sigma_1K_{1/2}(\sigma_1)\} + D_1\{I_{3/2}(\sigma_1)(\sigma_1^2+6) + 2\sigma_1I_{1/2}(\sigma_1)\} - \gamma^2[6B_2 + C_2\{K_{3/2}(\sigma_2)(\sigma_2^2+ \tag{20}$$

$$A_2 = 1 \tag{21}$$

where, $\gamma^2 = \frac{\widehat{\mu_2}}{\widehat{\mu_1}}$.

Solving the Equations (15)-(21), we get

$$A_1 = \frac{\gamma^2\{\sigma_2^2(1+\sigma_2)+\sigma_1^2(1+\sigma_2+\sigma_2^2)\}}{\sigma_1^2\sigma_2^2} \quad (22)$$

$$B_1 = -\frac{\gamma^2\{2\sigma_2^2(1+\sigma_2)+\sigma_1^2(1+\sigma_2+\sigma_2^2)\}}{\sigma_1^2\sigma_2^2} \quad (23)$$

$$C_1 = (3e^{\sigma_1}\sqrt{(2/\pi)}\,\gamma^2[\sigma_1\{\sigma_2^2(1+\sigma_2)+\sigma_1^2(1+\sigma_2+\sigma_2^2)\}\cosh(\sigma_1)+\{\sigma_2^2(1+\sigma_2)+\sigma_1^2(-1-\sigma_2+\sigma_2^2)\}\sinh(\sigma_1))/(\sigma_1^{(7} \quad (24)$$

$$D_1 = -(3\sqrt{(\pi/2)}\,\gamma^2[\{-\sigma_2^2(1+\sigma_2)-\sigma_1\sigma_2^2(1+\sigma_2)+\sigma_1^2(1+\sigma_2+\sigma_2^2)\}+\{\sigma_1^2(1+\sigma_2-\sigma_2^2)\}])/(\sigma_1^{(7/2)}\sigma_2^2\{\cosh(\sigma_1)+\sinh(\quad (25)$$

$$A_2 = 1 \quad (26)$$

$$B_2 = -\frac{3+3\sigma_2+\sigma_2^2}{\sigma_2^2} \quad (27)$$

$$C_2 = \frac{3e^{\sigma_2}\sqrt{\frac{2}{\pi}}}{\sqrt{\sigma_2}} \quad (28)$$

Thus all the coefficients have been determined, and hence we get the explicit expressions for the stream functions given by equations (9) and (10) in both the regions.

Evaluation of the drag force:

The drag force *experienced* by a porous sphere of radius a embedded in another porous medium can be evaluated by integrating the stresses over the porous sphere:

$$F = 2\pi a^2\int_0^\pi [T_{rr}^{(2)}\cos\theta - T_{r\theta}^{(2)}\sin\theta]_{r=a}\sin\theta\, d\theta \quad (29)$$

Since

$$T_{rr}^{(2)} = \frac{\hat{\mu}_2 U}{a} f(a)\cos\theta \quad (30)$$

$$T_{r\theta}^{(2)} = \frac{\hat{\mu}_2 U}{a} g(a)\sin\theta \quad (31)$$

Where

$$f(a) = \sigma_2^2 + \frac{1}{2(\sigma_2^2+12)B_2} + C_2\left\{4K_{\frac{3}{2}}(\sigma_2) - 2K'_{\frac{3}{2}}(\sigma_2)\right\} \quad (32)$$

$$g(a) = \frac{1}{2\left[6B_2 + C_2\left\{2K_{\frac{3}{2}}(\sigma_2) - 2K'_{\frac{3}{2}}(\sigma_2) + 2K''_{\frac{3}{2}}(\sigma_2)\right\}\right]} \quad (33)$$

Therefore inserting the values of equations (30) and (31) into equation (29), we have

$$F = 2\pi\hat{\mu}_2 aU\int_0^\pi [f(a)\cos^2\theta - g(a)\sin^2\theta]\sin\theta\, d\theta$$

$$= \frac{4}{3}\pi\hat{\mu}_2 aU[f(a) - 2g(a)]$$

$$= \frac{2}{3}\pi\hat{\mu}_2 aU\sigma_2^2\left[2 + B_2 - 2K_{\frac{3}{2}}(\sigma_2)C_2\right] \quad (34)$$

Where constant B_2 and C_2 are given by equation (27) and (28),respectively.

Also, the drag coefficient C_D can be defined as

$$C_D = \frac{F}{\frac{1}{2}\rho U^2 \pi a^2}$$

$$= \frac{\frac{2}{3}\pi\hat{\mu}_2 aU\sigma_2^2\left[2 + B_2 - 2K_{\frac{3}{2}}(\sigma_2)C_2\right]}{\frac{1}{2}\rho U^2 \pi a^2}$$

$$= \frac{\frac{8}{3}\sigma_2^2\left[2 + B_2 - 2K_{\frac{3}{2}}(\sigma_2)C_2\right]}{R_e}$$

Where $R_e = \frac{2aU}{\nu^{(2)}}$ and $\nu^{(2)} = \frac{\widehat{\mu_2}}{\rho}$ are the Reynolds number and kinematic viscosity of fluid, respectively.

Deduction of known result:

When $\sigma_2 \to \infty$, that is, $k_2 \to 0$, then the drag force F comes out as

$F = 6\pi\mu_2 Ua$

This is a well-known result for the drag force reported earlier by stokes(1851)for flow past a solid sphere in an unbounded fluid medium.

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