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Ternary semigroups in which prime ideals are maximal and primary ideals are prime and maximal

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ABSTRACT

In this paper, we study the structure of cancellative quasi commutative ternary semigroups. In fact we prove that if T is a cancellative quasi commutative ternary semigroup, then (1) T is a primary ternary semigroup, (2) proper prime ideals in T are maximal, (3) semiprimary ideals in T are primary, are equivalent. We obtain a characterization for semipseudo symmetric ternary semigroups with identity in which proper prime ideals are maximal and also we characterize semipseudo symmetric semigroups without identity in which proper prime idels are maximal and globally idempotent principal ideals from a chain. Further we characterize quasi commutative ternary semigroups containing cancellable elements in which proper prime ideals are maximal. Finally we study the ternary semigroups containing, identity with either one of the following properties. (1) Every nonzero primary ideal is prime as well as maximal, (2) every nonzero primary ideal is prime, (3) every nonzero ideal is prime.

1.Introduction

Anjaneyulu [1] made a study on primary ideals in semigroups. Later Anjanetulu [2] made a study on semigroups in which prime ideals are maximal. The study of ternary algebraic systems had been made by Lehmer [5], but earlier ternary structure was studied by Kerner [4] who give the idea of *n*-ary algebras. Aiyared Iampan [3] characteriae the relationship between the 0-minimal and maximal lateral ideals and the lateral 0-simple ternary semigroups. Los [6] studied some properties of ternary semigroup and proved that every ternary semigroup can be embedded in a semigroup. Shabir and Bashir [8] launched prime ideals in ternary semigroups. Sarala. Y, Anjaneyulu. A and Madhusudhana Rao. D [7] studied about globally idempotent ternary semigroups and proved that every maximal ideal of a globally idempotent ternary semigroup T is a prime ideal of T. In this paper we characterize quasi commutative ternary semigroup, semipseudo symmetric ternary semigroup and quasi commutative ternary semigroup containing cancellative elements, in which proper prime ideals are maximal and we characterize the ternary semigroup containing 0 and identity in which nonzero primary ideals are prime and maximal and also we study the ternary semigroup in which primary ideals are prime.

2. Preliminaries :

Definition 2.1 [7] : Let T be a non-empty set. Then T is said to be a ***Ternary semigroup*** if there exist a mapping from T×T×T to T which maps $(x_1, x_2, x_3) \to [x_1 x_2 x_3]$ satisfying the condition : $[(x_1x_2x_3)x_4x_5] = [x_1(x_2x_3x_4)x_5] = [x_1x_2(x_3x_4x_5)]$ $\forall\ x_i \in$ T, $1 \le i \le 5$.

Definition 2.2 [7] : A ternary semigroup T is said to be ***commutative*** provided for all $a,b,c \in$ T, we have $abc = bca = cab = bac = cba = acb$.

Definition 2.3 [7] : A ternary semigroup T is said to be ***quasi commutative*** provided for each $a, b, c \in$ T, there exists a natural number n such that abc = b^nac = bca = c^nba = cab = a^ncb.

Theorem 2.4 [7] **:** If T is a commutative ternary semigroup then T is a quasi commutative ternary semigroup.

Definition 2.5 [7] : An element a of a ternary semigroup T is said to be an ***identity*** provided $aat = taa = ata = t$ $\forall\ t \in$ T.

Example 2.6 [7] : Let Z_0^- be the set of all non-positive integers. Then with the usual ternary multiplication , Z_0^- forms a ternary semigroup with identity element -1.

Definition 2.7 [7] **:** An element a of a ternary semigroup T is said to be an ***idempotent*** element provided $a^3 = a$.

Definition 2.8 [7] **:** An element a of a ternary semigroup T is said to be a ***proper idempotent*** element provided a is an idempotent which is not the identity of T if identity exists.

Theorem 2.9 [7] **: An idempotent element *e* is an identity of a ternary semigroup T then it is unique.**

Notation 2.10 [7] : Let T be a ternary semigroup. If T has an identity, let T^1 = T and if T does not have an identity , let T^1 be the ternary semigroup T with an identity adjoined usually denoted by the symbol 1.

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Definition 2.11 [7] : An element a of a ternary semigroup T is said to be ***zero*** of T provided $abc = bac = bca = a \; \forall \; b,c \in$ T.

Notation 2.12 [7] **:** Let T be a ternary semigroup. if T has a zero, let T^0 = T and if T does not have a zero , let T^0 be the ternary semigroup T with zero adjoined usually denoted by the symbol 0.

Definition 2.13 [7] : A nonempty subset A of a ternary semigroup T is said to be ***ternary ideal*** or ***ideal*** of T if $b, c \in$ T, $a \in$ A implies $bca \in$ A, $bac \in$ A, $abc \in$ A.

Definition 2.12 [3] : A ternary semigroup T with zero is said to be 0-***simple*** provided it has no nonzero proper ideals and [TTT] $\neq \{0\}$.

Definition 4.13 : A ternary semigroup T is said to be an ***archimedian ternary semigroup*** provided for any $a, b \in$ T there exists an odd natural number n such that $a^n \in$ TbT.

Definition 2.14 [7]: An ideal A of a ternary semigroup T is said to be a ***maximal ideal*** provided A is a proper ideal of T and is not properly contained in any proper ideal of T.

Definition 2.15 [7] : An ideal A of a ternary semigroup T is said to be ***a principal ideal*** provided A is an ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by J (a) (or) < a >.

Definition 2.16 [7] : An ideal A of a ternary semigroup T is said to be a ***completely prime ideal*** of T provided $x, y, z \in$ T and $xyz \in$ A implies either $x \in$ A or $y \in$ A or $z \in$ A.

Definition 2.17 [7] : An ideal A of a ternary semigroup T is said to be a ***prime ideal*** of T provided X,Y,Z are ideals of T and XYZ $\subseteq$ A $\Rightarrow$ X $\subseteq$ A or Y $\subseteq$ A or Z $\subseteq$ A.

Definition 2.18 [7] : An ideal A of a ternary semigroup T is said to be a ***completely semiprime ideal*** provided $x \in$ T, $x^n \in$ A for some odd natural number n >1 implies $x \in$ A.

Definition 2.19 [7] : An ideal A of a ternary semigroup T is said to be ***semiprime ideal*** provided X is an ideal of T and $X^n \subseteq$ A for some odd natural number n implies X $\subseteq$ A.

Theorem 2.20 [7] **:** Every prime ideal of a ternary semigroup is semiprime.

Definition 2.21 [7] : If A is an ideal of a ternary semigroup T , then the intersection of all prime ideals of T containing A is called ***prime radical*** or simply ***radical*** of A and it is denoted by $\sqrt{A}$ or rad A.

Theorem 2.22 [7] **:** If A, B and C are any three ideals of a ternary semigroup T , then

i) A $\subseteq$ B $\Rightarrow \sqrt{A} \subseteq \sqrt{B}$

ii) if A ∩ B∩ C ≠ ∅ then $\sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$

iii) $\sqrt{\sqrt{A}} = \sqrt{A}$.

Theorem 2.23 [7] **: An ideal Q of ternary semigroup T is a Semiprime ideal of T if and only if** $\sqrt{Q}$ **= Q.**

Corollary 2.24 [7] **:** An ideal Q of a ternary semigroup T is a semiprime ideal if and only if Q is the intersection of all prime ideal of T contains Q.

Definition 2.25 [7] **:** An ideal A of a ternary semigroup T is said to be ***pseudo symmetric*** provided $x, y, z \in$ T, $xyz \in$ A implies $xsytz \in$ A for all $s, t \in$ T.

Definition 2.26 [7] **:** An ideal A in a ternary semigroup T is said to be ***semipseudo symmetric*** provided for any odd natural number n, $x \in$ T, $x^n \in$ A $\Rightarrow$ $< x >^n \subseteq$ A.

Theorem 2.27 [7] **:** Let A be a semipseudo symmetric ideal of a ternary semigroup T. Then the following are equivalent.

1) A_1=The intersection of all completely prime ideals of T containing A.

2) A_1^1 = The intersection of all minimal completely prime ideals of T containing A.

3) A_1^{11} = The minimal completely semiprime ideal of T relative to containing A.

4) A_2= {$x \in$ T : $x^n \subseteq$ A for some odd natural number n}

5) A_3= The intersection of all prime ideals of T containing A.

6) A_3^1= The intersection of all minimal prime ideals of T containing A.

7) A_3^{11} = The minimal semiprime ideal of T relative to containing A.

8) A_4 = {$x \in$ T : $<x>^n \subseteq$ A for some odd natural number n}.

Definition 2.28 [7] **:** A ternary semigroup T is said to be a ***semipseudo symmetric ternary semigroup*** provided every ideal of T is semipseudo symmetric.

Theorem 2.29 [7] **:** If T is a semipseudo symmetric ternary semigroup, then the following are equivalent.

1) T is a strongly archimedean semigroup.

2) T is an archimedean semigroup.

3) T has no proper completely prime ideals.

4) T has no proper completely semiprime ideals.

5) T has no proper prime ideals.

6) T has no proper semiprime ideals.

3. Ternary Semigroups In Which Prime Ideals Are Maximal:

Definition 3.1 : An ideal A of a ternary semigroup T is said to be a ***left primary ideal*** provided

(i) If X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and Y $\not\subseteq$ A, Z $\not\subseteq$ A then X $\subseteq$ √A.

(ii) √A is a prime ideal.

EXAMPLE 3.2 : Let T = { a, b, c, d } be a semigroup under the operation **.** given by

.	a	b	c	d
a	a	a	a	a
b	a	a	a	b
c	a	a	b	a
d	a	b	b	a

Define the ternary operation [ ] as $[xyz] = x(yz) = (xy)z$. Then (T, [ ]) is a ternary semigroup. Let A = {a, c}, B = {a, b}, C = {a, b, c} and D = {a, b, d }. Then A, B, C, D are all ideals of T. Now BAC $\subseteq$ D and A $\not\subseteq$ D, C $\not\subseteq$ D then B $\subseteq$ √D and √D is a prime ideal of T. Therefore D is a left primary ideal of T.

Definition 3.3 : An ideal A of a ternary semigroup T is said to be a ***lateral primary ideal*** provided

(i) If X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\not\subseteq$ A, Z $\not\subseteq$ A then Y $\subseteq$ √A.

(ii) √A is a prime ideal.

Example 3.4 : In the example 3.2, ABC $\subseteq$ D and A $\not\subseteq$ D, C $\not\subseteq$ D then B $\subseteq$ √D and √D is a prime ideal of T. Therefore D is a lateral primary ideal of ternary semigroup T.

Definition 3.5 : An ideal A of a semigroup T is said to be a ***right primary ideal*** provided

(i) If X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\nsubseteq$ A, Y $\nsubseteq$ A then Z $\subseteq$ √A.

(ii) √A is a prime ideal.

Example 3.6 : In the example 3.2, ACB $\subseteq$ D and A $\nsubseteq$ D, C $\nsubseteq$ D then B $\subseteq$ √D and √D is a prime ideal of T. Therefore D is a right primary ideal of ternary semigroup T.

Definition 3.7 : An ideal A of a ternary semigroup T is said to be a ***primary ideal*** provided A is a left primary ideal, a lateral primary ideal and a right primary ideal.

Example 3.8 : In example 3.2., the subset D is a primary ideal of ternary semigroup T.

Theorem 3.9 : An ideal A in a ternary semigroup T satisfies condition (i) of definition 3.1 iff $x, y, z \in$ T $< x > < y > <z> \subseteq$ A and $y, z \notin$ A, $x \subseteq$ √A.

Proof **:** Suppose that an ideal A of a ternary semigroup T satisfies the condition (i) of definition 3.1. Let $x, y, z \in$ T $\Rightarrow$ $< x > < y > < z > \subseteq$ XYZ and $y, z \notin$ A.

Since $y, z \notin$ A, $<y> \nsubseteq$ A, $< z > \nsubseteq$ A.

Then by assumption, $< x > < y > < z > \subseteq$ A and $<y> \nsubseteq$ A, $< z > \nsubseteq$ A $\Rightarrow$ $<x> \subseteq$ √A.

Therefore $x \in$ √A.

Conversely suppose that $x, y, z \in$ T, $< x > < y > < z > \in$ A and $y, z \notin$ A then $x \in$ √A.

Let X, Y, Z be three ideals of T such that XYZ $\subseteq$ A and Y $\nsubseteq$ A, Z $\nsubseteq$ A.

Suppose if possible X$\nsubseteq$ √A. Then there exists $x \in$ X such that $x \notin$ √A.

Since Y $\nsubseteq$ A, Z $\nsubseteq$ A, let $y \in$ Y, $z \in$ Z so that $y, z \notin$ A.

Now $< x > < y > < z > \subseteq$ XYZ $\subseteq$ A and $y, z \notin$ A $\Rightarrow x \in$ √A. It is a contradiction.

Therefore X $\subseteq$ √A. Therefore A satisfies the condition (i) of definition 3.1.

Theorem 3.10 : An ideal A in a ternary semigroup T satisfies condition (i) of definition 3.3, iff $< x > < y > < z > \subseteq$ A and $x, z \notin$ A $\Rightarrow$ $y \in$ √A.

Proof **:** The proof is similar to the proof of theorem 3.9.

Theorem 3.11: An ideal A in a ternary semigroup T satisfies condition (i) of definition 3.5, iff $< x > < y > < z > \subseteq$ A and $x, y \notin$ A $\Rightarrow$ $z \in$ √A.

Proof **:** The proof is similar to the proof of theorem 3.9.

Theorem 3.12 : Let S be a commutative semigroup and A be an ideal of S. Then the following conditions are equivalent.

1. A is primary ideal.
2. X, Y are two ideals of S, XYZ $\subseteq$ A and Y $\nsubseteq$ A, Z $\nsubseteq$ A then X $\subseteq$ √A.
3. ***x, y, z*** $\in$ **T, *xyz*** $\in$ **A, *y, z*** $\notin$ **A then *x*** $\in$ **√A.**

Proof **:** (1)$\Rightarrow$(2) : Suppose that A is a primary ideal. Then A is a left primary ideal.

So by definition 3.1, we get X,Y, Z are three ideals of T, XYZ$\subseteq$A, Y$\nsubseteq$A, Z $\nsubseteq$ A

$$\Rightarrow X \subseteq \sqrt{A}$$

(2) $\Rightarrow$(3): Suppose that X, Y, Z are three ideals of T, XYZ$\subseteq$A, Y$\nsubseteq$A, Z $\nsubseteq$ A $\Rightarrow$ X$\subseteq \sqrt{A}$.

Let $x, y, z \in$ T, $xyz \in$ A and $y, z \notin$ A. $xyz \in$ A$\Rightarrow$$< xyz > \subseteq$ A$\Rightarrow$$<x> <y> < z > \subseteq$A.

Also $y, z \notin$A$\Rightarrow$ $<y> \nsubseteq$ A and $< z > \nsubseteq$ A.

Now $<x><y> < z > \subseteq$ A and $<y>\nsubseteq$A, $< z > \nsubseteq$ A.

Therefore by assumption $<x> \subseteq \sqrt{A} \Rightarrow x \in$ √A.

(3) $\Rightarrow$(1) : Suppose that $x, y, z \in$ T, $xyz \in$ A and $y, z \notin$ A $\Rightarrow$ $x \in$ √A.

Let X, Y, Z be three ideals of T such that XYZ$\subseteq$A and Y $\nsubseteq$ A, Z $\nsubseteq$ A.

Y $\nsubseteq$ A, Z $\nsubseteq$ A $\Rightarrow$there exists $y \in$Y and $z \in$ Z such that $y, z \notin$ A.

Suppose if possible X $\nsubseteq \sqrt{A}$. Then there exists $x \in$X such that $x \notin$√A.

Now $xyz \in$ XYZ $\subseteq$ A.

Therefore $xyz \in$ A and $y, z \notin$ A, $x \notin$√A. It is a contradiction.

Therefore X$\subseteq \sqrt{A}$.

Let $x, y, z \in$T and $xyz \in$ √A. Suppose that $y, z \notin$√A.

Now $xyz \in$ √A$\Rightarrow$$(xyz)^m \in$ A$\Rightarrow$$x^m y^m z^m \in$ A for some odd natural number m.

Since $y, z \notin$√A, $y^m, z^m \notin$ A. Now $x^m y^m z^m \in$A, $y^m, z^m \notin$ A $\Rightarrow$ $x^m \in$ √A

$\Rightarrow x \in \sqrt{\sqrt{A}} = \sqrt{A}$. $\sqrt{A}$ is a completely prime ideal and hence $\sqrt{A}$ is a prime ideal. Therefore A is a left primary ideal. Similarly A is a lateral primary ideal and right primary ideal. Hence A is a primary ideal.

Note 3.13 : In an arbitrary ternary semigroup a left primary ideal is not necessarily a lateral primary ideal or a right primary ideal.

Example 3.14 : in example 3.2, D is a left primary ideal but neither lateral primary ideal nor right primary ideal.

Theorem 3.15 : Let T be a ternary semigroup with identity and let M be the unique maximal ideal in T. If √A = M for some ideal A in T, then A is a primary ideal.

Proof **:** Let $<x> <y> < z > \subseteq$A and $y, z \notin$ A. If $x \notin$ √A then $< x > \nsubseteq \sqrt{A}$ = M. Since M is the union of all proper ideals in T, we have $< x > =$ T $\Rightarrow$ $y, z \in < x >$ and hence $< y > = <x><y>< z > \subseteq$A. It is a contradiction. Therefore $x \in \sqrt{A}$. Clearly $\sqrt{A}$ = M is a prime ideal. Thus A is left primary. By symmetry it follows that A is lateral primary and right primary. Therefore A is a primary ideal.

Note 3.16 : If T has no identity, then the theorem 3.23 is not true, even if the ternary semigroup has a unique maximal ideal. In example 3.13, $\sqrt{< a >}$ = M where M = {a, b} is the unique maximal ideal. But $< a >$ is not a primary ideal.

Theorem 3.17 : If A is a semiprime ideal of a ternary semigroup T, then the following are equivalent.

1. A is a prime ideal.

2. A is a primary ideal.
3. A is a left primary ideal.
4. A is a lateral primary ideal.
5. A is a right primary ideal.
6. A is a semiprimary ideal.

Proof **:** (1) $\Rightarrow$(2) $\Rightarrow$(3) $\Rightarrow$(4) $\Rightarrow$ (5) and (2) $\Rightarrow$(3) $\Rightarrow$(4) $\Rightarrow$ (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$(1) : Suppose that A is a semiprimary ideal. Then $\sqrt{A}$ is a prime ideal. Since A is semiprime, A is the intersection of all prime ideals of T containing A. Therefore A = $\sqrt{A}$ is a prime ideal.

Theorem 3.18 : A ternary semigroup T is semiprimary iff prime ideals of T form a chain under set inclusion.

Proof **:** Suppose that T is a semiprimary semigroup. Let A, B and C be three prime ideals of T. Now $\sqrt{AI\ BI\ C}$ = $\sqrt{A}\cap\sqrt{B}\cap\sqrt{C}$ = $A\cap B\cap C$. Therefore $A\cap B\cap C$ is semiprime. By theorem 3.17, since T is a semiprimary semigroup it follows that $A\cap B\cap C$ is prime. Suppose that A $\nsubseteq$ B, A $\nsubseteq$ C, B $\nsubseteq$ A, B $\nsubseteq$ C and C $\nsubseteq$ A, C $\nsubseteq$ B.

Then there exists $x \in$ A\B, $x \in$ A\C, $y \in$ B\A, $y \in$ B\C and $z \in$ C\A, $z \in$ C\B.

Now < x > < y > < z > $\subseteq$ $A\cap B\cap C$ and $x, y, z \notin A\cap B\cap C$.

It is a contradiction. Therefore prime ideals of T form a chain.

Conversely suppose that prime ideals of T form a chain under set inclusion. For every ideal A, $\sqrt{A}$ = $\cap P_\alpha$, where intersection is over all prime ideals P_α containing A yields $\sqrt{A}$ = P_α for some α, so that A is a semiprimary ideal. Therefore T is a semiprimary semigroup.

Definition 3.19 : A ternary semigroup T is said to be ***left cancellative*** if for all $a, b, x, y \in$ T, $abx = aby \Rightarrow x = y$.

Definition 3.20 : A ternary semigroup T is said to be ***laterally cancellative*** if for all $a, b, x, y \in$ T, $axb = ayb \Rightarrow x = y$.

Definition 3.21 : A ternary semigroup T is said to be ***right cancellative*** if for all $a, b, x, y \in$ T, $xab = yab \Rightarrow x = y$.

Definition 3.22 : A ternary semigroup T is said to be ***cancellative*** if T is left cancellative, right cancellative and laterally cancellative.

Definition 3.23 : An element a of a ternary semigroup T is said to be ***invertible*** in T if there exists an element b in T such that $abx = bax = xab = xba = x$ for all $x \in$ T.

Definition 3.24 : A ternary semigroup T is said to be a ***ternary group*** if for $a, b, c \in$ T, the equations $abx = c$, $axb = c$ and $xab = c$ have solutions in T.

Theorem 3.25 : Let T be a ternary semigroup with identity. If (non-zero, assume this T has zero) proper prime ideals in T are maximal, then T is a primary ternary semigroup.

Proof **:** Since T contains identity, T has a unique maximal ideal M, which is the union of all proper ideals in T. If A is a (nonzero) proper ideal in T, then √A = M and hence by theorem 3.15, A is a primary ideal. If T has zero and if < *0* > is a prime ideal, then < *0* > is primary and hence T is primary. If < *0* > is not a prime ideal, then √< *0* > = M and hence by theorem 3.15, < *0* > is a primary ideal. Therefore T is a primary ternary semigroup.

Note 3.26 : If the ternary semigroup T has no identity, then from example 3.14, we remark that theorem 3.25, is not true even if the ternary semigroup has a unique maximal ideal. The converse of the theorem 3.25, is not true even if the semigroup is commutative.

Example 3.27 : Let T = {a, b, 1} be the ternary semigroup under the multiplication given in the following table.

Now T is a primary ternary semigroup in which the prime ideal < a > is not a maximal ideal.

Theorem 3.28 : Let T be a right cancellative quasi commutative ternary semigroup. If T is a primary ternary semigroup or a ternary semigroup in which semiprimary ideals are primary, then for any primary ideal Q, √Q is non maximal implies Q = √Q is prime.

Proof **:** Since √Q is non maximal, there exists an ideal A in T such that √Q ⊂ A ⊂ T. Let $a \in$ A\√Q and $b, c \in$ √Q. Now Q $\subseteq$ Q ∪ < abc > $\subseteq$ √Q. This implies by theorem 2.48, √Q $\subseteq$ √(Q ∪ < abc >) $\subseteq$ √(√Q) = √Q. Hence √(Q ∪ < abc >) = √Q. Thus by hypothesis Q ∪ < abc > is a primary ideal. Let $s, t \in$ T\A. Then for some natural number n, $asbtc = s^n abtc = s^n abct \in$ Q ∪ < abc >. Since $a \notin$ √Q = √(Q ∪ < abc >) and Q ∪ < abc > is a primary ideal, $sbtc \in$ Q ∪ < abc >. If $sbtc \in$ < abc > then $sbtc = rabtc$ for some $r \in T^1$ and hence by right cancellative property, we have $s = ra \in$ A, a contradiction. Thus $sbtc \in$ Q, which implies, since $s \notin \in$, $btc \in$ Q and hence √Q = Q. Therefore Q = √Q and so Q is prime.

Theorem 3.29 : Let T be a right cancellative quasi commutative ternary semigroup. If T is either a primary ternary semigroup of a ternary semigroup in which semiprimary ideals are primary, then proper prime ideals in T are maximal.

Proof **:** First we show that if P is a minimal prime ideal containing a principal ideal < d >, then P is a maximal ideal. Suppose P is not a maximal ideal.

Write M = T\P and A = {$x \in$ T : $xmn \in$ < d > for some $m, n \in$ M}.

Let $x \in$ A, $s, t \in$ T. $x \in$ A $\Rightarrow$ $xmn \in$ < d > $\Rightarrow$ $xmn = s_1 d t_1$ for some $s_1, t_1 \in$ T.

Now $stxmn = st(xmn) = st(s_1 d t_1) = (sts_1)dt_1 \in$ < d > $\in$ $stx \in$ A, similarly $sxt \in$ A and $xst \in$ A. Therefor A is an ideal of T.

If $x \in$ A, then $xmn \in$ < d > $\subseteq$ P . Since P is prime ideal and hence $x \in$ P. So A $\subseteq$ P.

Let $b \in$ P and suppose N = {$b^k mn$: $m, n \in$ M and k is a nonnegative odd interger}.

If $b^k mn$, $b^s pq$, $b^r uv \in$ N for $m, n, p, q, u, v \in$ M and $k, s,$ and r are nonnegative odd integers. Then $(b^k mn)(b^s pq)(b^r uv) = b^{k+s+r} mnpquv \in$ N.

Therefore N is a ternary subsemigroup of T containing M properly.

If $b \in$ P $\Rightarrow$ $bmn \in$ P $\Rightarrow$ $bmn \notin$ M and hence $bmn \in$ N and $bmn \notin$ M.

Since P is a minimal prime ideal containing < d >, M is a maximal ternary subsemigroup not meeting < d >. Since N contains M properly, we have N ∩ < d > ≠ ∅.

So there exist a odd natural number k such that $b^k mn \in \langle d \rangle \Rightarrow b^k \in A \Rightarrow b \in \sqrt{A}$.

Since P is prime, by theorem 2.19, P is semiprime and by theorem 2.23, $P = \sqrt{P}$.

Therefore $P \subseteq \sqrt{A} \Rightarrow P \subseteq \sqrt{A} \subseteq \sqrt{P} = P$. So $P = \sqrt{A}$.

By hypothesis A is a primary ideal. Since P is not a maximal ideal, we have by theorem 3.28, $\sqrt{A} = A \subseteq P = A$. Since $\langle d \rangle \subseteq P$ and $\langle d^3 \rangle \subseteq \langle d \rangle$.

Therefore $\langle d^3 \rangle \subseteq P$ and hence P is also a minimal prime ideal containing $\langle d^3 \rangle$.

Let $B = \{ y \in T : ymn \in \langle d^3 \rangle$ for some $m, n \in M\}$. As before , we have $B = P$.

Since $d \in P = A = B$, we have $dmn = std^3$ for some $s, t \in T^1$.

Since T is a quasi commutative ternary semigroup, $dmn = m^p nd = std^3$ for some natural number p. By right cancellative property $m^p n = std^2$, a contradiction.

Therefore P is maximal ideal. Now if P is any proper prime ideal, then for any $d \in P$, $\langle d \rangle$ is contained in a minimal prime ideal, which is maximal by the above and hence P is a maximal ideal.

Corollary 3.30 : If T is a cancellative commutative ternary semigroup such that either T is a primary ternary semigroup or in T an ideal A is primary if and only if $\sqrt{A}$ is a prime ideal, then the proper prime ideals in T are maximal.

Proof **:** The proof of this corollary is a direct consequence of theorem 3.29.

Theorem 3.31 : Let T be a right cancellative quasi commutative ternary semigroup with identity. Then the following are equivalent.

1) Proper prime ideals in T are maximal.
2) T is a primary ternary semigroup.
3) Semiprimary ideals in T are primary.
4) If x, y and z are not units in T, then there exists natural numbers n, m and p such that $x^n = yzs$, $y^m = xzt$ and $z^p = xyu$ for some $s, t, u \in T$.

Proof **:** Combining theorem 3.25, and 3.29, we have (1), (2) and (3) are equivalent.

1) $\Rightarrow$ 4) : Assume (1). Since T contains identity, T has a unique maximal ideal M, which is the only prime ideal in T. If x, y and z are not units.

If $\langle x \rangle \nsubseteq M$ then $\langle x \rangle = T \Rightarrow 1 \in \langle x \rangle \Rightarrow x$ is a unit, a contradiction and hence $x \in M$, similarly $y, z \in M$. Therefore $\sqrt{\langle x \rangle} = \sqrt{\langle y \rangle} = \sqrt{\langle z \rangle} = M$

$\Rightarrow y, z \in \sqrt{\langle x \rangle}$, $x, z \in \sqrt{\langle y \rangle}$ and $x, y \in \sqrt{\langle z \rangle} \Rightarrow x^n = yzs$, $y^m = xzu$ and $z^p = xyu$ for some $s, t, u \in T$.

4) $\Rightarrow$ 2) : Let A be any ideal in T and $xyz \in A$. Suppose that x, y, z are not units in T.

Let $y, z \notin A$, then $x^n = yzs \Rightarrow x^{n+2} = xxyzs \in A$. Therefore $x \in \sqrt{A}$.

Therefore A is left primary. Similarly A is lateral primary and right primary.

Therefore T is primary ternary semigroup.

Note 3.32 : If T has 0, then the theorem 3.31, is true by assuming nonzero proper prime ideals are maximal.

Theorem 3.33 : Let T be a right cancellative quasi commutative ternary semigroup not containing identity. Then the following are equivalent.

1) T is a primary ternary semigroup
2) Semiprimary ideals in T are primary
3) T has no proper prime ideals.
4) If $x, y \in T$, then there exists natural numbers n, m and p such that $x^n = yzs$, $y^m = xzt$ and $z^p = xyu$ for some $s, t, u \in T$.

Proof **:** (1) $\Rightarrow$ (2) Since T is primary ternary semigroup, then its every ideal is primary. Therefore semiprimary ideal is also primary.

(2) $\Rightarrow$ (3) : Assume (2). By theorem 3.29, proper prime ideals of T are maximal and hence if P is any prime ideal, then P is maximal. Let $a, b, c \in T \backslash P$. Suppose $abc \notin T \backslash P \Rightarrow abc \in P \Rightarrow$ either $a \in P$ or $b \in P$ or $c \in P$, a contradiction. Therefore $abc \in T \backslash P$. Clearly $T \backslash P$ satisfies associative property. Therefore $T \backslash P$ is ternary semigroup. Let $a, b \in T \backslash P$. Then $aaT \nsubseteq P$ and hence $P \cup aaT = T \Rightarrow b \in aaT \Rightarrow b = aax$ for some $x \in T$. If $x \in P$, then $b \in P$, a contradiction. Therefore $aax = b$ has a solution in $T \backslash P$. Similarly $yaa = b$ has a solution in $T \backslash P$ and hence $T \backslash P$ is a ternary group. Let e be the identity of the group $T \backslash P$. Now e is an idempotent in T and since S is a right cancellative ternary semigroup, then e is a left identity and lateral identity of T. Since T is a quasi commutative ternary semigroup, idempotents in T are commute and hence e is the identity of T, a contradiction, since T has no identity. Therefore T has no proper prime ideals.

3) $\Rightarrow$ 4) : Suppose T has no proper prime ideals. Then for any ideal A of T, $\sqrt{A} = T$.

Let $x, y, z \in T$. Now $\sqrt{\langle x \rangle} = \sqrt{\langle y \rangle} = \sqrt{\langle z \rangle} = T \Rightarrow y, z \in \sqrt{\langle x \rangle}$, $x, z \in \sqrt{\langle y \rangle}$ and $x, y \in \sqrt{\langle z \rangle} \Rightarrow y^m, z^p \in \langle x \rangle, x^n, z^p \in \langle y \rangle$ and $x^n, y^m \in \langle z \rangle$ for some odd natural numbers n, m, p $\Rightarrow x^n = yzs$, $y^m = xzu$ and $z^p = xyu$ for some $s, t, u \in T$.

4) $\Rightarrow$ 1) : Let A be any ideal of T. Let $xyz \in A$, Suppose that x, y, z are not units in T, then $x^n = yzs \Rightarrow x^{n+2} = xxyzs \in A \Rightarrow x \in \sqrt{A}$. Therefore A is left primary. Since T is quasi commutative ternary semigroup and hence A is lateral primary and right primary. Therefore A is primary and hence T is a primary ternary semigroup. This completes the proof of the theorem.

Theorem 3.34 : Let T be a right cancellative quasi commutative ternary semigroup. Then the following are equivalent.

1) T is a primary ternary semigroup.
2) Semiprimary ideals in T are primary.
3) Proper prime ideals in T are maximal.

Proof **:** The proof of this theorem is a direct consequence of theorem 3.32, and 3.33.

Corollary 3.35 : Let T be a cancellative commutative ternary semigroup. Then T is a primary ternary semigroup if and only if proper prime ideals in T are maximal. Furthermore T has no idempotents except identity, if it exists.

Proof **:** The proof of this corollary is a direct consequence of theorem 3.34.

Theorem 3.36 : Let T be a semipseudo symmetric ternary semigroup with identity. Then the following are equivalent.

1) Proper prime ideals in T are maximal.
2) T is either a simple ternary semigroup and so archimedian ternary semigroup or T has a unique prime ideal P such that T is a 0-simple extension of the archimedian ternary subsemigroup P. In either case T is a primary ternary semigroup and T has at most one globally idempotent principal ideal.

Proof **:** (1) $\Rightarrow$ (2) : Suppose proper prime ideals in T are maximal. If T is a simple ternary semigroup, then by theorem… T is an Archimedean ternary semigroup. If T is not a simple ternary semigroup, then T has a unique maximal ideal P, which is also the unique prime ideal. Since P is a maximal ideal in T, we have $T/P = T\backslash P \cup \{P\}$ is a 0-simple ternary semigroup. Let $a, b, c \in P$. Since P is the any prime ideal, then its intersection is also prime and hence $\sqrt{<a>} = \sqrt{<b>} = \sqrt{<c>} = P$. So by theorem …., $<a>^n \subseteq <b>$ for some odd natural number n. This implies $a^{n+2} \in PbP$. So P is an Archimedean ternary subsemigroup of T.

(2) $\Rightarrow$ (1) : Assume 2), Case-1 : Suppose T is simple. Therefore T has no proper prime ideals and hence there exist no proper ideal of T containing P $\Rightarrow$ P is maximal. Therefore 1) is true.

Case-2 : Suppose T is not simple. Then T has unique proper prime ideal P such that T is a 0-simple extension of P. Therefore T/P is 0-simple.

By theorem 3.25, T is a primary semigroup. Suppose $<a>$, $<b>$ and $<c>$ be three proper globally idempotent principal ideals. Then $\sqrt{<a>} = \sqrt{<b>} = \sqrt{<c>} = P$. So by theorem… $<a>^n = <b>$ for some natural number n. Since $<a>$ is globally idempotent, $<a> \subseteq <b>$. Similarly we can show that $<b> \subseteq <a>$. Therefore $<a> = <b>$.

Similarly we can show that $<b> = <c>$ and hence $<a> = <b> = <c>$.

Theorem 3.37: Let T be a semipseudo symmetric ternary semigroup without identity. Then the following are equivalent.

1) Proper prime ideals in T are maximal and globally idempotent principal ideals form a chain.

2) T is an archimedian ternary semigroup or there exists a unique prime ideal P in T and T is 0-simple extension of the archimedian ternary subsemigroup P.

3) Proper prime ideals in T are maximal and T has atmost two distinct globally idempotent principal ideals with one of its radical is T itself.

***Proof* : (**1) $\Rightarrow$ (2) : If T has no proper prime ideals, then by theorem 2.29, T is an Archimedean ternary semigroup. Suppose T has proper prime ideals. Let M and N be two proper prime ideals in T. By assumption M and N are maximal ideals in T and if $a \in T\backslash N \in a \notin M \in a^3 \notin M \in M \cup <a^3> = T \Rightarrow a \in <a^3> \Rightarrow a$ is semisimple and hence every element in $T\backslash N$ is semisimple. Similarly every element in $T\backslash M$ is semisimple. Let $a \in T\backslash M$ and $b \in T\backslash N$. Now a and b are semisimple elements and hence $<a>$ and $<b>$ are globally idempotent principal ideals. By hypothesis either $<a> \subseteq <b>$ are $<b> \subseteq <a>$. Suppose $<a> \subseteq <b>$. If $b \in M$, then $a \in M$, a contradiction. So $b \in T\backslash M$ and $a, b \in T\backslash M \Rightarrow M \cup <a> = M \cup <b> = T \Rightarrow <a> = <b>$. Similarly we can show that if $<b> \subseteq <a>$, then also $<a> = <b>$. From this we can conclude that $T\backslash M = T\backslash N$ and hence $M = N$. Thus T has a unique prime ideal. By an argument similar to theorem 3.36, We can prove that P is an Archimedean ternary subsemigroup of T.

(2) $\Rightarrow$ (3) : If T is an Archimedean ternary semigroup, then clearly by theorem 2.29, T has no proper prime ideals. Let $<a>$ and $<b>$ be two globally idempotent principal ideals. Now since T has no proper prime ideals, we have $\sqrt{<a>} = \sqrt{<b>} = T$. By theorem 2.27, $<a>^n \subseteq <b>$ and $<b>^m \subseteq <a>$ for some odd natural number n and m. Thus we have $<a> \subseteq <b>$ and $<b> \subseteq <a>$. So $<a> = <b>$. Suppose T has a unique prime ideal P such that T is a 0-simple extension of the Archimedean ternary subsemigroup P. Since T/P is a 0-simple ternary semigroup, we have P is a maximal ideal. Now for every $a, b \in T\backslash P$, we have $<a> = <b>$ and $\sqrt{<a>} = \sqrt{<b>} = T$. Let $<a>$ and $<b>$ be two globally idempotent principal ideals and $a, b \in P$. Now $\sqrt{<a>} = \sqrt{<b>} = P$ and hence $<a> = <b>$. Thus T has at most two proper globally idempotent principal ideals one of it's radical is T itself.

(3) $\Rightarrow$ (1) : Let $<a>$, $<b>$ are two globally idempotent principal ideals in T. Let $\sqrt{<b>} = T \Rightarrow a \in \sqrt{<b>} \Rightarrow <a> \subseteq \sqrt{<b>} \Rightarrow <a>^n \subseteq <b> \Rightarrow <a> \subseteq <b>$. Therefore globally idempotent principal ideals form a chain.

Theorem 3.39: Let T be a semipseudo symmetric ternary semigroup with $T \neq T^3$. Then T is a primary ternary semigroup in which proper prime ideals in T are maximal if and only if T is an Archimedean ternary semigroup.

***Proof* :** Let T be an Archimedean ternary semigroup. Then by theorem 2.29, T has no proper prime ideals. Hence it is trivially true that proper prime ideals are maximal. Let A be any ideal in T such that $<x><y><z>$ and $y, z \notin A$. Since T is an Archimedean ternary semigroup there exists a odd natural number n such that $x^n \in TyTzT \Rightarrow x^n \in <y><z>$ and $x^2 \in <x> \Rightarrow x^{n+2} \in <x><y><z>$. Now $x^{n+2} \in <x><y><z> \subseteq A$. So by theorem 2.27, $x \in \sqrt{A}$. Thus A is left primary. Similarly we can show that A is lateral primary as well as right primary. Therefore T is a primary ternary semigroup in which proper prime ideals are maximal.

Conversely suppose that T is a primary ternary semigroup in which proper prime ideals are maximal. Now T is a semiprimary ternary semigroup and hence by theorem 3.18, prime ideal in T form a chain. Let P and Q be two proper prime ideals. Therefore P, Q are maximal. Since prime ideals form a chain, then $P \subseteq Q$ or $Q \subseteq P$. Therefore $P = Q$ and hence T has a unique proper prime ideal which is also the unique maximal ideal. Now every element of $T\backslash P$ is semisimple and hence $T\backslash P \subseteq T^3$. Let $a, b \in T\backslash P$ and $x \in P$.

If $<a><b><x> \neq <x>$, then since T is a primary ternary semigroup and $x \notin <a><b><x>$, since prime ideal is unique, we have $a \in \sqrt{(<a><b><x>)} = P$, a contradiction. So $<a><b><x> = <x>$ for all $x \in P$ and hence by theorem 2.29, T is an Archimedean ternary semigroup.

4. Ternary semigroups in which primary ideals are prime and maximal

Theorem 4.1 : Let T be a ternary semigroup containing 0 and identity with the maximal ideal M. Then every nonzero primary ideal is prime as well as maximal if and only if T/M is a 0-simple ternary semigroup with either

(1) $M = (T\backslash M)\, a\, (T\backslash M)\, b\, (T\backslash M) \cup \{0\}$, $a, b \in M$ and $<a>^3 = 0$, $<b>^3 = 0$ or

(2) M is a 0-simple ternary semigroup.

***Proof* :** Suppose every nonzero primary ideal is prime and maximal. Since nonzero prime ideals are maximal, by theorem 3.23, T is a primary ternary semigroup. If $<0>$ is the maximal ideal of T, then the proof of this theorem is trivial. Suppose T has nonzero maximal ideal M. Since T is a primary ternary semigroup and every nonzero primary ideal is maximal, we have M is the only nonzero proper ideal in T.

Since M is a maximal ideal, T/M is a 0-simple ternary semigroup.

Now for every nonzero $a, b \in M$, $<a> = M$ and $<b> = M$.

Since M^3 is an ideal contained in M, either $M^3 = 0$ or $M^3 = M$.

If $M^3 = 0$, then for all $a, b, c \in M$, $<a><b><c> = 0$ and $<a>^3 = 0$, $<b>^3 = 0$ for all $a, b \in M$.

Since for all nonzero a, b, $c \in$ M, < a > = < b > = < c > = M, we have $c = fagbh$ for some f, g, $h \in$ T. If f or g or $h \in$ M, then by the above $c = 0$, a contradiction. So f, g, $h \in$ T\M.

Therefore (T\M) a (T\M) b (T\M) $\cup$ {0}, a, $b \in$ M and $< a >^3 = 0$, $< b >^3 = 0$.

If M^3 = M, then for every nonzero a, $b \in$ M, we have MaMbM = MTaTMTbTM = M^5 = M.

Therefore M is a 0-simple ternary semigroup.

Conversely if T/M is a 0-simple ternary semigroup with either M = (T\M) a (T\M) b (T\M) such that a, $b \in$ M and $< a >^3 = 0$, $< b >^3 = 0$ or M is a 0-simple ternary semigroup, then clearly M = < 0 > and T has no other ideals, or if M = < 0 >. Since T/M is 0-simple the < 0 > is maximal ideal of T.

Therefore T has no other nonzero ideals.

Suppose M $\neq$ < 0 >. Since T/M is 0-simple then M is a maximal ideal of T.

Let A is any nonzero proper ideal. Therefore A $\subseteq$ M. $a \in$ A $\Rightarrow a \in$ M $\Rightarrow$ < a > $\subseteq$ M.

M = (T\M) a (T\M) $\subseteq$ TaT $\subseteq$ < a > $\Rightarrow$ M $\subseteq$ < a >. Therefore < a > = M and hence M is the only nonzero ideal in T. Thus we have the conclusion.

Corollary 4.2: Let T be a ternary semigroup containing identity and not containing 0. Then every primary ideal is prime as well as maximal if and only if T is either a simple ternary semigroup or a 0-simple extension of a simple ternary semigroup.

***Proof* :** If T does not contain 0, then the case $M^3 = 0$ in the theorem 4.1, does not arise. Therefore the proof of this corollary is a direct consequence of theorem 4.1.

Theorem 4.3: Let T be a ternary semigroup containing 0 and identity with the maximal ideal M. Suppose that every nonzero primary ideal is prime. Then T/M is a 0-simple ternary semigroup such that either

1) M = (T\M) a (T\M) a (T\M) $\cup$ {0}, $a \in$ M and $< a >^3 = 0$ or
2) M^n = M for every odd natural number n.

***Proof* :** Suppose every non zero primary ideal is prime. If $M^3 = 0$. Let P be the any prime ideal. If $0 \in$ P $\Rightarrow M^3 \subseteq$ P $\Rightarrow$ M $\subseteq$ P. Clearly P $\subseteq$ M. Therefore M = P and hence M is the unique prime ideal in T. Now $\sqrt{< a >}$ = M for every nonzero $a \in$ M and thus < a > is primary by theorem 3.15, Then by hypothesis < a > is prime and hence < a > = M. Therefore the conclusion follows as in the proof of theorem 4.1,. Let $M^3 \neq 0$ and P be any proper prime ideal containing M^3**.** Therefore $M^3 \subseteq$ P $\Rightarrow$ M $\subseteq$ P. Clearly P $\subseteq$ M. Therefore M = P and hence $\sqrt{M^3}$ = M. By theorem 3.15, M^3 is a primary ideal and hence M^3 is a prime ideal by hypothesis. Thus M = M^3 and hence M = M^n for every odd natural number n.

Corolllary 4.4 : Let T be a ternary semigroup containing identity and not containing 0 in which primary ideals are prime. Then T is a 0-simple ternary semigroup extension of a globally idempotent ternary semigroup.

***Proof* :** The proof of this corollary is a direct consequence of theorem 4.3.

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