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Some classes of p-valent analytic functions involving certain integral operators

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ABSTRACT

In this paper we introduce some classes of p-valent analytic functions involving repeated Erdélyi-Kober fractional integral operators and investigate some of their properties specially inclusion relations for these classes. Some class preserving properties of an integral operator are also discussed.

1. Introduction

Let A(p) denote a class of p-valent functions of the form:

$$(1.1) \qquad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$$

($p \in N=\{1,2,...\}$) which are analytic in the unit disc $U := \{z \in C : |z| < 1\}$.

For a function $f \in A(p)$ given by (1.1) and $g \in A(p)$ given by:

$$(1.2) \qquad g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$$

a convolution (Hadmard product) of $f(z)$ and $g(z)$ is defined as:

$$(1.3) \qquad (f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k$$

and we say f(z) is subordinate of g(z) symbolically write as $f \pi g$ or $f(z) \pi g(z)$ if there exists a Schwarz function $w(z)$ in U such that

$$f(z) = g(w(z)), \; z \in U.$$

Let P denote the class of all functions ϕ which are analytic in U and $\phi(U)$ is convex with $\phi(0) = 1$ and $\text{Re}\{\phi(z)\} > 0, z \in U$.

Making use of the principle of subordination, several authors have studied subclasses $S_p^*(\alpha,\phi)$ and $K_p(\alpha,\phi)$ of A(p) for $\phi \in P$ and $0 \le \alpha < p$ which are defined as:

$$(1.4) \qquad S_p^*(\alpha,\phi) = \left\{ f : f \in A_p \text{ and } \frac{1}{p-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) \pi \, \phi(z), z \in U \right\}$$

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(1.5) $K_p(\alpha,\phi)=\Big\{f : f \in A_p$ and

$$\frac{1}{p-\alpha}\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)\pi\ \phi(z)\Big\},\ z\in U$$

If $\phi(z)=\frac{1+z}{1-z}$, $z \in U$, classes $S_p^*(\alpha,\phi)$ and $K_p(\alpha,\phi)$ are, respectively, called p-valently starlike and convex of order α. Further, if $\phi(z)=\frac{1+\beta z}{1-\beta z}$, $z \in U$, classes $S_p^*(\alpha,\phi)$ and $K_p(\alpha,\phi)$ are called p-valently starlike and convex of order α and type β.

Recently, fractional integral operators have found their applications in defining several classes of analytic functions in geometric function theory. In our investigation we consider certain subclasses of A(p) involving repeated Erdélyi-Kober integral operator [6] which is studied by Saigo et al. in [9] and is defined for integer $m \geq 1,\ \delta_i \geq 0,$ $\gamma_i \in R,\ \beta_i > 0, i = 1,.....,m$ as follows:

(1.6) $$I_{(\beta_i),m}^{(\gamma_i),(\delta_i)}f(z)=\left[\prod_{i=1}^{m} I_{\beta_i}^{\gamma_i,\delta_i}\right]f(z),\qquad \sum_{i=1}^{m}\delta_i > 0$$

$$= f(z),\quad \delta_1=\delta_2=....=\delta_m=0$$

where $I_\beta^{\gamma,\delta}$ is the Erdélyi-Kober integral operator [5] defined by

(1.7) $$I_\beta^{\gamma,\delta}f(z)=\frac{1}{\Gamma(\delta)}\int_0^1(1-t)^{\delta-1}t^{\gamma}f\left(zt^{\frac{1}{\beta}}\right)dt,\delta\in R_+$$

$$= f\ (z),\ \delta=0.$$

The image of power function z^k [6] under this operator is given as:

(1.8) $I_{(\beta_i),m}^{(\gamma_i),(\delta_i)}\left(z^k\right)=\lambda_k z^k$, with

(1.9) $$\lambda_k=\prod_{i=1}^{m}\frac{\Gamma\left(\gamma_i+1+\frac{k}{\beta_i}\right)}{\Gamma\left(\gamma_i+\delta_i+1+\frac{k}{\beta_i}\right)}>0$$

for each $k > \max_{1\leq i\leq m}\left[-\beta_i(\gamma_i+1)\right]$.

Thus a normalized repeated Erdélyi-Kober fractional integral operator $E_{(\beta_i),m}^{(\gamma_i),(\delta_i)}$ on $f \in A(p)$ is defined for integer $m \geq 1,\ \delta_i \geq 0,$ $\gamma_i \geq -1,\ \beta_i > 0,\ i = 1,....,m$ as:

(1.10) $$E_{(\beta_i),m}^{(\gamma_i),(\delta_i)}f(z)=(\lambda_p)^{-1}\ I_{(\beta_i),m}^{(\gamma_i),(\delta_i)}f(z).$$

The series expansion of (1.9) using (1.8) for $f \in A(p)$ of the form (1.1) is given by

(1.10) $$Ef(z)\equiv E_{(\beta_i),m}^{(\gamma_i),(\delta_i)}f(z)=z^p+\sum_{k=p+1}^{\infty}a_k\theta_k z^k$$

$$=\left\{z^p+\sum_{k=p+1}^{\infty}\theta_k z^k\right\}*f(z)$$

where,

$$\theta_k = \prod_{i=1}^{m} \frac{\Gamma\left(\gamma_i + 1 + \frac{k}{\beta_i}\right)}{\Gamma\left(\gamma_i + \delta_i + 1 + \frac{k}{\beta_i}\right)} \frac{\Gamma\left(\gamma_i + \delta_i + 1 + \frac{p}{\beta_i}\right)}{\Gamma\left(\gamma_i + 1 + \frac{p}{\beta_i}\right)}.$$

Further we consider an operator $E^j f(z)$ for $f \in A(p)$ of the form (1.1) which is defined as:

(1.11) $$E^j f(z) = \left\{ z^p + \sum_{k=p+1}^{\infty} \theta_k^j z^k \right\} * f(z)$$

where

$$\theta_k^j = \frac{\left(\gamma_j + 1 + \frac{k}{\beta_j}\right)}{\left(\gamma_j + 1 + \frac{p}{\beta_j}\right)} \theta_k$$ for some $j\,(1 \le j \le m)$ such that

$\delta_j > 0$.

It is easily verified from (1.10) and (1.11) that,

(1.12) $$\frac{z}{\beta_j}(Ef(z))' = \left(\gamma_j + 1 + \frac{p}{\beta_j}\right) E^j f(z) - (\gamma_j + 1) Ef(z)$$

which can also be written as :

(1.13) $$\frac{E^j f(z)}{z^p} = \frac{z}{\beta_j(\gamma_j + 1) + p} \left(\frac{Ef(z)}{z^p}\right)' + \frac{Ef(z)}{z^p}.$$

By specializing the parameters, the integral operator E defined in (1.10) reduces into various operators which were earlier studied by several authors. For example:

On setting $\gamma_i = b_i - 1 - p$, $\delta_i = c_i - b_i$, $\beta_i = 1$,

we get $\theta_k = \dfrac{(b_1)_{k-p} \ldots (b_m)_{k-p}}{(c_1)_{k-p} \ldots (c_m)_{k-p}}$

and E reduces to Dziok-Srivastava operator [2] for the class A(p).

Taking m = 2, $\gamma_1 = b - 1 - p$, $\gamma_2 = c - 1 - p$,

$\delta_1 = 1 - b$, $\delta_2 = d - c$, $\beta_1 = \beta_2 = 1$, we obtain

$\theta_k = \dfrac{(b)_{k-p}(c)_{k-p}}{(d)_{k-p}(k-p)!}$, and E reduces to Hohlov operator F(b, c; d) [4] for the class A(p) which further, on taking $b = p, c = \alpha + p, d = p$, gives $\theta_k = \dfrac{(\alpha + p)_{k-p}}{(k-p)!}$, and E reduces to Ruscheweyh derivative operator [8] for the class A(p).

Again, putting m=1, $\gamma_1 = b - 1 - p$, $\delta_1 = d - b$, $\beta_1 = 1$, we get $\theta_k = \dfrac{(b)_{k-p}}{(d)_{k-p}}$, and E reduces to Carlson-Shaffer type operator [1] introduced by Saitoh [10] for the class A(p).

Involving fractional integral operators $Ef(z)$ and $E^j f(z)$ defined by (1.10) and (1.11) respectively, we introduce following subclasses of A(p) as follows:

$$ES_p(\alpha, \phi) = \{f : f \in A(p) \text{ and } Ef \in S_p^*(\alpha, \phi)\}$$

$$ES^j{}_p(\alpha, \phi) = \{f : f \in A(p) \text{ and } E^j f \in S_p^*(\alpha, \phi)\}$$

$$EK_p(\alpha, \phi) = \{f : f \in A(p) \text{ and } Ef \in K_p(\alpha, \phi)\}$$

$E^jK_p(\alpha,\phi)=\{f : f\in A(p) \text{and} E^jf\in K_p(\alpha,\phi)\}$.

Also for $\phi\in P$ and $0\le\alpha<p$

$$EC_p(\alpha,\phi)=\left\{f : f\in A(p)\text{and}\ \frac{1}{1-\alpha}\left\{\frac{Ef(z)}{z^p}-\alpha\right\}\prec\phi(z), z\in U\right\}$$

$$EC^j{}_p(\alpha,\phi)=\left\{f : f\in A(p)\text{and}\ \frac{1}{1-\alpha}\left\{\frac{E^jf(z)}{z^p}-\alpha\right\}\prec\phi(z), z\in U\right\}$$

and for $0\le\lambda\le1$

$EQ_p^\lambda(\alpha,\phi)=\{f : f\in A(p)$and

$$\left.\frac{1}{1-\alpha}\left\{(1-\lambda)\frac{Ef(z)}{z^p}+\lambda\frac{E^jf(z)}{z^p}-\alpha\right\}\prec\phi(z), z\in U\right\}$$

Clearly, $f\in E^jS_p(\alpha,\phi)\Leftrightarrow E^jf\in S_p^*(\alpha,\phi)$ and

(1.14) $f(z)\in EK_p(\alpha,\phi)\Leftrightarrow\dfrac{zf'(z)}{p}\in ES_p(\alpha,\phi)$.

Inclusion relations, in Geometric function theory play important role. To obtain some of the inclusion relations between aforementioned classes we use Briot-Bouquet differential subordination method in the form of following lemma:

Lemma 1.1 [7] Let $\beta,\gamma\in C,\phi$ be convex in U with $\phi(0)=1$ and $\operatorname{Re}\{\beta\phi(z)+\gamma\}>0$, $z\in U$ and let $q(z)=1+q_1z+q_2z^2+\ldots\ldots$ be analytic in U. Then

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\prec\phi(z)\Rightarrow q(z)\prec\phi(z)\quad(z\in U).$$

Lemma 1.2 [3] Let h be an analytic and convex univalent in U. Let ϕ be analytic in U with $h(0)=\phi(0)=1.$ Then for $\gamma\in C$, $\operatorname{Re}\{\gamma\}\ge0$ and $\gamma\ne0$, $\phi(z)\prec q(z)=\dfrac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1}h(t)dt\prec h(z)$.

where q(z) is the best dominant.

2. Inclusion relations

Theorem 2.1 Let the operators E and E^j be defined, respectively, by (1.10) and (1.11) with $\operatorname{Re}\{\phi(z)\}<\dfrac{\beta_j(\gamma_j+1)+\alpha}{\alpha-p}$. Then

$$E^jS_p(\alpha,\phi)\subset ES_p(\alpha,\phi).$$

Proof: Let $f\in E^jS_p(\alpha,\phi)$ and set

$$(2.1)\qquad q(z)=\frac{1}{p-\alpha}\left(\frac{z(Ef(z))'}{Ef(z)}-\alpha\right)$$

which is analytic in U with q(0) = 1 and $q(z)\ne0$ for all $z\in U$. From (1.12) and (2.1), we obtain

$$(2.2)\qquad \frac{\left(\gamma_j+1+\frac{p}{\beta_j}\right)E^jf(z)}{Ef(z)}=\frac{1}{\beta_j}q(z)(p-\alpha)+\left(\gamma_j+1+\frac{\alpha}{\beta_j}\right).$$

Taking logarithmic differentiation on both sides of (2.2), we get

$$\frac{z(E^jf(z))'}{E^jf(z)}-\frac{z(Ef(z))'}{Ef(z)}=\frac{zq'(z)(p-\alpha)}{\beta_j\left\{\frac{q(z)(p-\alpha)}{\beta_j}+\left(\gamma_j+1+\frac{\alpha}{\beta_j}\right)\right\}}$$

or,

$$\frac{1}{p-\alpha}\left\{\frac{z(E^jf(z))'}{E^jf(z)}-\alpha\right\}=q(z)+\frac{zq'(z)}{q(z)(p-\alpha)+(\gamma_j+1)\beta_j+\alpha}$$

which on applying the hypothesis and Lemma 1.1, yields that

$$q(z)\prec \phi(z) \text{ in } U.$$

This proves Theorem 2.1.

Theorem 2.2 Let the operators E and E^j be defined by (1.10) and (1.11) respectively and $0\le\alpha<p,\ \phi\in P$ with $\operatorname{Re}\{\phi(z)\}<\frac{\beta_j(\gamma_j+1)+\alpha}{\alpha-p}$.

Then $E^jK_p(\alpha,\phi)\subset EK_p(\alpha,\phi)$.

Proof: On applying (1.14) and Theorem 2.1, we observe that

$$f(z)\in E^jK_p(\alpha,\phi)\Leftrightarrow \qquad E^jf(z)\in K_p(\alpha;\phi)$$

$$\Leftrightarrow \frac{z\left(E^jf(z)\right)'}{p}\in S_p^*(\alpha;\phi)$$

$$\Leftrightarrow E^j\left(\frac{zf'(z)}{p}\right)\in S_p^*(\alpha;\phi)$$

$$\Leftrightarrow \frac{zf'(z)}{p}\in E^jS_p(\alpha,\phi)\Rightarrow \frac{zf'(z)}{p}\in ES_p(\alpha,\phi)$$

$$\Leftrightarrow E\left(\frac{zf'(z)}{p}\right)\in S_p^*(\alpha;\phi)\Leftrightarrow \frac{z(Ef(z))'}{p}\in S_p^*(\alpha;\phi)$$

$$\Leftrightarrow Ef(z)\in K_p(\alpha;\phi)\Leftrightarrow f(z)\in EK_p(\alpha,\phi)$$

which proves Theorem 2.2.

Theorem 2.3 Let the operator E be defined in (1.10) with the same conditions on parameters and $0\le\alpha<p,\ \phi\in P$, then for $0\le\lambda\le1$,

$$EQ_p^\lambda(\alpha,\phi)\subseteq EC_p(\alpha,\phi).$$

Proof: Let $f\in EQ_p^\lambda(\alpha,\phi)$, and set

$$q(z)=\frac{1}{1-\alpha}\left\{\frac{Ef(z)}{z^p}-\alpha\right\}.$$

We have

$$h(z):=\frac{1}{1-\alpha}\left\{(1-\lambda)\frac{Ef(z)}{z^p}+\lambda\frac{E^jf(z)}{z^p}-\alpha\right\}\prec\phi(z).$$

Using (1.13) we get

$$h(z)=\frac{1}{1-\alpha}\left\{(1-\lambda)\frac{Ef(z)}{z^p}+\lambda\left(\frac{z}{\beta_j(\gamma_j+1)+p}\left(\frac{Ef(z)}{z^p}\right)'+\frac{Ef(z)}{z^p}\right)-\alpha\right\}$$

$$=\frac{1}{1-\alpha}\left\{\frac{Ef(z)}{z^p}-\alpha\right\}+\frac{\lambda z}{\beta_j(\gamma_j+1)+p}\left\{\frac{1}{1-\alpha}\left(\frac{Ef(z)}{z^p}-\alpha\right)\right\}'$$

$$=q(z)+\frac{\lambda zq'(z)}{\beta_j(\gamma_j+1)+p}\prec\phi(z).$$

On applying Lemma 1.2, it follows that $q(z)\prec\phi(z)$ in U, hence this proves the result of Theorem 2.3.

For $\lambda=1$, Theorem 2.3 gives following corollary.

Corollary 2.1 Let the operator E be defined in (1.10) and $0\le\alpha<p,\ \phi\in P$, then

$$E^jC_p(\alpha,\phi)\subset EC_p(\alpha,\phi).$$

Corollary 2.2 Let the operator E be defined in (1.10) and $0\le\alpha<p,\ \phi\in P$, then for

$0 \le \lambda_1 \le \lambda_2 \le 1$, $EQ_p^{\lambda_2}(\alpha,\phi) \subseteq EQ_p^{\lambda_1}(\alpha,\phi)$.

Proof: Let $f \in EQ_p^{\lambda_2}(\alpha,\phi)$. A simple computation gives

$$(2.8) \quad \frac{1}{1-\alpha}\left\{(1-\lambda_1)\frac{Ef(z)}{z^p} + \lambda_1 \frac{E^j f(z)}{z^p} - \alpha\right\}$$

$$= \left(1-\frac{\lambda_1}{\lambda_2}\right)\left\{\frac{1}{1-\alpha}\left(\frac{Ef(z)}{z^p} - \alpha\right)\right\} + \frac{\lambda_1}{\lambda_2}\frac{1}{(1-\alpha)}\left\{\left((1-\lambda_2)\frac{Ef(z)}{z^p} + \lambda_2 \frac{E^j f(z)}{z^p}\right) - \alpha\right\}.$$

By the hypothesis and Theorem 2.3, the right hand side of (2.8) is a convex combination of functions which are the subordinate of a convex function ϕ hence it is a subordinate of ϕ and $f \in EQ_p^{\lambda_1}(\alpha,\phi)$.

That proves Corollary 2.2.

Theorem 2.4 Let $f_1 \in EQ_p^{\lambda}(\alpha,\phi_1)$ and $f_2 \in EQ_p^{\lambda}(\alpha,\phi_2)$, then

$$\frac{E(f_1 * f_2) - \alpha z^p}{1-\alpha} \in EQ_p^{\lambda}(\alpha,\phi_1 * \phi_2).$$

Proof: we have

$$(2.9) \quad \frac{1}{1-\alpha}\left\{(1-\lambda)\frac{Ef_1(z)}{z^p} + \lambda \frac{E^j f_1(z)}{z^p} - \alpha\right\} \prec \phi_1(z) \quad (z \in U).$$

and by Theorem 2.3 we have

$$(2.10) \quad \frac{1}{1-\alpha}\left\{\frac{Ef_2(z)}{z^p} - \alpha\right\} \prec \phi_2(z) \quad (z \in U).$$

We know [3] that if in a unit disk, $f \prec F$ and $g \prec G$ for convex functions F and G, then $f * g \prec F * G$.

Thus, by (2.9) and (2.10) we get

$$\frac{1}{1-\alpha}\left[(1-\lambda)\left\{\frac{E\left\{\frac{E(f_1 * f_2) - \alpha z^p}{1-\alpha}\right\}}{z^p}\right\} + \lambda\left\{\frac{E^j\left\{\frac{E(f_1 * f_2) - \alpha z^p}{1-\alpha}\right\}}{z^p}\right\} - \alpha\right] \prec (\phi_1 * \phi_2)(z).$$

This proves Theorem 2.4.

3. Integral Operator $R_c(f)$

In this section, we examine some class preserving properties of an integral operator $R_c(f)$ defined by

$$(3.1) \quad R_c(f) = \frac{(c+p)}{z^c}\int_0^z t^{c-1} f(t)\,dt \quad (f \in A(p), c > -p).$$

Theorem 3.1 Let the operator E be defined in (1.10) and $0 \le \alpha < p$, $\phi \in P$, then

$$R_c : ES_p(\alpha,\phi) \to ES_p(\alpha,\phi).$$

Proof: To prove the theorem, let $f \in ES_p(\alpha,\phi)$ and let,

$$q(z) = \frac{1}{p-\alpha}\left(\frac{z(ER_c(f))'}{ER_c(f)} - \alpha\right)$$

which is analytic in U with q(0) = 1 and $q(z) \ne 0$ for all $z \in U$. From (3.1), we obtain,

$$\frac{z(ER_c(f))'}{ER_c(f)} = (c+p)\frac{(Ef(z))}{ER_c(f)} - c$$

or,

$$\frac{1}{p-\alpha}\left\{\frac{z(ER_c(f))'}{ER_c(f)}-\alpha\right\}=\frac{1}{p-\alpha}\left\{(c+p)\frac{(Ef(z))}{ER_c(f)}-c-\alpha\right\}$$

or,

(3.2) $$\frac{(c+p)(Ef(z))}{ER_c(f)}=q(z)(p-\alpha)+c+\alpha .$$

By using logarithmic differentiation on both sides of (3.2), we get

$$\frac{1}{p-\alpha}\left\{\frac{z(Ef(z))'}{Ef(z)}-\alpha\right\}=\frac{zq'(z)}{q(z)(p-\alpha)+c+\alpha}+q(z)\,\pi\,\phi(z)\cdot$$

Hence by the hypothesis and Lemma 1.1, we conclude that $q(z)\,\pi\,\phi(z)$ in U, which implies that $R_c(f)\in ES_p(\alpha,\phi)$. This proves Theorem 3.1.

Theorem 3.2 Let the operator E be defined in (1.10) and $0\le\alpha<p,\ \ \phi\in P$, then

$$R_c : EK_p(\alpha,\phi)\to EK_p(\alpha,\phi).$$

Proof: On applying (1.14) and Theorem 3.1 it follows that,

$$f(z)\in EK_p(\alpha,\phi)\Leftrightarrow \frac{zf'(z)}{p}\in ES_p(\alpha,\phi)$$

$$\Rightarrow R_c\left(\frac{zf'(z)}{p}\right)\in ES_p(\alpha,\phi)$$

$$\Leftrightarrow \frac{z(R_cf(z))'}{p}\in ES_p(\alpha,\phi)$$

$$\Leftrightarrow R_cf(z)\in EK_p(\alpha,\phi),$$

which proves Theorem 3.2.

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