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The Hankel type transform of Gevrey Ultra-distibutions

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ABSTRACT

In this paper we have the space $H_w^{\alpha,\beta}$ and some properties of this space are studied. It is shown that the conventional Hankel type transform $\boldsymbol{h}_{\alpha,\beta}$ is an automorphism of $H_w^{\alpha,\beta}$. The generalized Hankel type transform of Gevrey ultradistributions is defined and it is established that the generalized Hankel type transform is an automorphism of $\left(H_w^{\alpha,\beta}\right)'$. Multiplication on $H_w^{\alpha,\beta}$ and convolution on $\left(H_w^{\alpha,\beta}\right)'$ are investigated.

1. Introduction :

In the recent past many authors have extended Hankel transformation.

$$(\boldsymbol{h}_\mu \phi)(y) = \int_0^\infty \phi(x)\sqrt{xy}\, J_\mu(xy)\, dx \; , \tag{1.1}$$

$0 < y < \infty, \qquad \mu \geq -\frac{1}{2}$ to distributions belonging to H'_μ on $I = (0,\infty)$, where J_μ is the Bessel function of the first kind and order μ. Zemanian [15] has considered these transformations in his monograph. Waphare [14] has investigated Hankel type transformation

$$\left(\boldsymbol{h}_{\alpha,\beta}\, \phi\right)(s) = \int_0^\infty (st)^{\alpha+\beta} J_{\alpha-\beta}(st)\, \phi(t)\, dt \tag{1.2}$$

and has been extended to distributions belonging to the dual space $H'_{\alpha,\beta}$ consisting of all complex valued infinitely differentiable functions ϕ defined on I satisfying

$$\rho_{m,k}^{\alpha,\beta}(\phi) = \sup_{x\in I} \left| x^m \left(x^{-1} D_x\right)^k x^{2\beta-1} \phi(x)\right| < \infty, \tag{1.3}$$

for all $m, k \in \mathbf{N}_0$.

The generalized Hankel type transformation $\boldsymbol{h}'_{\alpha,\beta}$ is defined as the adjoint of $\boldsymbol{h}_{\alpha,\beta}$ through the relation

$$\langle \boldsymbol{h}'_{\alpha,\beta} f, \phi\rangle = \langle f, \boldsymbol{h}_{\alpha,\beta}\, \phi\rangle, \tag{1.4}$$

where $(\alpha-\beta) \geq -\frac{1}{2}, \qquad f \in H'_{\alpha,\beta}$ and $\phi \in H_{\alpha,\beta}$.

The classical Hankel type convolution transform of f and g belonging to the class

$$L^1_{\alpha,\beta} = \left\{ f : \|f\|_{\alpha,\beta} = \int_0^\infty |f(x)|\, x^{2\alpha}\, dx < \infty,\ (\alpha-\beta) \geq -\frac{1}{2}\right\}$$

is defined by

$$(f \# g)(x) = \int_0^\infty f(y)\, (\tau_x g)(y)\, dy, \qquad x \in I, \tag{1.5}$$

where

$$(\tau_x g)(y) = \int_0^\infty g(z)\, D_{\alpha,\beta}(x,y,z)\, dz, \qquad x, y \in I \tag{1.6}$$

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and for $x, y, z \in I$,

$$D_{\alpha,\beta}(x,y,z) = \int_0^\infty t^{2\beta-1} j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(yt)\, j_{\alpha-\beta}(zt)\, dt \tag{1.7}$$

$$= \begin{cases} \dfrac{(xyz)^{2\beta}[z^2-(x-y)^2]^{-2\beta}\,[(x+y)^2-z^2]^{-2\beta'}}{2^{\alpha-5\beta}\,\Gamma(2\alpha)}, & x,y,z \in I,\ |x-y|<z<x+y \\ 0 \quad ; & x,y,\ z \in I,\ 0<z<|x-y| \text{ or } x+y<z<\infty \end{cases}$$

with

$$j_{\alpha-\beta}(xt) = (xt)^{\alpha+\beta} J_{\alpha-\beta}(xt). \tag{1.8}$$

It follows from the definition of $D_{\alpha,\beta}(x,y,z)$ that

$$\int_0^\infty j_{\alpha-\beta}(xt)\, D_{\alpha,\beta}(x,y,z)\, dz = t^{2\beta-1} j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(yt). \tag{1.9}$$

The theory and applications of the Hankel convolution transform can be found in [1], [2], [5], [7], [11], [14].

The Hankel convolution transfer defined by (1.5) was extended to distributions in $H'_{\alpha,\beta}$ by Betancor and Marrero [1], [2].

Ultra distributions have been introduced by Beurling [3], Bjorck [4], and Roumieu [12] as generalizations of Schwartz distributions. A unification of Beurling Bjorck theory and Roumieu theory has been given by Komatsu [8]. The Hankel transform of ultradistributions in Roumieu setting has been given by Pathak and Pandey [10]. The purpose of the present paper is to introduce Gevrey type ultradistributions which are suitable for the study of Hankel type transform and Hankel type convolution transform.

In this paper, a test function space $H_w^{\alpha,\beta}$, generalizing the Zemanian space $H_{\alpha,\beta}$ is defined. It is shown that the conventional Hankel type transform (1.2) is an automorphism of $H_w^{\alpha,\beta}$. For $w(x) = \log(1+x)$ it reduces to $H_{\alpha,\beta}$ and for $w(x) = x^a\ (0<a<1)$, $H_w^{\alpha,\beta}$ is a Gevrey space of test functions. The generalized Hankel type transform of ultradistributions belonging to $\left(H_w^{\alpha,\beta}\right)'$ is defined by the adjoint operator method (1.4) and it is found that the generalized Hankel type transform is also an automorphism of $\left(H_w^{\alpha,\beta}\right)'$. Multiplication on $H_w^{\alpha,\beta}$ and convolution on $\left(H_w^{\alpha,\beta}\right)'$ are investigated.

2. $H_w^{\alpha,\beta}$ *and* $\left(H_w^{\alpha,\beta}\right)'$ *spaces*: Let w be a continuous real valued function defined on $I = (0,\infty)$ possessing the following properties:

(a) $0 \le w(s+t) \le w(s) + w(t)$, $\quad$ *for all* $s,t \in I$,

(b) $\displaystyle\int_0^\infty \frac{w(s)}{1+s^2}\, ds < \infty$,

(c) $w(s) \ge a + b\log(1+s)$, for some real $a, b > 0$.

We denote by $\mathcal{M}$ the set of all continuous real valued functions satisfying (a), (b) and (c). From (c) it follows that

$$x \le e^{-\frac{a}{b}} e^{\frac{w(x)}{b}}, \qquad x > 0. \tag{2.1}$$

For each real number $(\alpha-\beta)$, the space $H_w^{\alpha,\beta}(I)$ is defined as follows.

A complex valued $C^\infty -$ function ϕ on I is said to belong to the space $H_w^{\alpha,\beta}(I)$ if

$$\eta_{\lambda,k}^{\alpha,\beta}(\phi) = \sup_{x\in I}\left|e^{\lambda w(x)}\,(x^{-1}D_x)^k\left[x^{2\beta-1}\phi(x)\right]\right| < \infty \tag{2.2}$$

for all non-negative real numbers λ and non-negative integers k.

$H_w^{\alpha,\beta}(I)$ is clearly a linear space. The topology of $H_w^{\alpha,\beta}$ is generated by the seminorms $\left\{\eta_{\lambda,k}^{\alpha,\beta}\right\}$. Following technique used in [15, p.131], it can be proved that $H_w^{\alpha,\beta}$ is a Frechet space.

From definitions (1.3), (2.2) and the inequality $x^m \leq e^{\lambda w(x)}$ for $\lambda b \geq m$, it follows that $H_w^{\alpha,\beta} \subseteq H_{\alpha,\beta}$. It is also clear that $D(I) \subset H_w^{\alpha,\beta}(I) \subset E(I)$. Since $D(I)$ is a dense subspace of $E(I)$, then $H_w^{\alpha,\beta}(I)$ is dense in $E(I)$. Hence $E'(I) \subset \left(H_w^{\alpha,\beta}\right)'(\mathrm{I})$, the dual of $H_w^{\alpha,\beta}(I)$, called the space of Gevrey ultradistributions. Since $H_w^{\alpha,\beta} \subset H_{\alpha,\beta}$, the following properties given by Zemanian [15] hold in the present case also when $w \in \mathcal{M}$. We use the following definitions [14]

$$N_{\alpha,\beta} = x^{2\alpha} D x^{2\beta-1}, \qquad M_{\alpha,\beta} = x^{2\beta-1} D x^{2\alpha},$$
$$\Delta_{\alpha,\beta} = M_{\alpha,\beta} N_{\alpha,\beta} = x^{2\beta-1} D x^{4\alpha} D x^{2\beta-1}$$
$$= (2\beta-1)(4\alpha+2\beta-2) x^{4(\alpha+\beta-1)} + 2(2\alpha+2\beta-1) \times x^{4\alpha+4\beta-3} D_x + x^{2(2\alpha+2\beta-1)} D_x^2 .$$

If we take $\alpha = \frac{1}{4} + \frac{\mu}{2}$, $\quad \beta = \frac{1}{4} - \frac{\mu}{2}$, we obtain

$S_\mu = D_x^2 + \frac{(1-4\mu^2)}{4x^2}$, the operator studied in Zemanian [15].

Lemma 2.1:

(i) The operation $\phi \to M_{\alpha,\beta}\phi$ is a continuous linear mapping of $H_w^{\alpha,\beta,1}$ into $H_w^{\alpha,\beta}$.

(ii) The operation $\phi \to N_{\alpha,\beta}\phi$ is a continuous linear mapping of $H_w^{\alpha,\beta}$ into $H_w^{\alpha,\beta,1}$.

(iii) The operation $\phi \to \Delta_{\alpha,\beta}\phi$ is a continuous linear mapping of $H_w^{\alpha,\beta}$ into itself.

(iv) If q is an even integer, then $H_w^{\alpha,\beta,q} \subset H_w^{\alpha,\beta}$.

Using (2.1), the following result can be easily proved.

Lemma 2.2: Differentiation is a continuous operator of $H_w^{\alpha,\beta,1}$ into $H_w^{\alpha,\beta}$.

3. The generalized Hankel type transformation:

The conventional Hankel type transform $h_{\alpha,\beta}$, $(\alpha-\beta) \geq -\frac{1}{2}$ defined by (1.2) exists for every $\phi \in H_w^{\alpha,\beta} \subset L^1(0,\infty)$. Further more, we have

Theorem 3.1: For $(\alpha-\beta) \geq -\frac{1}{2}$, the conventional Hankel type transform $h_{\alpha,\beta}$ is an automorphism of $H_w^{\alpha,\beta}$.

Proof: Let $\Phi(y) = \left(h_{\alpha,\beta}\phi\right)(y)$, where $\phi \in H_w^{\alpha,\beta}$. The following facts are well known [15, p. 139].

$$h_{\alpha,\beta,1}(-x\phi) = N_{\alpha,\beta}\, h_{\alpha,\beta}\,\phi \tag{3.1}$$

$$h_{\alpha,\beta,1}\left(N_{\alpha,\beta}\,\phi\right) = -y\; h_{\alpha,\beta}\,\phi . \tag{3.2}$$

Applying (3.1) k times and (3.2) m-times and then using Zemanian's identity [15, p. 141], we obtain

$$(-1)^{k+m} y^m \left(y^{-1}\frac{d}{dy}\right)^k y^{2\beta-1}\left(h_{\alpha,\beta}\phi\right)(y)$$
$$= \int_0^\infty z^{4\alpha+2k+m}\left(x^{-1}\frac{d}{dx}\right)^m \left[x^{2\beta-1}\phi(x)\right](xy)^{-(\alpha-\beta+k)} J_{\alpha-\beta+k+m}(xy)\,dx$$

so that

$$(-1)^k \sum_{m=0}^{\infty} (-1)^m \frac{(Ay)^m}{m!}\left(y^{-1}\frac{d}{dy}\right)^k y^{2\beta-1}\left(h_{\alpha,\beta}\phi\right)(y)$$
$$= \sum_{m=0}^{\infty}\frac{A^m}{m!}\int_0^\infty x^{4\alpha+2k+m}\left(x^{-1}\frac{d}{dx}\right)^m\left[x^{2\beta-1}\phi(x)\right](xy)^{-(\alpha-\beta+k)} J_{\alpha-\beta+k+m}(xy)\,dx,$$

where $A > 0$. Also, by property

$\int_0^\infty \frac{w(s)}{1+s^2}\,ds < \infty, \qquad for\ \epsilon > 0$ there exists a constant $c(\epsilon)$

such that $w(s) < \epsilon s + c(\epsilon)$. Hence

$$e^{vw(\epsilon)} \le e^{v\epsilon s + vc(\epsilon)} \le e^{vc(\epsilon)} \sum_{m=0}^{\infty} \frac{(v\epsilon)^m}{m!} s^m .$$

Now, for any choice of v and k we have

$$\eta_{v,k}^{\alpha,\beta}(\Phi) = \underset{s}{\text{Sup}} \left| e^{vw(s)} \left(s^{-1}\frac{d}{ds}\right)^k s^{2\beta-1} \Phi(s) \right|$$

$$\le \underset{s}{Sup} \left| e^{vc(\epsilon)} \sum_{m=0}^{\infty} \frac{(v\epsilon)^m}{m!} s^m \left(s^{-1}\frac{d}{ds}\right)^k s^{2\beta-1} \Phi(s) \right|$$

$$= e^{vc(\epsilon)} \underset{s}{Sup} \left| \sum_{m=0}^{\infty} \frac{(v\epsilon)^m}{m!} \int_0^{\infty} x^{4\alpha+2k+m} \left(x^{-1}\frac{d}{dx}\right)^m \times \left[x^{2\beta-1}\phi(x)\right] (xs)^{-(\alpha-\beta+k)} J_{\alpha-\beta+k+m}(xs)\, dx \right|.$$

As $(\alpha-\beta) \ge -\frac{1}{2}$, $(xs)^{-(\alpha-\beta+k)} J_{\alpha-\beta+k+m}(xs)$ is bounded on $0 < x, \quad s < \infty$ by the constant $B_{k,m}$. Let N be an integer no less than $\alpha - \beta + k + {(m+1)}/_{2}$. Then

$x^{4\alpha+2k+m} < (1+x^2)^N \; for \;\; x > 0.$

So that

$$\eta_{v,k}^{\alpha,\beta}(\Phi) \le e^{vc(\epsilon)} \sum_{m=0}^{\infty} \frac{(v\epsilon)^m}{m!} \int_0^{\infty} (1+x^2)^{N+1} \left| \left(x^{1}\frac{d}{dx}\right)^m x^{2\beta-1} \phi(x) \right| \times B_{k,m} \frac{1}{1+x^2}\, dx$$

$$\le \frac{\pi}{2} e^{vc(s)} \sum_{m=0}^{\infty} \frac{(v\epsilon)^m}{m!} B_{k,m} \sum_{r=0}^{n+1} \binom{N+1}{r} Sup \left| x^{2r} \left(x^{-1}\frac{d}{dx}\right)^m x^{2\beta-1} \phi(x) \right|.$$

Since

$e^{w(x)} \ge e^{a+b\log(1+x)}$

then

$$e^{\frac{2rw(x)}{b}} \ge e^{\frac{2ar}{b}} (1+x)^{2r} \ge e^{\frac{2ar}{b}} x^{2r} ;$$

so that

$$\left| x^{2r} \left(x^{-1}\frac{d}{dx}\right)^m x^{2\beta-1} \phi(x) \right| \le \left| e^{-2ar} e^{\frac{2rw(x)}{b}} \left(x^{-1}\frac{d}{dx}\right)^m x^{2\beta-1} \phi(x) \right|$$

$$= e^{-\frac{2ar}{b}} \eta_{\frac{2r}{b},m}^{\alpha,\beta}(\phi)$$

$$\le e^{-\frac{2ar}{b}} \eta_{\frac{2(N+1)}{b},m}^{\alpha,\beta}(\phi) .$$

Now choosing

$$\epsilon < \left(v^m B_{k,m} \eta_{\frac{2(N+1)}{b},m}^{\alpha,\beta}(\phi) \right)^{-1/_m}, \qquad (m \ge 1),$$

we have for some $A' > 0$,

$$\eta_{v,k}^{\alpha,\beta}(\Phi) \le \frac{\pi}{2} e^{vc(\epsilon)} A' \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{r=0}^{N+1} \binom{N+1}{r} e^{-\frac{2ar}{b}}$$

$$= \frac{\pi}{2} e^{vc(\epsilon)} A' e \sum_{r=0}^{N+1} \binom{N+1}{r} e^{-\frac{2ar}{b}} < \infty.$$

This proves that $\Phi \in H_w^{\alpha,\beta}$ and that the linear mapping $\boldsymbol{h}_{\alpha,\beta}$ is also continuous from $H_w^{\alpha,\beta}$ into $H_w^{\alpha,\beta}$. Since $H_w^{\alpha,\beta} \subset L^1(0,\infty)$, when $(\alpha-\beta) \geq -\frac{1}{2}$ we can apply the classical inversion theorem and the fact that $\boldsymbol{h}_{\alpha,\beta}^{-1} = \boldsymbol{h}_{\alpha,\beta}$ to this case and conclude that $\boldsymbol{h}_{\alpha,\beta}$ is one-to-one. Thus $\boldsymbol{h}_{\alpha,\beta}$ is an automorphism on $H_w^{\alpha,\beta}$.
Thus proof is completed.

The generalized Hankel type transformation $\boldsymbol{h}'_{\alpha,\beta}$ on $\left(H_w^{\alpha,\beta}\right)'$ is defined to be the adjoint of $\boldsymbol{h}_{\alpha,\beta}$ on $H_w^{\alpha,\beta}$. More precisely, for any $\phi \in H_w^{\alpha,\beta}$ and $f \in \left(H_w^{\alpha,\beta}\right)'$, we have
$\langle \boldsymbol{h}'_{\alpha,\beta} f, \phi \rangle = \langle f, \boldsymbol{h}_{\alpha,\beta}\,\phi \rangle$.
By using Theorem 3.1, we immediately obtain the following.

Theorem 3.2: For any $(\alpha-\beta) \geq -\frac{1}{2}$, the generalized Hankel type transformation $\boldsymbol{h}'_{\alpha,\beta}$ is an automorphism of $\left(H_w^{\alpha,\beta}\right)'$.

4. Multiplication and convolution on $H_w^{\alpha,\beta}$:

We denote by T_m the space of all $C^\infty - functions\ \phi(x)$, $0 < x < \infty$ such that for each non-negative integer m, there exists a non-negative integer $k = k(m)$ for which

$$e^{-kw(x)} \left| \left(x^{-1}\frac{d}{dx}\right)^m \phi(x) \right|$$

is bounded.

Here T_m is the space of multipliers for $H_w^{\alpha,\beta}$. The following results will be used in the sequel.
If $f, g \in L^1_{\alpha,\beta}(0,\infty)$, then from [1, p. 285] we have

$$\left(\boldsymbol{h}_{\alpha,\beta}(\tau_x f)\right)(t) = t^{2\beta-1} j_{\alpha-\beta}(tx) \left(\boldsymbol{h}_{\alpha,\beta} f\right)(t), \qquad t \in I \tag{4.1}$$

and

$$\left(\boldsymbol{h}_{\alpha,\beta}(f \# g)\right)(t) = t^{2\beta-1} \left(\boldsymbol{h}_{\alpha,\beta} f\right)(t) \left(\boldsymbol{h}_{\alpha,\beta} g\right)(t), \qquad t \in I, \tag{4.2}$$

our aim in this section is to study Hankel type convolution on $H_w^{\alpha,\beta}$.

Theorem 4.1: If $f \in H_w^{\alpha,\beta}(I)$ and $x^{2\alpha} g \in H_w^{\alpha,\beta}(I)$, then $f\,g \in H_w^{\alpha,\beta}(I)$.

Proof: For non-negative integer k and non-negative real number λ, we have by definition (2.2),

$$\eta_{\lambda,k}^{\alpha,\beta}(fg) = \underset{x\in I}{Sup} \left| e^{\lambda w(x)} \left(x^{-1}\frac{d}{dx}\right)^k \left[x^{2\beta-1} f(x)\, g(x)\right] \right|$$

$$= \underset{x\in I}{Sup} \left| e^{\lambda w(x)} \left(x^{-1}\frac{d}{dx}\right)^k \left[x^{2\beta-1} f(x)\, x^{2\beta-1}(x^{2\alpha} g(x))\right] \right|.$$

Now by using Leibnitz theorem, we obtain

$$\eta_{\lambda,k}^{\alpha,\beta}(fg) \leq \sum_{r=0}^{k} \binom{k}{r} = \underset{x}{Sup} \left| e^{\lambda w(x)} \left(x^{-1}\frac{d}{dx}\right)^r \left(x^{2\beta-1} f(x)\right) \right|$$

$$\times \underset{x}{Sup} \left| \left(x^{-1}\frac{d}{dx}\right)^{k-r} \left[x^{2\beta-1}(x^{2\alpha} g)\right] \right| < \infty .$$

Hence $f\,g \in H_w^{\alpha,\beta}(I)$. Thus proof is completed.

Theorem 4.2: For every $x \in I = (0,\infty)$, the mapping $\phi \to \tau_x \phi$ is continuous from $H_w^{\alpha,\beta}$ into itself.

Proof: Let $\phi \in H_w^{\alpha,\beta}(I)$. Then $\left(\boldsymbol{h}_{\alpha,\beta}\,\phi\right)(t) \in H_w^{\alpha,\beta}(I)$. By definitions (1.2) and (1.6), we have

$$\mathbf{h}_{\alpha,\beta}\,(\tau_x\phi)\,(t) = \int_0^\infty (\tau_x\phi)\,(y)\,j_{\alpha-\beta}\,(ty)\,dy$$
$$= \int_0^\infty j_{\alpha-\beta}\,(ty)\left[\int_0^\infty \phi\,(z)\,D_{\alpha,\beta}\,(x,y,z)\,dz\right]dy$$
$$= \int_0^\infty \phi\,(z)\,dz\int_0^\infty j_{\alpha-\beta}\,(ty)\,D_{\alpha,\beta}\,(x,y,z)\,dy.$$

Now by making use of (1.9) we can obtain

$$\mathbf{h}_{\alpha,\beta}\,(\tau_x\,\phi)\,(t) = \int_0^\infty \phi(z)t^{2\beta-1}j_{\alpha-\beta}(tx)j_{\alpha-\beta}(tz)\,dz = t^{2\beta-1}\,j_{\alpha-\beta}(tx)\left(\mathbf{h}_{\alpha,\beta}\,\phi\right)(t).$$

Now we show that

$$t^{2\beta-1}\,j_{\alpha-\beta}\,(tx)\,\in\,T_m.$$

We have

$$\left(t^{-1}\frac{d}{dt}\right)^m\left[t^{2\beta-1}\,j_{\alpha-\beta}\,(tx)\right] = x^{\alpha+\beta}\left(t^{-1}\frac{d}{dt}\right)^m\left[t^{-(\alpha-\beta)}J_{\alpha-\beta}(tx)\right]$$
$$= (-1)^m\,x^{2\alpha+2m}\,(tx)^{-(\alpha-\beta+m)}J_{\alpha-\beta+m}(tx);$$

so that there exists $p > 0$ such that

$$\left|e^{-pw(t)}\left(t^{-1}\frac{d}{dt}\right)^m t^{2\beta-1}\,j_{\alpha-\beta}\,(tx)\right| < \infty \quad \text{for every } x \in I.$$

Hence $t^{2\beta-1}\,j_{\alpha-\beta}\,(tx)\in T_m$ for fixed $x \in I$. But

$$\left(\mathbf{h}_{\alpha,\beta}\,\phi\right)\in H_w^{\alpha,\beta}, \text{ then } t^{2\beta-1}\,j_{\alpha-\beta}\,(tx)\left(\mathbf{h}_{\alpha,\beta}\,\phi\right)(t)\in H_w^{\alpha,\beta}.$$

As $\mathbf{h}_{\alpha,\beta}$ is an automorphism of $H_w^{\alpha,\beta}$, therefore $\tau_x\,\phi \in H_w^{\alpha,\beta}$, and the mapping $\phi \to \tau_x\,\phi$ is continuous from $H_w^{\alpha,\beta}$ into itself.

This completes the proof.

Theorem 4.3: If $f,g \in H_w^{\alpha,\beta}\,(I)$, then

Proof: By using (2.2), we have

$$\eta_{\lambda,\kappa}^{\alpha,\beta}\left(\mathbf{h}_{\alpha,\beta}\,(f\#g)\right) = \sup_{s\in I}\left|e^{\lambda w(s)}\left(s^{-1}\frac{d}{ds}\right)^k s^{2\beta-1}\,h_{\alpha,\beta}\,(f\#g)\,(s)\right|.$$

Now using (4.2), we obtain

$$\eta_{\lambda,k}^{\alpha,\beta}\left(\mathbf{h}_{\alpha,\beta}\,(f\#g)\right) = \sup_{s\in I}\left|e^{\lambda w(s)}\left(s^{-1}\frac{d}{ds}\right)^k s^{2\beta-1}\,s^{2\beta-1}\left(h_{\alpha,\beta}\,f\right)(s)\left(h_{\alpha,\beta}\,g\right)(s)\right|.$$

One of the applications of Leibnitz theorem gives

$$\eta_{\lambda,k}^{\alpha,\beta}\left(\mathbf{h}_{\alpha,\beta}\,(f\#g)\right) \le \sum_{r=0}^{k}\binom{k}{r}\eta_{\lambda,k}^{\alpha,\beta}\,(f)\,\eta_{0,k-r}^{\alpha,\beta}\,(g) < \infty.$$

Thus $\mathbf{h}_{\alpha,\beta}\,(f\#g)\in H_w^{\alpha,\beta}\,(I)$. As $\mathbf{h}_{\alpha,\beta}$ is an automorphism of $H_w^{\alpha,\beta}\,(I)$, therefore .

This completes the proof.

5. Hankel type convolution on $\left(H_w^{\alpha,\beta}\right)'$:

In this section we study Hankel type convolution on $\left(H_w^{\alpha,\beta}\right)'$.

Definition 5.1 : For $\phi \in H_w^{\alpha,\beta}\,(I)$ and $f \in \left(H_w^{\alpha,\beta}\right)'$ the convolution of f and ϕ is defined by

$$(f\#\phi)\,(x) = \langle f(y),\quad (\tau_x\,\phi)\,(y)\rangle,\qquad x\in I\,. \tag{5.1}$$

Since for every $\psi \in H_w^{\alpha,\beta}$ generates an ultradistribution belonging to $\left(H_w^{\alpha,\beta}\,(I)\right)'$, we have

$\langle \psi, \tau_x \phi \rangle = \int_0^{\infty} \psi(y) (\tau_x \phi)(y) \, dy$;

so that the classical Hankel type convolution is the special case of the generalized Hankel type convolution (5.1). The following lemma will be useful in the sequel. Moreover $f \in E'(I)$, (5.1) holds.

Lemma 5.2: If $f \in E'(I)$, then $t^{2\beta-1} h'_{\alpha,\beta} f \in T_m$.

Proof: Let $f \in E'(I)$. Choose $\rho(x) \in D(I)$ such that $\rho(x) = 1$.

on a neighbourhood k of the support f. Then

$\left(h'_{\alpha,\beta} f\right)(t) = \langle f(x), \quad \rho(x) j_{\alpha-\beta}(tx) \rangle$,

so that

$$\left(t^{-1}\frac{d}{dt}\right)^m \left[t^{2\beta-1} \left(h'_{\alpha,\beta} f\right)(t)\right] = \langle f(x), \left(t^{-1}\frac{d}{dt}\right)^m \left[t^{2\beta-1}\rho(x) j_{\alpha-\beta}(tx)\right] \rangle.$$

We then have

$$\left(t^{-1}\frac{d}{dt}\right)^m \left[t^{2\beta-1} \rho(x) j_{\alpha-\beta}(tx)\right] = \rho(x) \left(t^{-1}\frac{d}{dt}\right)^m \left[t^{2\beta-1} (xt)^{\alpha+\beta} j_{\alpha-\beta}(tx)\right]$$
$$= (-1)^m \rho(x) x^{2\alpha+2m} (tx)^{-(\alpha-\beta+m)} J_{\alpha-\beta+m}(tx),$$

so that

$$\left(t^{-1}\frac{d}{dt}\right)^m \left[t^{2\beta-1} \left(h'_{\alpha,\beta} f\right)(t)\right]$$
$$= \langle f(x), \quad (-1)^m \rho(x) \, t^{-(\alpha-\beta+m)} x^{2\alpha+2m} x^{-(\alpha-\beta+m)} J_{\alpha-\beta+m}(tx) \rangle.$$

As $f \in E'(I)$, there exists a positive constant M and non-negative integer r such that

$$\left|\langle f(x), \quad (-1)^m \rho(x) \, t^{-(\alpha-\beta+m)} x^{2\alpha+2m} x^{-(\alpha-\beta+m)} J_{\alpha-\beta+m}(tx) \rangle\right|.$$

$$\le t^{-(\alpha-\beta+m)} M \max_{0\le k\le r} \operatorname{Sup}_{x\in k} \left|D_x^k \left[\rho(x) x^{2\alpha+2m} x^{-(\alpha-\beta+m)} J_{\alpha-\beta+m}(tx)\right]\right|$$
$$= t^{-(\alpha-\beta+m)} M \max_{0\le k\le r} \operatorname{Sup} \times \left|\sum_{i=0}^{k} \binom{k}{i} D_x^i \left[x^{-(\alpha-\beta+m)} J_{\alpha-\beta+m}(tx)\right]\right|$$
$$\le M' \sum_{i=0}^{r} \binom{r}{i} \operatorname{Sup} \times \left|t^i (xt)^{-(\alpha-\beta+m)} J_{\alpha-\beta+m+i}(tx)\right|$$
$$\le M'' \sum_{i=0}^{r} \binom{r}{i} t^i \le M''(1+t)^r \; ;$$

so that

$$\left|e^{-pw(t)} \left(t^{-1}\frac{d}{dt}\right)^m t^{2\beta-1} \left(h'_{\alpha,\beta} f\right)(t)\right| \le M'' e^{-pw(t)} (1+t)^r < \infty.$$

Hence, $t^{2\beta-1} \left(h'_{\alpha,\beta} f\right)(t) \in T_m$. Thus proof is completed.

Theorem 5.3: For any $g \in H_w^{\alpha,\beta}(I)$ and $f \in E'(I)$ we have

and

$$\left[h_{\alpha,\beta}(f \# g)\right](t) = t^{2\beta-1} \left(h'_{\alpha,\beta} f\right)(t) \left(h_{\alpha,\beta} g\right)(t).$$

Proof: Let $f \in E'(I)$ and $g \in H_w^{\alpha,\beta}(I)$. Then as in [11, p. 1341], we have

$\left[h_{\alpha,\beta}(f \# g)\right](t) = \langle f(y), \quad \left[h_{\alpha,\beta}(\tau_y g)\right](t) \rangle$.

An application of (4.1) yields

$$\left[h_{\alpha,\beta}(f \# g)\right](t) = \langle f(y), \quad t^{2\beta-1} j_{\alpha-\beta}(ty) \rangle \left(h_{\alpha,\beta} g\right)(t)$$
$$= \langle f(y), j_{\alpha-\beta}(ty) \, t^{2\beta-1} \left(h_{\alpha,\beta} g\right)(t) \rangle$$

$$= t^{2\beta-1}\left(h'_{\alpha,\beta} f\right)(t)\left(h_{\alpha,\beta} g\right)(t).$$

Now by Lemma 5.2, $t^{2\beta-1}\left(h'_{\alpha,\beta} f\right)(t) \in T_m$ so that $h_{\alpha,\beta}(f\#g)(t) \in H_w^{\alpha,\beta}(I)$.

Since $h_{\alpha,\beta}$ is an automorphism of $H_w^{\alpha,\beta}$, therefore

Thus proof is completed.

Definition 5.4: For $f \in \left(H_w^{\alpha,\beta}\right)'$ and $g \in E'$, the Hankel type convolution is defined by

$$< f\#g, \quad \phi > = < f, g\#\phi >, \quad for\ all\ \phi \in H_1w^1(\alpha,\beta)\ .$$

Theorem 5.5: If $f \in \left(H_w^{\alpha,\beta}\right)'$ and $g \in E'$, then and

$$\left[h_{\alpha,\beta}(f\#g)\right](t) = t^{2\beta-1}\left(h'_{\alpha,\beta} f\right)(t)\left(h'_{\alpha,\beta}\, g\right)(t).$$

Proof: Let $\{\phi_v\}$ be a sequence of functions in $H_w^{\alpha,\beta}$ that converges to zero in $H_w^{\alpha,\beta}$. Then by Definition 5.4,

$$< f\#g, \quad \phi_1 v > = < f, g\#\square\ \phi\square_1 v >.$$

As $g \in E'$ and $\phi_v \in H_w^{\alpha,\beta}$, therefore by Theorem 5.3 we have

and

$$\left| e^{\lambda w(x)}\left(x^{-1}\frac{d}{dx}\right)^k x^{2\beta-1}(g\#\phi_v)\right| \leq \infty.$$

Since $\phi_v \to 0$ in $H_w^{\alpha,\beta}$, therefore in $H_w^{\alpha,\beta}$, so that

That is is continuous on $H_w^{\alpha,\beta}$. Similarly we can prove linearity. Hence

Moreover,

$$\langle\left[h'_{\alpha,\beta}(f\#g)\right](t), \quad h_{\alpha,\beta}\,\phi\,(t)\rangle = \langle(f\#g)(x), \quad \phi(x)\rangle = \langle f(x), (g\#\phi)(x)\rangle$$

$$= \langle\left(h'_{\alpha,\beta} f\right)(t), \quad h'_{\alpha,\beta}(g\#\phi)(t)\rangle$$

$$= \langle\left(h'_{\alpha,\beta} f\right)(t), \quad t^{2\beta-1}\left(h'_{\alpha,\beta}\, g\right)(t)\left(h'_{\alpha,\beta}\,\phi\right)(t)\rangle$$

$$= \langle t^{2\beta-1}\left(h'_{\alpha,\beta} f\right)(t)\left(h'_{\alpha,\beta}\, g\right)(t), \left(h_{\alpha,\beta}\,\phi\right)(t)\rangle\ ,$$

so that

$$\left[h'_{\alpha,\beta}(f\#g)\right](t) = t^{2\beta-1}\left(h'_{\alpha,\beta} f\right)(t)\left(h'_{\alpha,\beta}\, g\right)(t).$$

Thus the proof is completed.

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