Available online at www.elixirpublishers.com (Elixir International Journal)

Advances in Pure Mathematics

Elixir Adv. Pure Math. 60 (2013) 16183-16188

Elixir ISSN: 2229-712X

Annulus for the zeros of the polynomial with perturbed monotonic coefficients

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ARTICLE INFO

Article history:
Received: 19 May 2013;
Received in revised form:
17 June 2013;
Accepted: 4 July 2013;

Keywords
Polynomial, zeros,
Eneström – Kakeya theorem,
Generalizations.

ABSTRACT

In this paper we establish some more generalizations of Eneström – Kakeya theorem by taking the case when the monotonic coefficients are perturbed and hence finding annulus for the polynomial. Besides many consequences our results considerably improve the bounds in some cases as well.

Introduction

The fallowing classical result known as Eneström – Kakeya theorem [5,6] is famous in the theory of distribution of zeros of polynomials.

Theorem **A.** If $P(z)=\sum_{r=0}^{n} a_r z^r$ is a polynomial of degree n ,such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0..$$

then $P(z)$ has all its zeros in the disk $|z| \leq 1$

In the literature attempts have been made to extend and generalize the Enestrom - Kakeya theorem . Joyal, Labelle and Rahman [4] extended it to the polynomials with general monotonic coefficients by showing that, if the coefficients of the polynomial satisfy the condition

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$$

then all the zeros of $P(z)$ are contained in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$$

Aziz and zargar [1] relaxed the hypothesis of Theorem **A** and proved :

Theorem **B.** If $P(z)=\sum_{r=0}^{n} a_r z^r$ a polynomial of degee n such that for some $\lambda \geq 1$,

$$\lambda a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$$

Then $P(z)$ has all its zeros in the disk.

$$|z+\lambda-1| \leq \frac{(a_n - a_0 + |a_0|}{|a_n|}$$

On the other hand Y Choo [2] extended this theorem to the polynomials with perturbed monotonic coefficients, and proved the following:

Theorem C

Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be the nth-order polynomial such that for some λ and k with $1 \leq k \leq n$

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if $a_{n-k-1} \geq a_{n-k}$ then all the zeros of $P(z)$ lie in the disk $R_{11} \leq |z| \leq R_{12}$

where

$$R_{11} = \frac{a_0}{(|a_n| + a_n) + (\lambda - 1)(|a_{n-k}| + a_{n-k}) - a_0}$$

And R_{12} is the positive root of the equation

$$R^{k+2} - \delta_1 R^{k+1} - |\gamma_1| = 0$$

$\gamma_1 = \dfrac{((- 1)a_{n-k}}{a_n}$ and $\delta_1 = \dfrac{a_n + ((- 1)a_{n-k} - a_0 + |a_0|}{|a_n|}$.

If $a_{n-k} \geq a_{n-k+1}$, then all the zeros of P(z) lie in the disk $R_{21} \leq |z| \leq R_{22}$

$$R_{21} = \frac{a_0}{(|a_n| + a_n) + (1 - \lambda)(|a_{n-k}| + a_{n-k}) - a_0}$$

And R_{22} is the positive root of the equation

$$R^{k} - \delta_2 R^{k-1} - |\gamma_2| = 0$$

$\gamma_2 = \dfrac{(1 - ()a_{n-k}}{a_n}$ and $\delta_2 = \dfrac{a_n + (1 - ()a_{n-k} - a_0 + |a_0|}{|a_n|}$.

In this theorem we generalize the result of Y. Choo and prove the fallowing result

Theorem 1

Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be the nth-order polynomial if $t_1 > t_2 \geq 0$ can be found such that for some λ and k with $1 \leq k \leq n$

$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_r \geq 0 \qquad r = 2,3, \ldots n : r \neq n-k$

$\lambda a_{n-k} t_1 t_2 + a_{n-k+1}(t_1 - t_2) - a_{n-k+2} \geq 0$ if $a_{n-k-1} \geq a_{n-k}$

if $a_{n-k-1} \geq a_{n-k}$ then all the zeros of $P(z)$ lie in the disk $R_{11} \leq |z| \leq R_{12}$

where

$$R_{11} = \frac{a_0 t_1 t_2}{(|a_n| + a_n) t_1^{n+2} + (\lambda - 1)(|a_{n-k}| + a_{n-k}) t_1^{n-k+1} - a_0 \ t_1 t_2}$$

And R_{12} is the positive root of the equation

$$R^{k+2} - \delta_1 R^{k+1} - |\gamma_1| = 0$$

$\gamma_1 = \dfrac{((- 1)t_1 t_2 a_{n-k}}{a_n}$ and $\delta_1 = \dfrac{a_n + ((- 1)a_{n-k} \frac{t_2}{t_1^k} - \frac{t_2}{t_1^k} a_0 + \frac{t_2}{t_1^k |a_0|}}{|a_n|}$.

If $a_{n-k} \geq a_{n-k+1}$, then all the zeros of P(z) lie in the disk $R_{21} \leq |z| \leq R_{22}$

$$R_{21} = \frac{a_0 t_1 t_2}{(|a_n| + a_n) t_1^{n+2} + (1 - \lambda)(|a_{n-k}| + a_{n-k}) t_1^{n-k+1} - a_0 \ t_1 t_2}$$

And R_{22} is the positive root of the equation

$$R^k - \delta_2 R^{k-1} - |\gamma_2| = 0$$

$\gamma_2 = \dfrac{(1-()t_1 t_2 a_{n-k}}{a_n}$ and $\delta_2 = \dfrac{a_n + \frac{(1-()1}{t_1^{k+1}} a_{n-k} - \frac{t_2}{t_1^k} a_0 + \frac{t_2}{t_1^k |a_0|}}{|a_n|}$.

Remark 1 if we put $t_1 = t_2 = 1$ we get theorem D

Proof of the proof: Consider a polynomial

$$F(z) = (t_1 + z)(t_2 - z)P(z)$$

$$= \{t_1 t_2 + (t_1 - t_2)z - z^2\}(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)$$

$$= -a_n z^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\} z^{n+1} + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\} z^n$$

$$+ \cdots + \{a_{n-k} t_1 t_2 + a_{n-k+1}(t_1 - t_2) - a_{n-k+2}\} z^{n-k}$$

$$+ \cdots + \{a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0\} z^2 + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\} z + a_0 t_1 t_2$$

if $a_{n-k-1} \geq a_{n-k}$ then $a_{n-k+1} \geq a_{n-k}$ and $F(z)$ can be written as

$$= -a_n z^{n+2} - (\lambda - 1) a_{n-k} t_1 t_2 z^{n-k} + \{a_n(t_1 - t_2) - a_{n-1}\} z^{n+1}$$

$$+ \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\} z^n$$

$$+ \cdots + \{\lambda a_{n-k} t_1 t_2 + a_{n-k-1}(t_1 - t_2) - a_{n-k-2}\} z^{n-k}$$

$$+ \cdots + \{a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0\} z^2 + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\} z + a_0 t_1 t_2$$

For $|z| > t_1$

$|F(z)| \geq |a_n z^{(n+2)} + (\lambda - 1)\, a_{(n-k)}\, t_1 t_2 z^{(n-k)}| - |z|^{(n+1)} \{a_n (t_1 - t_2) - a_{(n-1)} + (a_n t_1 t_2 + a_{(n-1)} (t_1 - t_2) - a_{(n-2)})/|z| +$

$\geq |a_n z^{(n+2)} + (\lambda - 1)\, a_{(n-k)}\, t_1 t_2 z^{(n-k)}| - |z|^{(n+1)} \{a_n (t_1 - t_2) - a_{(n-1)} + a_n t_2 + a_{(n-1)} - a_{(n-2)}\ t_2/t_1 + \cdots + \lambda a_{(n-k)}\ t_2/$

$$\geq |a_n z^{n+2} + (\lambda - 1) a_{n-k} t_1 t_2 z^{n-k}| - |z|^{n+1} \left\{ a_n t_1 + (\lambda - 1) a_{n-k} \frac{t_2}{t_1^k} - a_0 \frac{t_2}{t_1^n} + \frac{|a_0| t_2}{t_1^n} \right\}$$

$$> 0$$

if

$$\left| |z|^{k+2} + \frac{(\lambda - 1) a_{n-k} t_1 t_2}{a_n} \right| > |z|^{k+1} \left\{ \frac{a_n t_1 + (\lambda - 1) a_{n-k} \frac{t_2}{t_1^k} - a_0 \frac{t_2}{t_1^n} + \frac{|a_0| t_2}{t_1^n}}{a_n} \right\}$$

$$||z|^{(k+2)} + (_1 1 | > |z|^{(k+1)}\ \delta_1$$

Where

$\gamma_1 = \dfrac{((-1)t_1 t_2 a_{n-k}}{a_n}$ and $\delta_1 = \dfrac{a_n + ((-1)a_{n-k} \frac{t_2}{t_1^k} - \frac{t_2}{t_1^k} a_0 + \frac{t_2}{t_1^k |a_0|}}{|a_n|}$.

This inequality holds if

$$||z|^{(k+2)} - |(_1 1 | > |z|^{(k+1)}\ \delta_1$$

Hence all the zeros of $P(z)$ with modulus greater than t_1 lie in the disk $|z| \le R_{12}$ where R_{12} is the greatest positive root of the equation.

$$R^{k+2} - \delta_1 R^{k+1} - |\gamma_1| = 0$$

But the zeros of $P(z)$ with modulus less than or equal to t_1 are already contained in the disk $|z| \le R_{12}$

For the inner bound consider the polynomial

$$F(z) = (t_1 + z)(t_2 - z)P(z)$$

$$= \{t_1 t_2 + (t_1 - t_2)z - z^2\}(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)$$

$$= -a_n z^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\}z^{n+1} + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\}z^n$$

$$+ \cdots + \{a_{n-k} t_1 t_2 + a_{n-k+1}(t_1 - t_2) - a_{n-k+2}\}z^{n-k}$$

$$+ \cdots + \{a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0\}z^2 + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\}z + a_0 t_1 t_2$$

$$= a_0 t_1 t_2 + f(z)$$

where

$$f(z) = -a_n z^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\}z^{n+1} + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\}z^n$$

$$+ \cdots + \{a_{n-k} t_1 t_2 + a_{n-k+1}(t_1 - t_2) - a_{n-k+2}\}z^{n-k}$$

$$+ \cdots + \{a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0\}z^2 + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\}z$$

If $|z| < t_1$

$$|f(z)| \le |a_n| t_1^{n+2} + (\lambda - 1)t_1^{n-k+1} t_2 |a_{n-k}| + a_n t_1^{n+2} - a_n t_1^{n+1} t_2 + a_{n-1} t_1^{n+1} - a_{n-1} t_1^n t_2 -$$

$$-a_{n-2} t_1^n + \cdots + \lambda a_{n-k} t_1^{n-k+1} t_2 + a_{n-k-2} t_1^{n-k+1} t_2 - a_{n-k-2} t_1^{n-k+1} t_2 - a_{n-k-2} t_1^{n-k} + \cdots$$

$$+ \quad a_2 t_1^3 t_2 + a_1 t_1^3 - a_1 t_1^2 t_2 - a_0 t_1^2 + a_1 t_1^2 t_2 + a_0 t_1^2 - a_0 t_1 t_2$$

$$|f(z)| \le |a_n| t_1^{n+2} + (\lambda - 1)t_1^{n-k+1} t_2 |a_{n-k}| + a_n t_1^{n+2} + (\lambda - 1[\square)a]_{n-k} t_1^{n-k+1} t_2 - a_0 t_1 t_2$$

Since $f(0) = 0$ it follows from Schwarz lemma that

$$|f(z)| \le \{(|a_n| + a_n[\square)t]_1^{n+2} + (\lambda - 1)t_1^{n-k+1} t_2(|a_{n-k}| + a_{n-k}) - a_0 t_1 t_2\}|z|: \text{ for } |z| < t_1$$

Hence

$$|P(z)| \ge a_0 t_1 t_2 - |f(z)|$$

$$\ge a_0 t_1 t_2 - \{(|a_n| + a_n[\square)t]_1^{n+2} + (\lambda - 1)t_1^{n-k+1} t_2(|a_{n-k}| + a_{n-k}) - a_0 t_1 t_2\}|z|: \text{ for } |z| < t_1$$

□□□□□□□0

$$|z| < \frac{a_0 t_1 t_2}{(|a_n| + a_n[\square)t]_1^{n+2} + (\lambda - 1)t_1^{n-k+1} t_2(|a_{n-k}| + a_{n-k}) - a_0 t_1 t_2} = R_{11}$$

Consequently $P(z)$ does not vanish in $|z| < R_{11}$
. Since $R_{11} < t_1$, the first part of the theorem is proved.

The second part can be proved similarly if $a_{n-k} \ge a_{n-k+1}$ then $a_{n-k} \ge a_{n-k+1}$ and $F(z)$ can be written as

$$F(z) = -a_n z^{n+2} - (1 - \lambda)a_{n-k} t_1 t_2 z^{n-k+2} + \{a_n(t_1 - t_2) - a_{n-1}\}z^{n+1}$$

$$+\{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\}z^n$$

$$+\cdots+\{a_{n-k+2}t_1 t_2 + a_{n-k+1}(t_1 - t_2) - \lambda a_{n-k}\}z^{n-k}$$

$$+\cdots+\{a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0\}z^2 + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\}z + a_0 t_1 t_2$$

For $|z| > t_1$

$$\geq \left|a_n z^{n+2} + (1-\lambda)a_{n-k}t_1 t_2 z^{n-k+2}\right| - |z|^{n+1}\left\{a_n(t_1 - t_2) - a_{n-1} + a_n t_2 + a_{n-1} - a_{n-2}\frac{t_2}{t_1} + \cdots + \lambda a_{n-k}\frac{1}{t_1^{k+1}} + \cdots + a_1\frac{1}{t_1^{n-1}} - a_0\frac{t_2}{t_1^n} + \frac{|a_0|t_2}{t_1^n}\right\}$$

$$\geq \left|a_n z^{n+2} + (\lambda - 1)a_{n-k}t_1 t_2 z^{n-k}\right| - |z|^{n+1}\left\{a_n t_1 + (1-\lambda)a_{n-k}\frac{1}{t_1^{k+1}} - a_0\frac{t_2}{t_1^n} + \frac{|a_0|t_2}{t_1^n}\right\}$$

$$> 0$$

if

$$\left||z|^{k+2} + \frac{(1-\lambda)a_{n-k}t_1 t_2}{a_n}\right| > |z|^{k+1}\left\{\frac{a_n t_1 + (1-\lambda)a_{n-k}\frac{1}{t_1^{k+1}} - a_0\frac{t_2}{t_1^n} + \frac{|a_0|t_2}{t_1^n}}{a_n}\right\}$$

$$||z|↑k + (ı2\,| > |z|↑(k-1)\ \delta_2$$

Where

$\gamma_2 = \dfrac{(1-()t_1 t_2 a_{n-k}}{a_n}$ and $\delta_2 = \dfrac{a_n + \frac{(1-()1}{t_1^{k+1}}a_{n-k} - \frac{t_2}{t_1^k}a_0 + \frac{t_2}{t_1^k|a_0|}}{|a_n|}$

This inequality holds if

$$||z|↑k - |(ı2\,| > |z|↑(k-1)\ \delta_2$$

Hence all the zeros of $P(z)$ with modulus greater than t_1 lie in the disk $|z| \leq R_{22}$ where R_{22} is the greatest positive root of the equation.

$$R^k - \delta_2 R^{k-1} - |\gamma_2| = 0$$

But the zeros of $P(z)$ with modulus less than or equal to t_1 are already contained in the disk $|z| \leq R_{22}$

For the inner bound consider the polynomial

$$F(z) = (t_1 + z)(t_2 - z)P(z)$$

$$= \{t_1 t_2 + (t_1 - t_2)z - z^2\}(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$$

$$= -a_n z^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\}z^{n+1} + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\}z^n$$

$$+\cdots+\{a_{n-k}t_1 t_2 + a_{n-k+1}(t_1 - t_2) - a_{n-k+2}\}z^{n-k}$$

$$+\cdots+\{a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0\}z^2 + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\}z + a_0 t_1 t_2$$

$$= a_0 t_1 t_2 + f(z)$$

where

$$f(z) = -a_n z^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\}z^{n+1} + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\}z^n$$

$$+\cdots+\{a_{n-k}t_1t_2+a_{n-k+1}(t_1-t_2)-a_{n-k+2}\}z^{n-k}$$

$$+\cdots+\{a_2t_1t_2+a_1(t_1-t_2)-a_0\}z^2+\{a_1t_1t_2+a_0(t_1-t_2)\}z$$

If $|z|<t_1$

$$|f(z)| \le |a_n|t_1^{n+2}+(1-\lambda)t_1^{n-k+1}|a_{n-k}|+a_nt_1^{n+2}-a_nt_1^{n+1}t_2+a_{n-1}t_1^{n+1}-a_{n-1}t_1^nt_2-$$

$$-a_{n-2}t_1^n+\cdots+\lambda a_{n-k}t_1^{n-k+2}+a_{n-k-2}t_1^{n-k+1}t_2-a_{n-k-2}t_1^{n-k+1}t_2-a_{n-k-2}t_1^{n-k}+\cdots$$

$$+\quad a_2t_1^3t_2+a_1t_1^3-a_1t_1^2t_2-a_0t_1^2+a_1t_1^2t_2+a_0t_1^2-a_0t_1t_2$$

$$|f(z)| \le |a_n|t_1^{n+2}+(1-\lambda)t_1^{n-k+1}|a_{n-k}|+a_nt_1^{n+2}+(1-\lambda)a_{n-k}t_1^{n-k+2}-a_0t_1t_2$$

Since $f(0)=0$ it follows from Schwarz lemma that

$$|f(z)| \le \{(|a_n|+a_n)t_1^{n+2}+(1-\lambda)t_1^{n-k+2}(|a_{n-k}|+a_{n-k})-a_0t_1t_2\}|z|: \text{ for } |z|<t_1$$

Hence

$$|P(z)| \ge a_0t_1t_2-|f(z)|$$

$$\ge a_0t_1t_2-\{(|a_n|+a_n)t_1^{n+2}+(1-\lambda)t_1^{n-k+2}(|a_{n-k}|+a_{n-k})-a_0t_1t_2\}|z|: \text{ for } |z|<t_1$$

□□□□□□0

$$|z|<\frac{a_0t_1t_2}{(|a_n|+a_n)t_1^{n+2}+(1-\lambda)(|a_{n-k}|+a_{n-k})t_1^{n-k+2}-a_0\ t_1t_2}=R_{21}$$

Consequently $P(z)$ does not vanish in $|z|<R_{21}$. Since $R_{21}<t_1$, the second part of the theorem is proved.

References

[1]A.Aziz and B. A.Zargar, Some extensions of Eneström - Kakeya theorem, Glasnik Matematički **31**(1996),239-244

[2]Y.Choo, On the Generalizations of Eneström – Kakeya theorem,Int.Journal of Math. Analysis,5(**20)(**2011),983-995.

[3] N.K.Govil and Q.I.Rahman, On the Eneström - Kakeya theorem,Tohoku Math.J ,**20** (1968),126-136

[4] A.Joyal,G.Labelle and Q.I.Rahman,On the location of zeros of polynomials, Canad. Math.Bull.,**10**(1967), 55-63

[5]M.Marden, Geometry of polynomials,Math.Surveys,No.3;Amer.Math.Society, (R.I:Providence) ,(1966)

[6] Q .I . Rahman and G.Schmeisser, Analytic theory of polynomials, Oxford University Press,New York,(2002).