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On pathos adjacency line graph of a binary tree

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ABSTRACT

In this communication, the concept of Pathos Adjacency Line graph *PAL(T)* of a binary tree *T* is introduced. We decompose *PAL(T)* of *T* in to an edge disjoint complete bipartite subgraphs and then give the reconstruction of *T*. We also present a characterization of those graphs whose pathos adjacency line graphs are planar, outerplanar, maximal outerplanar, minimally non-outerplanar and Eulerian.

Introduction

The line graph of a graph *G*, denoted by *L(G)*, is the graph whose vertices are the edges of *G* with two vertices of *L(G)* are adjacent whenever the corresponding edges of *G* are adjacent. The concept of pathos of a graph *G* was introduced by Harary [1], as a collection of minimum number of edge disjoint open paths whose union is *G*. The path number of a graph *G* is the number of paths in pathos. A Binary Tree *T* is a tree in which each vertex has at most two children. The order (size) of *T* is the number of vertices (edges) in it. The path number of a binary tree *T* is equal to α, where *2α* is the number of odd degree vertices of *T*. The edge degree of an edge *uv* of a binary tree *T* is the sum of the degrees of *u* and *v* [2]. A graph *G* is planar if it can be drawn in the plane in such a way that any intersection of two distinct edges occurs only at a vertex of the graphs. A graph *G* is called outerplanar if *G* has an embedding in the plane in such a way that each vertex bounds the infinite face. An outerplanar graph *G* is maximal outerplanar if no edge can be added without losing its outer planarity. If *G* is a planar graph, then the inner vertex number $i(G)$ of a graph *G* is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of *G* in the plane [3]. A graph is said to be minimally non-outerplanar if $i(G)$=1[4]. All other undefined terminology will confirm with that in [5].

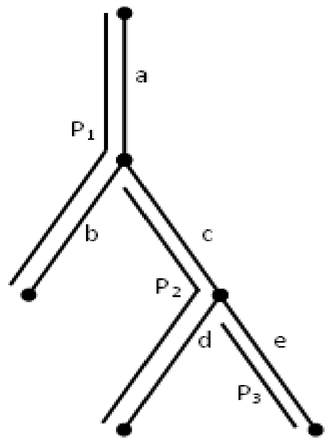
The pathos adjacency line graph *PAL(T)* of a binary tree *T* is defined as a graph, in which;

a) *V(PAL(T))* is the union of the set of edges and paths of pathos of *T* in which two vertices are adjacent if and only if the corresponding edges of *T* are adjacent and edges lies on the corresponding path P_i of pathos.

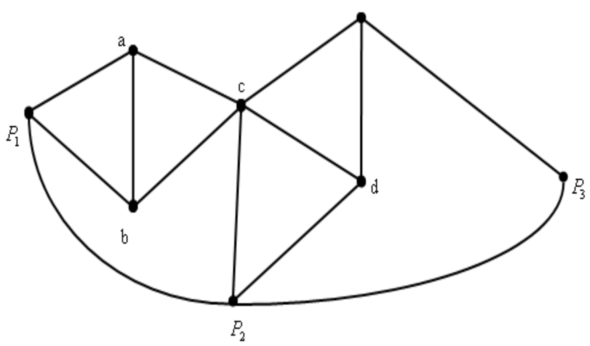
b) With reference to the binary tree *T*, the pathos vertex $P_m(v_i, v_j)$ is adjacent to $P_n(v_k, v_l)$ in *PAL(T)* if and only if the pathos P_m and P_n have the common vertex v_c in *T* such that there exists an edge between v_i and v_l through v_c.

Since the system of pathos for a binary tree is not unique, the corresponding pathos adjacency line graph is also not unique. In the following figure, *T* is a binary tree and its *PAL(T)* is shown.

Binary Tree T:



PAL(T):



We need the following results to prove further results:

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Theorem [A][2]: The pathos line graph *PL(T)* of a tree *T* is maximal outerplanar if and only if *T* is a Path.

Theorem [B][5]: If *G* is a *(p, q)* graph whose vertices have degree d_i, then *L(G)* has *q* vertices and q_L edges, where $q_L = -q + \frac{1}{2}\sum d_i^{\,2}$.

Theorem [C][5]: A connected graph *G* is Eulerian if and only if each vertex in *G* has even degree.

Theorem [D][5]: A graph G is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$.

***PAL(T)* Decomposition and Reconstruction of T:**

We recall a graph a complete bipartite (m,n)-graph on m+n vertices is the simple graph $G = K_{m,n}$, where

$$V(K_{m,n}) = \{u_1, u_2, \ldots\ldots, u_m\} \cup \{v_1, v_2, \ldots\ldots, v_n\},$$
$$E(K_{m,n}) = \{\{u_i, v_j\}; 1 \le i \le m, 1 \le j \le n\}.$$

that is, the vertices are of two types, where every pair of vertices of different types are adjacent, but no two vertices of the same type are adjacent. We have the following cases.

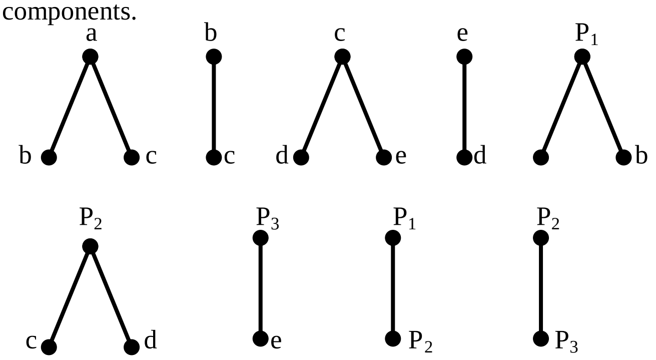
Case 1: Here we decompose *L(T)* of *PAL(T)* into an edge disjoint complete bipartite subgraphs as follows. Let $\{v_1, v_2, \ldots\ldots, v_n\}$ are the vertices of subgraph *L(T)* of *PAL(T).* Then two vertices $v_i, v_j (v_i \ne v_j)$ are adjacent in *PAL(T)* decomposition if they are adjacent in *L(T)* such that they not form a cycle in *L(T)* and the edge forming a cycle in *L(T)* is taken as a separate component.

Case 2: The pathos vertex corresponding to pathos of T and vertices corresponding to edges of T are adjacent in *PAL(T)* decomposition if they are adjacent in *PAL(T).*

Case 3: Two pathos vertices P_i , P_j($P_i \ne P_j$) are adjacent in *PAL(T)* decomposition if they are adjacent in *PAL(T).* Hence, *PAL(T)* consists of mutually edge disjoint complete bipartite subgraphs.

Conversely, let T' be the graph of the type which is described above. We can construct the binary tree *T* which has T' as its pathos adjacency line graph as follows.

We first consider the line graph *L(T)* decomposition of *PAL(T).* Let the vertices $\{v_1', v_2', \ldots\ldots, v_n'\}$ of *L(T)* decomposition are the edges in T. Then if two vertices $v_i', v_j' (v_i' \ne v_j')$ are adjacent in decomposition, then the corresponding edges v_i', v_j' are adjacent in *T*. We finally consider the *PAL(T)* decomposition components $L_i's$ having vertices corresponding to pathos and edges of T. Then, draw the pathos in T along each of the pendant vertices corresponding to edges of T in decomposition components.



Hence, we have the theorem.

Theorem 1: *PAL(T)* is the pathos adjacency line graph of some binary tree *T* if and only if *PAL(T)* can be partitioned into mutually edge disjoint complete bipartite subgraphs.

Theorem 2: If *G* is a *(p, q)* graph, where the vertices have the degree d_i, then its *PAL(T)* has (q+ k) vertices and $\left|q_{PAL(T)}\right| = \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2} + (P-1)$, where k being path number, *P* is the number of pathos in *T*.

Proof: By the definition, the number of vertices in *PAL(T)* is (q+ k).

The number of edges in *PAL(T)* is the sum of edges in *L(G)*, the number of edges lies on the path P_i of pathos of *G* which is *q* and the number of adjacency of pathos of *G* which is *(P-1).*

Hence, by Theorem [B] we have,

$$\left|q_{PAL(T)}\right| = -q + \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2} + q + (P-1)$$

$$\Rightarrow \left|q_{PAL(T)}\right| = \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2} + (P-1)$$

Theorem 3: For any binary tree *T* on $p \ge 2$ vertices, *PAL(T)* is always a planar.

Proof: Since $\Delta(T) \le 3$, We have the following cases.

Case 1: Suppose *T* is K_2. Then *PAL(T)* is K_2, which is a planar.

Case 2: Suppose *T* is a path P_n on $n > 2$ vertices. Then it forms a path of length *(n-2)* in *L(T).* Also T has exactly one path of pathos. The edges joining vertices of *L(T)* from the corresponding pathos vertex gives *PAL(T)* in which each block is K_3. Hence *PAL(T)* is planar.

Case 3: Suppose T is not a path P_n. Then there exists at least one vertex of degree three in T. Then each block of *L(T)* is either K_2 or K_3. The edges joining these blocks from the pathos vertices are adjacent to at most two vertices of each block of *L(T).* This gives a planar *PAL(T).*

Theorem 4: Pathos adjacency line graph *PAL(T)* of a binary tree *T* is an outerplanar if and only if $\Delta(T) \le 2$.

Proof: Suppose *PAL(T)* is an outerplanar. Assume that T has a vertex of degree 3. Then *L(T)* has only one block as K_3. The number of path of pathos in T is exactly two. The edges joining vertices of K_3 from the pathos vertices are adjacent to at most two vertices of K_3. Finally, the adjacency of pathos vertices gives *PAL(T)* such that $i(PAL(T)) \ge 1$, a contradiction.

Conversely, suppose every vertex *p* of T lies on at most two edges. Then, each edge in *T* lies on exactly one path of pathos. By definition, it forms a path of length exactly *(p-2)* in *L(T).* The edges joining vertices of *L(T)* from the pathos vertex gives *PAL(T)* in which each region is a triangle. By Theorem [D], *PAL(T)* is an outerplanar.

Theorem 5: Pathos adjacency line graph *PAL(T)* of a binary tree *T* is maximal outerplanar if and only if $\Delta(T) \le 2$.

Proof: Suppose *PAL(T)* is maximal outerplanar. Then, *PAL(T)* is connected.

Hence, *T* is connected. If *PAL(T)* is K_2, then T is also K_2. Let *T* be a connected binary tree with $p \geq 2$ vertices, *q*-edges and the path number *k*. Then, *PAL(T)* has *(q+ k)* vertices.

Since each edge in *T* lies on exactly one path of pathos, *P-1=0*. Hence, by Theorem [2], $\left|q_{PAL(T)}\right| = \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2}$ Then, by Theorem [A], *PAL(T)* is maximal outerplanar.

Theorem 6: Pathos adjacency line graph *PAL(T)* of a binary tree *T* is minimally non-outerplanar if and only if *T* has unique vertex of degree 3.

Proof: Suppose *PAL(T)* is minimally non-outerplanar. Assume that there exists at least two vertices of degree 3 in T. Then each block in *L(T)* is either K_2 or K_3. Any pathos vertex of *T* is adjacent to at most two vertices of each block of *L(T)*. Also, the adjacency of pathos vertices gives *PAL(T)* such that $i(PAL(T)) > 1,$ a contradiction.

Conversely, suppose *T* has a unique vertex of degree three. Then *L(T)* has only one block as K_3. Also, the number of path of pathos in T is exactly two. Each pathos vertex is adjacent to at most two vertices of K_3. Finally, the adjacency of pathos vertices gives *PAL(T)* such that $i(PAL(T)) = 1.$ Hence, *PAL(T)* is minimally non-outerplanar.

Theorem 7: Pathos adjacency line graph *PAL(T)* of a binary tree *T* is Eulerian if and only if *T* is a path P_n on *n=3* vertices.

Proof: Suppose *PAL(T)* is Eulerian. Assume that T is a path P_n on $n \geq 4$ vertices. Then it forms a path of length *(n-2)* in *L(T)* in which degree of each vertex except end vertices is two. Also, T has exactly one path of pathos. The edges joining vertices of *L(T)* from the pathos vertex increases the degree of even vertices of *L(T)* by one in *PAL(T).*

By Theorem [C], *PAL(T)* is non-Eulerian, a contradiction.

Conversely, suppose T is a path P_n on *n=3* vertices. Then each edge in T lies on exactly one pathos. By the definition of *L(T)* it forms K_2.The edges joining vertices of K_2 from the pathos vertex forms K_3 in *PAL(T)*. Then, by Theorem [C], *PAL(T)* is Eulerian.

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