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The Hahn sequence space of fuzzy numbers defined by Orlicz function

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ABSTRACT
In this paper the Orlicz space of Hahn sequence of fuzzy numbers is introduced. Some properties of this sequence space like solidness, symmetricity, convergence free are studied. Some inclusive relations involving this sequence space are also obtained.

1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by zadeh [14] and subsequently several authors have discussed various aspects of theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers was introduced by Matloka [6]. Nandha [8] has studied the space of all absolutely p-summable convergent sequences of fuzzy numbers and shown that they are all complete metric spaces. Later on sequence of fuzzy numbers have been discussed by Dutta [4]Murseleen [7], Nuray and savas [9] ,Talo and Basar[10] and many others.

Chandrasekhara Rao[2] introduced Hahn sequence space. It is defined as follows.

$$h = \left\{ x = (x_k) : \sum_k k\,|x_k - x_{k-1}|\; converges\ and\ \lim_{k\to\infty} x_k = 0 \right\}$$

The above space is a Banach space normed by

$$\|x\| = \sum_{k=1}^{\infty} k\,|x_k - x_{k+1}|$$

This space was further developed by chandrasekhara Rao and N.Subramanian[3].

Lindenstrauss and Tzafriri [5] investigated Orlicz sequence spaces in more detail. An Orlicz function is a function M : $[0, \infty] \to [0, \infty]$ which is continuous, non-decreasing and convex with M(0) = 0, M(x) > 0, for x > 0 and $M(x) \to \infty$ as $x \to \infty$. If the convexity of the Orlicz function is replaced by sub-additivity then this function is called modulus function. An Orlicz function satisfies the inequality $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Sargent [12] introduced the space m (ϕ) and studied some properties of this space. Later on it was studied from sequence point of view and some matrix classes were characterized by Rath and Tripathy [11] and others. In this article we introduce the space $h_F(M, \phi, p)$ of sequences of fuzzy numbers defined by orlicz function.

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2. Definitions and preliminaries

We begin with giving some required definitions and statements of theorems, propositions and lemmas. A fuzzy number is a fuzzy set on the real axis i.e. a mapping u:R $\to$ [0,1] which satisfies the following four conditions.

(i) u is normal i.e. there exists an $x_0 \in R$ such that u (x_0) =1

ii) u is fuzzy convex i.e. u $[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all x, y $\in$ R and for all $\lambda \in [0,1]$

iii) u is upper semi continuous

iv) The set $[u]_0 = \overline{\{x \in R : u(x) > 0\}}$ is compact (Zadeh [1]) where $\overline{\{x \in R : u(x) > 0\}}$ denotes the closure of the set $\{x \in R : u(t) > 0\}$ in the usual topology of R. We denote the set of all fuzzy numbers on R by E′ and called it as the space of fuzzy numbers. The λ -level set $[u]_\lambda$ of u $\in$ E′ is defined by $[u]_\lambda = \begin{cases} \{t \in R : u(t) \geq \lambda\}, & (0 < \lambda \leq 1) \\ \overline{\{t \in R : u(t) > \lambda\}}, & \lambda = 0 \end{cases}$

The set $[u]_\lambda$ is a closed bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by

$[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$

$\mathbb{R}$ can be embedded in E′. Since each r $\in \mathbb{R}$ can be regarded as a fuzzy number $\bar{r}$ defined by

$$\bar{r}(x) = \begin{cases} 1, & (x = r) \\ 0, & (x \neq r) \end{cases}$$

Let u, v, w $\in$ E′ and k $\in \mathbb{R}$ .The operations addition , scalar multiplication and product defined on E′ by

$u + v = w \Leftrightarrow [w]_\lambda = [u]_\lambda + [v]_\lambda$ for all $\lambda \in [0, 1]$

$\Leftrightarrow w^-(\lambda) = [u^-(\lambda), v^-(\lambda)]$ and $w^+(\lambda) = [u^+(\lambda), v^+(\lambda)]$ and for all $\lambda \in [0, 1]$

$[ku]_\lambda = k[u]_\lambda$ for all $\lambda \in [0, 1]$

and

$u v = w \Leftrightarrow [w]_\lambda = [u]_\lambda [v]_\lambda$ for all $\lambda \in [0, 1]$

where it is immediate that ,

$$w^-(\lambda) = \min\{u^-(\lambda)v^-(\lambda),\ u^-(\lambda)v^+(\lambda),\ u^+(\lambda)v^-(\lambda),\ u^+(\lambda)v^+(\lambda)\}$$

and

$$w^+(\lambda) = \max\{u^-(\lambda)v^-(\lambda),\ u^-(\lambda)v^+(\lambda),\ u^+(\lambda)v^-(\lambda),\ u^+(\lambda)v^+(\lambda)\}$$

for all $\lambda \in [0, 1]$.

$$u/v = w \Leftrightarrow [w]_\alpha = [u]_\alpha / [v]_\alpha \ \text{for all } \alpha \in [0,1]$$

$$= [u^-(\alpha), u^+(\alpha)] \bullet \left[\frac{1}{v^-(\alpha)}, \frac{1}{v^+(\alpha)}\right]$$

$$= \min\left[\left\{\frac{[u]^-(\alpha)}{[v]^+(\alpha)}, \frac{[u]^-(\alpha)}{[v]^-(\alpha)}, \frac{[u]^+(\alpha)}{[v]^+(\alpha)}, \frac{[u]^+(\alpha)}{[v]^-(\alpha)}\right\}, \max\left\{\frac{[u]^-(\alpha)}{[v]^+(\alpha)}, \frac{[u]^-(\alpha)}{[v]^-(\alpha)}, \frac{[u]^+(\alpha)}{[v]^+(\alpha)}, \frac{[u]^+(\alpha)}{[v]^-(\alpha)}\right\}\right]$$

Let W be the set of all closed and bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$ ie) A= $[\underline{A}, \overline{A}]$

Define the relation d on W by

$d(A, B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}$

Then it can be observed that d is a metric on W and (W, d) is a complete metric space (Nanda [8]) . Now we can define the metric D on E′ by means of a Hausdroff metric d as

$D(u, v) = \sup d([u]_\lambda, [v]_\lambda) = \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}$

(E’, D) is a complete metric space .One can extend the natural order relation on the real line to intervals as follows .

$A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$

The partial order relation on E′ is defined as follows.

$u \leq v \Leftrightarrow [u]_\lambda \leq [v]_\lambda \Leftrightarrow u^-(\lambda) \leq v^-(\lambda)$ *and* $u^+(\lambda) \leq v^+(\lambda)$ for all $\lambda \in [0, 1]$

An absolute value $|u|$ of a fuzzy number u is defined by

$$|u|(t) = \begin{cases} \max\{u(t), u(-t)\}, & (t \geq 0) \\ 0, & (t < 0) \end{cases}$$

λ -level set $[\![u]\!]_\lambda$ of the absolute value of u$\in$E′ is in the form$[\![u]\!]_\lambda$
where

$$|u|^-(\lambda) = \max\{0, u^-(\lambda), -u^+(\lambda)\}$$
$$|u|^+(\lambda)) = \max\{|u^-(\lambda)|, |u^+(\lambda)|\}$$

The absolute value $|uv|$ of u , v $\in$ E′ satisfies the inequalities (Talo [8])

$$|uv|^-(\lambda) \le |uv|^+(\lambda) \le \max\{|u^-(\lambda)|\,|v^-(\lambda)|, \quad |u^-(\lambda)|\,|v^+(\lambda)|, \quad |u^+(\lambda)|\,|v^-(\lambda)|, \quad |u^+(\lambda)|\,|v^+(\lambda)|\}$$

u$\in$ E′ is a non-negative fuzzy number if and only if u(x)=0 for all x<0. It is immediate that u$\ge$0 if u is a non negative fuzzy number.
One can see that

$$D(u,\bar{0}) = \sup_{\lambda\in[0,1]} \max\{|u^-(\lambda)|, |u^+(\lambda)|\} = \max\{|u^-(\lambda)|, |u^+(\lambda)|\}.$$

Proposition 2.1 Let u,v,w$\in$E′ and k$\in$R . Then
(i) (E′, D) Is a complete Metric space
(ii) D(ku,kv)=|k| D(u,v)
(iii) D(u+v,w+v)=D(u,w)
(iv) D(u+v,w+z)$\le$D(u,w)+D(v,z)
(v) $|D(u,\bar{0}) - D(v,\bar{0})| \le D(u,v) \le D(u,\bar{0}) + D(v,\bar{0})$.

Lemma 2.2 The following statements hold (Talo [10])
(i) $D(uv,\bar{0}) \le D(u,\bar{0}) D(v,\bar{0})$ for all u , v $\in$ E′
(ii) If $u_k \to u$ as $k\to\infty$ then D (u_k, $\bar{0}$) $\to$ D (u , 0) as $k\to\infty$ where (u_k) $\in$ w (F) .
In the sequel, we require the following Definitions and lemmas.

Definition 2.3 A sequence u = (u_k) of fuzzy numbers is a function u from the set N into the set E′ .The fuzzy number u_k denotes the value of the function at k$\in$ ℕ and is called the k^{th} term of the sequence. Let w(F) denote the set of all sequences of fuzzy numbers.

Definition 2.4 A sequence (u_k) $\in \omega(F)$ is called convergent with limit u$\in$E′ if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in$ N such that D (u_k ,u) < ε for all k$\ge$ n_0 .

Theorem 2.5 [7]. Let (u_k),(v_k) $\in$ w (F) with $u_k \to a, v_k \to b$ as $k \to \infty$.Then,

i. $u_k + v_k \to a + b$ as $k \to \infty$
ii. $u_k - v_k \to a - b$ as $k \to \infty$
iii. $u_k v_k \to ab$ as $k \to \infty$
. iv. $u_k\ u_k / v_k \to a/b$ as $k \to \infty$ where $0 \notin [v_k]_0$ for all $k \in N$ and $0 \notin [b]_0$.

Definition 2.6 A sequence (u_k) $\in$ w(F) is called bounded if and only if the set of all fuzzy numbers consisting of the terms of the sequence (u_k) is a bounded set.
That is to say that a sequences (u_k) $\in$w(F) is said to be bounded if and only if there exist two fuzzy numbers m and M such that m$\le$ u_k $\le$M for all k$\in$N .

Definition 2.7 Let u = (u_k) be a sequence then s(u) denotes the set of all permutations of the elements of (u_k).
i.e. s(u) = {($u_{\pi(k)}$): π is a permutation of N }. A sequence space E is said to be symmetric if s(u) $\subset$ E for all u$\in$E
A sequence space E is said to be monotone if E contains the canonical pre-images of all its step-spaces

Lemma 1. A sequence space E is monotone whenever it is solid.
Let $\wp_s$ be the class of all subsets of N those do not contain more than s number of elements. Throughout {ϕ_s} is a non-decreasing sequence of positive real numbers such that $n\phi_{n+1} \le (n+1)\phi_n$ for all n$\in$N.
The space m (ϕ) introduced by sargent [12] is defined as,

$$m(\phi) = \left\{(x_k) \in w : \| x \|_{m(\phi)} = \sup_{s\ge1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} | x_k | < \infty\right\}$$

Lindenstrauss and Tzafrir [5] used the notion of Olicz function and introduced the sequence space

$$\ell_M = \left\{(x_k) \in w : \sum_{k=1}^{\infty} m\left(\frac{| x_k |}{\rho}\right) < \infty \ \ for\ some\ \rho > 0 \right\}$$

The space ℓ_M becomes a Banachspace with the norm defined by

$$||(x_k)|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} m\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $m(x) = x^p$, $1 \le p < \infty$ In [1]T.Balasubramanian and A.Pandiarani introduced Hahn sequence space of fuzzy numbers.It was defined as follows

Let A denote the matrix $A = (a_{nk})$ defined by

$$a_{n_k} = \begin{cases} n(-1)^{n-k}, & n-1 \le k \le n \\ 0 & 1 \le k \le n-1 \quad or\ k > n \end{cases}$$

Define the sequence $y = (y_k)$ which will be frequently used as the A–transform of a sequence $x = (x_k)$

ie) $y_k = (Ax)_k = k (x_k - x_{k-1})\ k \ge 1$

We introduce the sequence spaces h (F) as the set of all sequences such that the A – transforms of them are in ℓ (F) that is

$$h(F) = \left\{ u = (u_k) \in w(F) : \sum_{k=1}^{\infty} D\left[(Au)_k, \bar{0}\right] < \infty \ and \ \lim_{k\to\infty} D[u_k, \bar{0}] = 0 \right\}$$

In this article we introduce the following sequence space.

$$h_F (M, \phi, p) = \left\{ u = (u_k) \in w(F) : \sup_{s\ge1,\, \sigma\in\rho_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[M\left(\frac{D[Au_k, \bar{0}]}{\rho}\right)\right]^{\rho} < \infty \right\}$$

for some $\rho>0$ and $\left\{ \lim_{k\to\infty} D[u_k, \bar{0}] = 0 \right\}, 1 \le p < \infty$

3. Main Results

Therorem 3.1 The set h_F (M, ϕ, p) is a complete metric space with the metric

$$g (u, v) = \sup_k D(u_k, v_k) + \inf \left\{ \rho > 0 : \sup_{s\ge1,\, \sigma\in\rho_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[M\left(\frac{D[(Au)_k (Av)_k]}{\rho}\right)\right]^{\rho} \le 1 \right\}$$

for some $\rho > 0$, $0 < p < \infty$ and $u, v \in h_F (M, \phi, p)$

Proof It is easy to show that $h_F(M, \phi, p)$ is a metric space with the metric g Let $(u^{(i)})$ be a Cauchy sequence in h_F (M, ϕ, p) such that $u^{(i)} = (u_n^{(i)})_{n=1}^{\infty}$

Let $\varepsilon > 0$ be given. For a fixed $x_0>0$, choose $r > 0$ such that $M\left(\frac{rx_0}{2}\right) \ge 1$. Then there exists a positive integer $n_o(\varepsilon)$ such that $g(u^{(i)}, u^{(j)}) < \frac{\varepsilon}{r(x_0)}$ for all i, j $\ge n_o$

By the definition of g, we get

$$\sup_n D(u_n^{(i)}, v_n^{(i)}) + \inf \left\{ \rho > 0 : Sup_{s\ge1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[M\left(\frac{D[(Au)_k^{(i)} (Au)_k^{(j)}]}{\rho}\right)\right]^{\rho} \le 1 \right\} < \varepsilon \text{ for all i, j} \ge n_o \qquad (3.1)$$

Which implies that $\sup_n D(u_n^{(i)}, v_n^{(i)}) < \varepsilon$ for all i, j $\ge n_0$ and we get

$$D(u_n^{(i)}, v_n^{(i)}) < \varepsilon \text{ for all i, j} \ge n_o,\ n = 1, 2, 3, \ldots\ldots\ldots\ldots \qquad (3.2)$$

Hence $\sup_n (u_n^{(i)})$ is a Cauchy sequence in E'. So it is convergent in E'

Let $\lim_{i\to\infty} k_n^{(i)} = u_n$, for n = 1, 2, 3, …………

$$\text{Now} \quad \sup_{s\ge1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[M\left(\frac{D[(Au)_k^{(i)}, (Au)_k^{(j)}]}{\rho}\right)\right]^{p} \le 1 \quad \text{for all i, j} \ge n_0 \qquad (3.3)$$

For s = 1 and σ varying over $\wp_s$ we get

$$\sum_{k\in\sigma} \left[M\left(\frac{D[(Au)_k^{(i)}, (Au)_k^{(j)}]}{g(u^{(i)}, u^{(j)})}\right)\right]^{p} \le \phi \quad \text{, for all i, j} \ge n_0$$

Which implies $M\left[\frac{D[(Au)_k^{(i)},(Au)_k^{(j)}]}{g(u^{(i)},u^{(j)})}\right] \le \phi^{1/p} \le M\left(\frac{rx_0}{2}\right)$ for all i,j≥n_0

using the continuity of M, we get

$$D[(Au)_k^{(i)},(Au)_k^{(j)}] \le \left(\frac{rx_0}{2}\right) g(u^{(i)},u^{(j)}) \quad \text{for all } i,j \ge n_o$$

$$\Rightarrow \quad D[(Au)_k^{(i)},(Au)_k^{(j)}] < \frac{rx_0}{2}\cdot\frac{\varepsilon}{rx_0} = \varepsilon/2 \quad \text{for all } i,j \ge n_o$$

Which implies that $\left(Au_k^{(i)}\right)$ is a Cauchy sequence in E' .Since E' is complete, it is convergent.Let $\lim_i Au_k^{(i)} = Au_k \in E'$ for each k∈N. We have to prove $\lim_i u^i = u$ and u∈$h_F(M, \phi, p)$

Using the continuity of M, we get from (3.3)

$$\sup_{s\ge1,\,\sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma}\left[M\left(\frac{D(Au_k^{(i)},Au_k)}{\rho}\right)\right]^p \le 1$$

For some ρ > 0 and i ≥ n_o.

Now taking the infinum of such ρ^s and using (3.1)

$$\text{Inf}\left\{\rho>0: \sup_{s\ge1,\,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{D\,Au_k^{(i)},Au_k^{(j)}}{\rho}\right)\right]^p \le 1\right\} < \varepsilon$$ for all i ≥ n_o Hence we get.

$$\sup_n D\left(u_n^{(i)},u_n\right) + \inf\left\{\rho>0: \sup_{s\ge1,\,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{D\,Au_k^{(i)},Au_k}{\rho}\right)\right]^p \le 1\right\}$$

< ε + ε = 2ε for all i ≥ n_o which implies that

$\rho(u^{(i)}, u) < 2\varepsilon$ for all i ≥ $n_o \Rightarrow \lim_i u^i = u.$

Now, $\rho(u,\bar{0}) \le \rho(u^{(i)},u) + \rho(u^{(i)},\bar{0}) < \varepsilon + M$ for all n ≥ n_o (∈)

i.e. $\rho(u,\bar{0})$ is finite which implies that u∈$h_F(M, \phi, p)$

Hence the space $h_F(M, \phi, p)$ is a complete metric space.

Proposition 3.2 $h_F(M, \phi, p) \subseteq h_F(M, \Psi, p)$ if and only if $\sup_n\left(\frac{\phi_s}{\psi_s}\right) < \infty$ for $1 \le \rho < \infty$ and for the sequences (ϕ_s) and (Ψ_s) of real numbers.

Proof. First, suppose that $\sup_{s\ge1}\left(\frac{\phi_s}{\psi_s}\right) = k < \infty$

Then we have $\phi \le k\Psi_s$

Now if $(u_k) \in h_F(M, \phi, p)$ then

$$\sup_{s\ge1,\,\sigma\in\wp_s}\frac{1}{\phi_s}\sum\left[M\left(\frac{D(Au_k,\bar{0})}{\rho}\right)\right]^p < \infty$$

$$\Rightarrow \quad \sup_{s\ge1,\,\sigma\in\wp_s}\frac{1}{k\psi_s}\sum_{k\in\sigma}\left[M\left(\frac{D(Au_k,\bar{0})}{\rho}\right)\right]^p < \infty$$

i.e., $(u_k) \in h(M, \Psi, p)$. Hence $h_F(M, \phi,p) \subseteq h_F(M, \Psi, p)$ conversely, suppose that $h_F(M, \phi, p) \subseteq h_F(M, \Psi, p)$.

We should prove that $\sup_{s\ge1}\left(\frac{\phi_s}{\psi_s}\right) = \sup_{s\ge1}(\eta_s) < \infty \text{ where } \eta_s = \frac{\phi_s}{\psi_s}$ suppose that $\sup_{s\ge1}(\eta_s) = \infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that $\lim_{i\to\infty}(\eta_{s_i}) = \infty$. Then for $(u_k) \in h_F(M, \phi, p)$ we have,

$$\sup_{s\ge1,\,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{D(Au_k,\bar{0})}{\rho}\right)\right]^p < \infty$$

$$\ge \sup_{s\ge1,\,\sigma\in\wp_s}\frac{\eta_{si}}{\phi_{si}}\sum_{k\in\sigma}\left[M\left(\frac{D(Au_k,\bar{0})}{\rho}\right)\right]^p < \infty$$

i.e. $$\sup_{s\ge1,\,\sigma\in\wp_s}\frac{1}{\psi_s}\sum_{k\in\sigma}\left[M\left(\frac{D(Au_k,\bar{0})}{\rho}\right)\right]^p < \infty$$

Which implies that $(u_k) \notin h_F(M, \phi, p)$, a contradiction. This completes the proof.

Corollary 1. $h_F(M, \phi, p) = h_F(M, \Psi, p)$ if and only if $\sup_{s\geq 1}(\eta_s) < \infty$ and $\sup_{s\geq 1}(\eta_s^{-1}) < \infty$
Where $\eta_s = \frac{\phi_s}{\varphi_s}$, for $1 \leq p < \infty$.

Theorem 3.3 $\ell_p^F(M, \phi) \subseteq h_F(m, \phi, p) \subseteq \ell_\infty^F(M, F)$

Proof. Take $M(x) = x$, p for $1 \leq p < \infty$ and $\phi_n = 1$ for all $n \in N$

We get that $h_F(M, \phi, p) = l_p^F(M)$. So the first inclusion is proved. Next, suppose that, $(u_k) \in h_F(M, \phi, p)$.
This implies that

$$\sup_{s\geq 1, \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left\{ M\left[\frac{D(Au_k, \bar{0})}{\rho}\right]^p \right\}^{1/p} = k < \infty$$

For s = 1, $M\left[\frac{D(Au_k, \bar{0})}{\rho}\right] \leq k\phi$, $k \in \sigma$ which implies that

$$\sup_{k\geq 1} M\left[\frac{D(Au_k, \bar{0})}{\rho}\right] < \infty$$

Thus we have $u_k \in l_\infty^F(M)$. This completes the proof.

Proposition 3.4 $h_F(M, \phi, p) = l_p^F(M)$ if and only if

$$\sup_{s\geq 1}(\phi_s) < \infty, \sup_{s\geq 1}(\phi_s^{-1}) < \infty$$

The proof can be obtained by putting $\Psi_n = 1$ for all $n \in N$ in corollary1

Corollary 2 $h_F(M, \phi, p) = l_p^F(M)$ if $\lim_{s\to\infty}\left(\frac{\phi_s}{s}\right) > 0,$ for $1 \leq p < \infty$

Theorem 3.5 The sequence space $h_F(M, \phi, p)$ is not solid.
Proof The proof follows from the following example.
Example: Let p = 2. and $\phi_s = s$ for all $s \in N$
Let $M(x) = |x|$ for all $x \in (0, \infty)$ and define.

$$u_k = \begin{cases} \bar{k} & for\ k \leq n \\ \bar{0} & otherwise \end{cases}$$

Then $D(Au_k, \bar{0}) = 0$
And $\sup_{s\geq 1, \sigma\in\varphi_s} \frac{1}{s} \sum_{k\in\sigma} \left[M\left(\frac{D(Au_k, \bar{0})}{\rho}\right)\right]^p < \infty$ for some $\rho > 0$
Therefore $(u_k) \in h_F(M, \phi, p)$

Now consider the sequence of fuzzy numbers.

$$v_k = \begin{cases} \bar{1} & for\ k\ even \\ -\bar{1} & for\ k\ odd \\ \bar{0} & otherwise \end{cases}$$

Then $\sum \left[M\left(\frac{D(Au_k, \bar{0})}{\rho}\right)\right]^p = \infty$ for some $\rho > 0$

Hence $(v_k) \notin h_F(M, \phi, p)$
Also $D(v_k, \bar{0}) \leq D(u_k, \bar{0})$
Thus $h_F(M, \phi, F)$ is not holid.

Theorem 3.6 The sequence space $h_F(M, \phi, p)$ is not symmetric.
Proof The proof is given by the following example. Consider the sequence of fuzzy numbers given by

$$u_k = \begin{cases} \bar{1} \; for \; k \leq n \\ \bar{0} \; otherwise \end{cases}$$

Then $D(Au_k, \bar{0}) = 0$.

Let $p = 1$, $M(x) = |x|$ and $\phi_s = s$ for all $s \in N$.

Then $\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \left[M\left(\frac{D(Au_k, \bar{0})}{\rho} \right) \right]^p < \infty$

Therefore $(uk) \in h_F(M, \phi, p)$

Let (v_k) be the re-arrangement of (u_k) such that

$v_n = u_n$

$v_k = \bar{1}$ for k odd

$= \bar{0}$ otherwise.

$\sum D(A_k, \bar{0}) = \infty$

Thus $(v_k) \notin h_F(M, \phi, p)$.

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