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Example for the stability of reciprocal type functional equation controlled by product of different powers in singular case

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ABSTRACT

In this paper, we present a counter-example for the stability of reciprocal type functional equation controlled by product of different powers in singular case.

1. Introduction

An inquisitive question that was given a serious thought by S.M. Ulam [12] concerning the stability of group homomorphisms gave rise to the stability problem of functional equations. The laborious intellectual strivings of D.H. Hyers [4] did not go in vain because he was the first to come out with a partial answer to solve the question posed by Ulam on Banach spaces. In course of time, the theorem formulated by Hyers was generalized by T. Aoki [1] for additive mappings and by Th.M. Rassias [11] for linear mappings by taking into consideration an unbounded Cauchy difference. The findings of Th.M. Rassias have exercised a delectable influence on the development of what is addressed as the generalized Hyers-Ulam-Rassias stability of functional equations. A generalized and modified form of the theorem evolved by Th.M. Rassias was advocated by P. Gavruta [3] who replaced the unbounded Cauchy difference by driving into study a general control function within the viable approach designed by Th.M. Rassias. In 1982, a generalization of the result of D.H. Hyers was proved by J.M. Rassias using weaker conditions controlled by a product of different powers of norms [6]. The above stability involving a product of different powers of normed is called as **Ulam-Gavruta-Rassias (or UGR) stability** by Bouikhalene and Elqorachi [2], Sibaha et.al. [10], Nakmahachalasint [5], K. Ravi and M. Arunkumar [8], K. Ravi and B.V. Senthil Kumar [9]. In [7], J.M. Rassias proved the following result.

Theorem 1.1. *Let* X *be a normed space with the norm* $\|\cdot\|_1$ *and let* Y *be a Banach space with the norm* $\|\cdot\|_2$. *Assume in addition that* $f: X \to Y$ *is a mapping such that* $f(tx)$ *is continuous in* t *for each fixed* x. *If there exist* $a, b, \leq a+b<1,$ *and* $c_2 \geq 0$ *such that*

$$\|f(x+y)-f(x)-f(y)\| \leq c_2 \|x\|_1^a \|y\|_1^b \tag{1.1}$$

for all $x, y \in X$ *for which the second member of (1.1) is defined, then there exists a unique linear mapping* $L: X \to Y$ *such that*

$$\|f(x)-L(x)\| \leq c\|x\|_1^{a+b} \tag{1.2}$$

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for all $x \in X$, *where* $c = \frac{c_2}{2 - 2^{a+b}}$.

In the same paper, J.M. Rassias posed the following question:

What is the situation in the above Theorem 1.1 in the case $a + b = 1$?

A clever counter-example has been given by P. Gavruta [3] by proving the following theorem.

Theorem 1.2. *Let be* $0 < a < 1$. *Then there is a function* $f: \mathbb{R} \to \mathbf{R}$ *and a constant* $c_2 > 0$ *such that*

$$|f(x+y) - f(x) - f(y)| \leq c_2 |x|^a |y|^{1-a} \qquad (1.3)$$

for all $x, y \in \mathbf{R}$ *and*

$$sup_{x\neq 0} \frac{(\square|f(x) - T(x))}{|x|} = +\infty \qquad (1.4)$$

for every additive mapping $T: \mathbb{R} \to \mathbb{R}$.

In 2010, K. Ravi and B.V. Senthil Kumar [9] investigated the well-known Ulam-Gavruta-Rassias stability for the 2-dimensional Rassias reciprocal functional equation

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \qquad (1.5)$$

by proving the following theorem.

Theorem 1.3. *Let* X *be the space of non-zero real numbers. Let* $f: X \to \mathbf{R}$ *be a mapping and there exist* $a, b: \rho = a + b \neq -1$ *and* $c_2 \geq 0$ *such that*

$$\left| f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)} \right| \leq c_2 |x|^a |y|^b \qquad (1.6)$$

for all $x, y \in X$, *then there exists a unique reciprocal mapping* $r: X \to \mathbf{R}$ *such that*

$$|f(x) - r(x)| \leq \frac{2c_2}{|1 - 2^{\rho+1}|} |x|^\rho \qquad (1.7)$$

for all $x \in X$.

Inspired by the brilliant counter-example constructed by P. Gavruta in Theorem 1.2 to the question of J.M. Rassias, in this paper, we present a counter-example to the singular case $\rho = a + b = -1$ in Theorem 1.3.

2. A COUNTER-EXAMPLE TO THE SINGULAR CASE IN THEOREM 1.3

We present here a counter-example to the singular case $\rho = a + b = -1$ in Theorem 1.3 by proving the following theorem.

Theorem 2.1. *Let* a *be such that* $-1 < a < 0$. *Then there is a function* $f: [2,4] \to \mathbf{R}$ *and a constant* $c_2 > 0$ *such that*

$$|Dg(x,y)| \leq c_2 x^a y^{-1-a} \qquad (2.1)$$

for all $x, y \in [2,4]$ *and*

$$\max_{x\in[2,4]} \frac{|g(x) - r(x)|}{\frac{1}{x}} = \varepsilon \qquad (2.2)$$

for a reciprocal mapping $r:[2,4]\to \mathbb{R}$ *with* $r(x)=\frac{1}{x}$ *and a small positive quantity* $\varepsilon\left(=\left|\frac{1}{ln2}-1\right|\right)$.

Proof. Let us define $f:[2,4]\to \mathbb{R}$ by

$$f(x)=\frac{1}{xlnx}, \text{for all } x\in[2,4].$$

Then

$$\max_{x\in[2,4]}\frac{|g(x)-r(x)|}{\frac{1}{x}}=\max_{x\in[2,4]}\frac{\left|\frac{1}{xlnx}-\frac{1}{x}\right|}{\frac{1}{x}}$$

$$=\max_{x\in[2,4]}\left|\frac{1}{lnx}-1\right|$$

$$=\left|\frac{1}{ln2}-1\right|=\varepsilon.$$

Hence the relation (2.2) follows. We have to prove (2.1) is true.

$$|Dg(x,y)|=\left|\frac{1}{(x+y)\ln(x+y)}-\frac{\frac{1}{xlnx}\frac{1}{ylny}}{\frac{1}{xlnx}+\frac{1}{ylny}}\right|$$

$$=\left|\frac{1}{(x+y)\ln(x+y)}-\frac{1}{xlnx+ylny}\right|$$

$$=\left|\frac{(x+y)\ln(x+y)-(xlnx+ylny)}{(xlnx+ylny)(x+y)\ln(x+y)}\right|$$

$$=\left|\frac{xy\left(\frac{1}{x}+\frac{1}{y}\right)\ln(x+y)-xy\left(\frac{1}{y}lnx+\frac{1}{x}lny\right)}{(xlnx+ylny)(x+y)\ln(x+y)}\right|$$

$$=\left|\frac{\left(\frac{1}{x}+\frac{1}{y}\right)\ln(x+y)-\frac{1}{y}lnx-\frac{1}{x}lny}{(xlnx+ylny)\left(\frac{x+y}{xy}\right)\ln(x+y)}\right|$$

$$\le\left|\left(\frac{1}{x}+\frac{1}{y}\right)\ln(x+y)-\frac{1}{y}lnx-\frac{1}{x}lny\right|$$

$$=\left|\frac{1}{x}ln\left(\frac{x+y}{y}\right)+\frac{1}{y}ln\left(\frac{x+y}{x}\right)\right|$$

$$=\left|\frac{1}{x}ln\left(1+\frac{x}{y}\right)+\frac{1}{y}ln\left(1+\frac{y}{x}\right)\right|.$$

We have to prove that there exists a constant $c_2>0$ such that

$$\frac{1}{x}ln\left(1+\frac{x}{y}\right)+\frac{1}{y}ln\left(1+\frac{y}{x}\right)\le c_2x^ay^{-1-a} \tag{2.3}$$

for all $x,y\in[2,4]$. Taking $\frac{y}{x}=t$, the inequality (2.3) is equivalent to the inequality

$$t^{1+a}ln\left(1+\frac{1}{t}\right)+t^a\ln(1+t)\le c_2.$$

By using L'Hospital rule, we have

$$\lim_{t\to 0}t^a\ln(1+t)=\lim_{t\to 0}\frac{\ln(1+t)}{t^{-a}}$$

$$= \lim_{t\to 0} \frac{\frac{1}{1+t}}{-at^{-a-1}}$$

$$= -\frac{1}{a}\lim_{t\to 0} \frac{t^{a+1}}{1+t} = 0$$

and

$$\lim_{t\to\infty} t^a \ln(1+t) = \lim_{t\to\infty} \frac{\ln(1+t)}{t^{-a}}$$

$$= -\frac{1}{a}\lim_{t\to\infty} \frac{t^{a+1}}{1+t}$$

$$= -\frac{1}{a}\lim_{t\to\infty} \frac{(a+1)t^a}{1}$$

$$= -\left(\frac{a+1}{a}\right)\lim_{t\to\infty} t^a = 0.$$

Since the function $t \to t^a \ln(1+t)$ is continuous on $(0,\infty)$, it follows that there is a constant $c = \frac{c_2}{2} > 0$ such that

$$t^a \ln(1+t) \le \frac{c_2}{2}. \tag{2.4}$$

Also,

$$\lim_{t\to 0} t^{1+a} \ln\left(1+\frac{1}{t}\right) = \lim_{t\to 0} \frac{\ln\left(1+\frac{1}{t}\right)}{t^{-1-a}}$$

$$= \lim_{t\to 0} \frac{\frac{1}{1+\frac{1}{t}}\left(-\frac{1}{t^2}\right)}{(-1-a)t^{-a-2}}$$

$$= \frac{1}{a+1}\lim_{t\to 0} \frac{t^{a+2}}{t^2+t}$$

$$= \frac{1}{a+1}\lim_{t\to 0} (a+2)\frac{t^{a+1}}{2t+1} = 0$$

and

$$\lim_{t\to\infty} t^{1+a} \ln\left(1+\frac{1}{t}\right) = \lim_{t\to\infty} \frac{\ln\left(1+\frac{1}{t}\right)}{t^{-1-a}}$$

$$= \frac{1}{a+1}\lim_{t\to\infty} \frac{(a+2)t^{a+1}}{2t+1}$$

$$(a+2) = \lim_{t\to\infty} \frac{t^a}{2} = 0.$$

Since the function $t \to t^{a+1} \ln\left(1+\frac{1}{t}\right)$ is continuous on $(0,\infty)$, it follows that there is a constant $c = \frac{c_2}{2} > 0$ such that

$$t^{a+1} \ln\left(1+\frac{1}{t}\right) \le \frac{c_2}{2}. \tag{2.5}$$

From the inequalities (2.4) and (2.5), we find that the inequality (2.3) holds and hence (2.1) also holds. Therefore, there exists a reciprocal function $r(x) = \frac{1}{x}$, for all $x \in [2,4]$ which shows that the functional equation (1.5) is still stable for the singular case $\rho = a + b = -1$ in Theorem 1.3.

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