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Cofinitely quasi-injective modules

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ABSTRACT

In this paper cofinitely quasi-injective modules defined. Let M and N be $R-$modules. Let $g: K \to N$ be a monomorphism from any $R-$module K such that $g(K)$ is cofinite submodule of N. Then M is called cofinitely $N-$injective module if any homomorphism $f: K \to M$ can be extended to an $R-$homomorphism $h: N \to M$. An $R-$module M is called cofinitely quasi injective, if M is cofinitely M injective module.

Introduction:

Throughout this paper R will denote an arbitrary ring with unity and all $R-$modules are unitary left $R-$modules. Let M be an $R-$module. $N \leq M$ will mean that N is a submodule of M. Let M and N be two $R-$modules. Then M is called $N-$injective in $R-$Mod with an exact row can be extended commutatively by a module homomorphism $N \to M$. If M is $N-$injective for every $N \in R-$Mod, then M is called injective in $R-$Mod. We define a cofinitely quasi-injective module which is the generalization of cofinitely injective modules. Here we discuss some properties of cofinitely quasi injective modules and when cofinitely injective module will not be an injective module. Cofinitely injective module defined by Hiremath in 1978. An $R-$module M is said to be cofinitely injective if it is injective with respect to short exact sequences of $R-$modules of the form $0 \to A \to B \to C \to 0$, where A is cofinitely generated. An $R-$module M is said to be cofinitely generated if M has finitely generated essential socle. A submodule K of M is called cofinite submodule, if M/K is finite. The socle and Jacobson radical of M are denoted by $Soc(M)$ and $J(M)$. A non zero module M is said to be semi-simple if it is expressible as sum of simple submodules. If M is semi-simple then $Soc(M) = M$.

Here we define a different meaning of cofinitely injective module.

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Let M and N be $R-$modules. Let $g:K \to N$ be a monomorphism from any $R-$module K such that $g(K)$ is cofinite submodule of N. Then M is called cofinitely $N-$injective module if any homomorphism $f:K \to M$ can be extended to an $R-$homomorphism $h:N \to M$. Obviously, N is cofinitely $M-$injective if $i:T \to M$ is inclusion and T a cofinite submodule of M, and any homomorphism from T to N can be extended to a homomorphism from M to N. An $R-$module M is called cofinitely quasi injective, if M is cofinitely M injective module.

Cofinitely quasi-injective modules

Lemma 1 Let $f:M \to N$ and $g:N \to M$ be $R-$module homomorphism.

If $g \circ f = I_M$, then $N = \operatorname{Im}(f) \oplus Ker(g)$.

Proof: See [7, 11.10(1)].

Theorem 2 Let N be an $R-$module and $\{M_i \mid i \in I\}$ a family of $R-$modules. Then the product $\Pi_{i \in I} M_i$ is cofinite $N-$injective if and only if M_i is cofinitely $N-$injective for each $i \in I$.

Proof: Similar to the proof of [7, 16.1.(1)].

Corollary 3 Let M be cofinitely quasi injective $R-$module. Then every direct summand of M is cofinitely $M-$injective.

Theorem 4 Let M be a cofinitely quasi injective $R-$module. Then M is cofinitely $M/L-$injective for every $L \le M$.

Proof: Consider the diagram.

$$\begin{tikzcd} 0 \arrow[r] & K/L \arrow[r, "i"] \arrow[d, "f"] & M/L \\ & M & \end{tikzcd}$$

Where $i:K/L \to M/L$ be the inclusion with K/L a cofinite submodule of M/L and $f:K/L \to M$ be the homomorphism. Then since $M/K \cong \dfrac{M/L}{K/L}$ and $\dfrac{M/L}{K/L}$ is finitely generated, M/K is also finitely generated, so K is cofinite submodule of M. Define $p: K \to K/L$ by $k \to k+L$.

Then by hypothesis the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "f \circ p"'] & M \arrow[dl, "g"] \\ & M & \end{tikzcd}$$

can be extended commutatively by an $R-$module homomorphism $g: M \to M$. Let $h: M/L \to M$, $n+L \to g(n)$. Clearly $h: M/L \to M$ is an $R-$module homomorphism for $k+L \in K/L$, $(h \circ i)(k+L) = h(k+L) = g(k) = f(p(k)) = f(k+L)$, and then $h \circ i = f$.

Thus the diagram

$$\begin{tikzcd} 0 \arrow[r] & K/L \arrow[r, "i"] \arrow[d, "f"'] & M/L \arrow[dl, "h"] \\ & M & \end{tikzcd}$$

is commutative, and so M is cofinitely $M/L-$injective.

Corollary 5 Let M be a cofinitely $N-$injective $R-$module. Then M is cofinitely $N/L-$injective for every $L \leq N$.

Corollary 6 If M is cofinitely quasi injective, then M is cofinitely K injective for every homomorphic image K of M.

Proof: Clear from the (Theorem 4)

Theorem 7 If M is cofinite quasi injective $R-$module and L a cofinite submodule of M. Then M is cofinitely $L-$injective module.

Proof: Consider a diagram with K a cofinite submodule of L.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i_K"] \arrow[d, "f"] & L \\ & M & \end{tikzcd}$$

By $K \le L$, $M/L \cong \dfrac{M/K}{L/K}$. Since $\dfrac{M/K}{L/K}$ and L/K are finitely generated, M/K is also finitely generated. Thus K is cofinite submodule of M.

By hypothesis the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i_K"] \arrow[d, "f"'] & M \arrow[dl, "g"] \\ & M & \end{tikzcd}$$

Can be extended commutatively by an $R-$module homomorphism $g : M \to M$. Let $h = g \circ i_L$. Then for every $k \in K$,

$f(k) = g(i_K(k)) = g(i_L(i_K(k))) = h(i_K(k)) = (h \circ i_K)(k)$, and then $f = h \circ i_K$.

Hence M is cofinitely $L-$injective.

Corollary 8 Let M be a cofinitely quasi injective $R-$module and let $0 \to L' \to M \to L'' \to 0$ be any exact sequence with L'' finitely generated. Then M is cofinitely L' and $L''-$injective.

Proof: Clear from the (Theorem 4) and (Theorem 7).

Corollary 9 For $R-$module M the following statements are equivalent.

1. M is cofinitely quasi injective.
2. M is cofinitely $M/L-$injective for every $L \le M$.
3. M is cofinitely $L-$injective for every cofinite submodule L of M.

Proof: Clear from the (Theorem 4) and (Theorem 7).

Theorem 10 Let M be an $R-$module and $U_1 \le U_2 \le .....$ an ascending chain of $R-$modules. If M is cofinitely U_n-injective for all $n \in \text{¥}^{+}$, then M is cofinitely $\bigcup_{i=1}^{N} U_i -$injective for every $N \in \text{¥}^{+}$.

Proof: See [5, 2.10].

Corollary 11 Let M be a Noetherian $R-$module and $U_1 \le U_2 \le .....$ an ascending chain of submodules of M. If M is cofinitely U_n-injective for all $n \in \text{¥}^{+}$, then M is cofinitely quasi injective module.

Proposition 12 Let M be an $R-$module. Then M is injective in $R-$Mod if and only if M is cofinitely $N-$injective for every $N \in R-$Mod.

Proof: $(\Rightarrow)$ clear

$(\Leftarrow)$ Since M is cofinitely $N-$injective for every $N \in R-$Mod, M is co-

finitely $R-$injective, since R is finitely generated, M is $R-$injective, then by [7, 16.4], M is injective in $R-$Mod.

Corollary 13 Let M be an $R-$module. Then M is quasi injective if and only if M is cofinitely quasi injective.

Lemma 14 Let M be an $R-$module. Then the following statements are equivalent.

1. M is semi simple.
2. Every $R-$module is $M-$injective.
3. Every $R-$module is $M-$projective.

Proof: See [7, 20.2].

Lemma 15 Let M be an $R-$module. Then the following statements are equivalent.

1. Every cofinite submodule of M is a direct summand.
2. M is cofinitely quasi injective.

Proof: $(1.) \Rightarrow (2.)$ Let M be an $R-$module and $i: K \to M$ be any injection where K is cofinite submodule of M. By hypothesis K is a direct summand of M. Let $M = K \oplus T$ and let $h: M = K \oplus T \to M$ be the mapping $k+t \to f(k)$, $(k \in K, t \in T)$. Clearly h is an $R-$module homomorphism. For $k \in K$, Let $f: K \to M$ be any homomorphism.

$(h \mathrm{o} i)(k) = h(i(k)) = h(k) = h(k+0) = f(k)$, and then $h \mathrm{o} i = f$ thus M is cofinitely quasi injective.

$(2.) \Rightarrow (1.)$ Let K be a cofinite submodule of M. By hypothesis K is cofinitely $M-$injective. Where $i: K \to M$ be the injection and $I_K: K \to K$ be any identity homomorphism then I_K can be extended to a homomorphism $h: M \to K$. Then $h \mathrm{o} i = I_K$, and

by (Lemma 1), $M = i(K) \oplus Ker(h) = K \oplus Ker(K)$.

Lemma 16 Let M be an $R-$module and $Rad(M)=M$, Then M is cofinitely quasi injective.

Proof: Clear since M has no non zero finitely generated factor module.

Proposition 17 Let M be a non zero $R-$module. If every cofinitely quasi injective $R-$module is quasi injective then $Rad(M)\neq M$.

Proof: Assume $Rad(M)=M$. Then by lemma 16, M is cofinitely quasi injective. Hence $R-$module M is quasi injective, and then by lemma 14, M is semi simple. This contradicts $Rad(M)=M$.

A ring R is called a max ring if every non zero $R-$module has a maximal submodule.

Corollary 18 If every cofinitely quasi injective $R-$module is quasi injective for $M\in R-$Mod, then R is max ring.

Proof: Clear from the (Proposition 17).

Lemma 19 Let M be an $R-$module which is not semi simple and for which every cofinite submodule of M is a direct summand. Then there exist an $R-$module P such that P is cofinitely $M-$injective but not $M-$injective.

Proof: Clear for the (Lemma 14) and (Lemma 15).

Lemma 20 Let H be a non local hollow $R-$module which is not semi-simple and let L be a semi-simple $R-$module. Let $M=H\oplus L$. Then there exist an $R-$module X such that X is cofinitely $M-$injective but not $M-$injective.

Proof: Let K be any cofinite submodule of M . Then there exist $m_1m_2......m_k\in M$ such that

$$M/K=\langle m_1+K,m_2+K.........m_k+K\rangle=(\langle m_1,m_2,.........m_k\rangle+K)/K.$$

Then $M=\langle m_1,m_2,......m_k\rangle+K$. Let $m_i=h_i+l_i$ for $h_i\in H,l_i\in L$ and $1\leq i\leq k$.

Then $\langle m_1,m_2,.........m_k\rangle\leq\langle h_1,h_2,..........h_k\rangle+\langle l_1,l_2,.........l_k\rangle$, and then

$$M=\langle h_1,h_2,.........h_k\rangle+\langle l_1,l_2,.........l_k\rangle+K.$$

Since H is nonlocal hollow module, H is not finitely generated. Hence, $\langle h_1,h_2,.........h_k\rangle$ is a proper submodule of H, and $\langle h_1,h_2,........h_k\rangle=H$. In this case $M=\langle l_1,l_2,......l_k\rangle+K$, and since $\langle l_1,l_2,......l_k\rangle\leq L$, $M=K+L$.

Since L is semi-simple, then $K \cap L$ is a direct summand of L. Let $L = (K \cap L) \oplus T$. Then $M = K + ((K \cap L) \oplus T) = K \oplus T$.

Hence K is a direct summand of M. That is every cofinite submodule of M is direct summand. Since M is not semi-simple, then there exist an $R-$module X such that X is cofinitely $M-$injective but not $M-$injective.

Example 21 Let ¢ and ¤ be ¢ $-$modules. Then ¢ is cofinitely ¤ $-$injective since $Rad($¤$)=$¤ , but ¢ is not cofinitely quasi injective module.

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