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Some results involving Gould-hopper polynomials

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ABSTRACT

In this paper, we obtain successive differentiation and change of argument associated with Gould-Hopper polynomials. We also derived generalized Curzon's integral and linear generating relations.

Introduction

The Gould-Hopper polynomials $g_n^m(x,y)$ [3,p.58(6.2)] defined in the year 1962, are the generalization of classical Hermite polynomials $H_n(x)$ [7,p.187(2)]. The $g_n^m(x,y)$ is defined by [3;10,p.76(6)] in the form:

$$g_n^m(x,y)=\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{n!}{k!(n-mk)!}y^k\,x^{n-mk} \tag{1.1}$$

$$=x^n\;{}_mF_0\left[\begin{matrix}\Delta(m;-n)\;\;;\\ \text{------};\end{matrix}\;\; y\left(-\frac{m}{x}\right)^m\right] \tag{1.2}$$

where $n\in\mathbb{N}_0$, $m\in\{2,3,4,\ldots\}$ and $\Delta(m;-n)$ abbreviates the array of m parameters given by

$$\frac{-n}{m},\frac{-n+1}{m},\frac{-n+2}{m},\ldots,\frac{-n+m-1}{m}$$

and $[x]$ denotes the greatest integer function. It is being understood that the set $\Delta(0;\lambda)$ is empty.

When $m=2, y=-1$ and replacing x by $2x$ in (1.1) and (1.2), these polynomials reduce to the classical Hermite polynomials given by

$$H_n(x)=g_n^2(2x,-1)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\,n!(2x)^{n-2k}}{k!(n-2k)!}$$

$$=(2x)^n\;{}_2F_0\left[\begin{matrix}\frac{-n}{2},\frac{-n+1}{2}\;\;;\\ \text{--------};\end{matrix}\;\; -\frac{1}{x^2}\right] \tag{1.3}$$

$$g_n^2\left(x,-\frac{1}{4}\right)=\frac{H_n(x)}{2^n} \tag{1.4}$$

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The equation (1.1) can be derived from the following generating relation:

$$\sum_{n=0}^{\infty} g_n^m(x,y)\frac{t^n}{n!} = \exp(xt + yt^m) \tag{1.5}$$

Curzon's Integral for $P_n(x)$ [7,p.199(Q.No.4A); see also 2] is given by

$$\int_0^\infty \exp(-t^2)\,t^n\, H_n(xt)\,\mathrm{d}t = \frac{n!\sqrt{\pi}}{2} P_n(x); \quad n \in \mathbb{N}_0 \tag{1.6}$$

where $P_n(x)$ is the Legendre's polynomial of first kind[7,p.166(4),p.183(Q.No.3)]

$$P_n(x) = \frac{\left(\frac{1}{2}\right)_n (2x)^n}{n!}\; {}_2F_1\left[\begin{array}{ccc}\Delta(2;-n)\ ; & & \\ \frac{1}{2}-n & ; & -\frac{1}{x^2}\end{array}\right] \tag{1.7}$$

Differential formula and change of argument

If $m, n \in \mathbb{N}_0,\ p \in \mathbb{N}$ such that $m \le n,$ then

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} g_n^p(x,y) = (-1)^m (-n)_m\, g_{n-m}^p(x,y) \tag{2.1}$$

$$g_n^p(-x,y) = (-1)^n\, g_n^p(x,(-1)^p\, y) \tag{2.2}$$

Proofs:

If we expand $g_n^p(x,y)$ in series form and apply suitable successive differential formula $\frac{\mathrm{d}^m}{\mathrm{d}x^m} x^q = \frac{q!\,x^{q-m}}{(q-m)!}$ such that $m \le q$ and $m,\ q \in \mathbb{N}_0,$ after simplification we get R.H.S. of (2.1).

If we replace x by $-x$ in series representation of $g_n^p(x,y),$ after simplification we get (2.2).

When $p = 2, y = \frac{-1}{4}$ in (2.1) and in view of the relation (1.4), we get

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} H_n(x) = (-2)^m (-n)_m\, H_{n-m}(x) \tag{2.3}$$

Generalization of curzon's integral (1.6)

$$\int_0^\infty \exp(-t^p)\,t^q\, g_n^m(xt^\ell, yt^s)\,\mathrm{d}t$$

$$= \frac{x^n}{p}\Gamma\left(\frac{q+\ell n+1}{p}\right) {}_2\Psi_0^*\left[\begin{array}{ll}(-n,m),\left(\frac{q+\ell n+1}{p},\frac{s-\ell m}{p}\right)\ ; & y\left(\frac{-1}{x}\right)^m \\ \text{-----------------};\end{array}\right] \tag{3.1}$$

where $m \in \mathbb{N},\ n \in \mathbb{N}_0;\ p,q,\ell,s,\frac{s-\ell m}{p} \in \mathbb{R}^+$ such that $q+\ell n+1 > (\ell m - s)\left[\frac{n}{m}\right].$

$$\int_0^\infty \exp(-t^p)\,t^q\, g_n^m(xt^\ell, yt^s)\,\mathrm{d}t$$

$$= \frac{x^n}{p}\Gamma\left(\frac{q+\ell n+1}{p}\right) {}_U F_0 \left[\begin{matrix} \Delta(m,-n), \Delta\left(\frac{s-\ell m}{p}, \frac{q+\ell n+1}{p}\right) & ; & y\left(\frac{-m}{x}\right)^m \left(\frac{s-\ell m}{p}\right)^{\frac{s-\ell m}{p}} \\ \text{------------------} & ; & \end{matrix}\right] \quad (3.2)$$

$$U = \frac{mp-\ell m+s}{p}$$

where $m \in \mathbb{N},\ n \in \mathbb{N}_0;\ p,q,\ell,s \in \mathbb{R}^+; \frac{s-\ell m}{p} \in \mathbb{N}$ such that $q+\ell n+1 > (\ell m - s)\left[\frac{n}{m}\right]$.

$$\int_0^\infty \exp(-t^p)\, t^q\, g_n^m(xt^\ell, yt^s)\, \mathrm{d}t$$

$$= \frac{x^n}{p}\Gamma\left(\frac{q+\ell n+1}{p}\right) {}_m F_V \left[\begin{matrix} \Delta(m,-n) & ; & y\left(\frac{-m}{x}\right)^m \left(\frac{s-\ell m}{p}\right)^{\frac{s-\ell m}{p}} \\ \Delta\left(\frac{\ell m-s}{p}, \frac{p-q-\ell n-1}{p}\right) & ; & \end{matrix}\right] \quad (3.3)$$

$$V = \frac{\ell m-s}{p}$$

where $m \in \mathbb{N},\ n \in \mathbb{N}_0;\ p,q,\ell,s \in \mathbb{R}^+; \frac{s-\ell m}{p} \in \mathbb{N}, \frac{q+\ell n+1}{p} \notin \mathbb{Z}$ such that $q+\ell n+1 > (\ell m - s)\left[\frac{n}{m}\right]$.

Proofs:

If we replace $g_n^m(xt^\ell, yt^s)$ by its series representation, integrate term by term and use the definition of Gamma function with suitable convergence condition, in view of the definition of Fox-Wright generalized hypergeometric function ${}_p\Psi_q^*$ \$ [10,p.179(Q.No.34(iii))] we obtain the right hand side of (3.1) under the stated convergence conditions.

Similarly under the stated convergence conditions, applying algebraic properties of Pochhammer symbol we get (3.2) and (3.3).

Linear generating relations for gould-hopper polynomials

If $x,\ y \in \mathbb{R}$ and $m \in \mathbb{N}$, then

$$\sum_{n=0}^{\infty} \frac{g_{2n}^{2m}(x,y)\,t^n}{(2n)!} = \cosh(x\sqrt{t})\exp(yt^m) \quad (4.1)$$

$$\sum_{n=0}^{\infty} \frac{g_{2n+1}^{2m}(x,y)\,t^n}{(2n+1)!} = t^{-1/2}\sinh(x\sqrt{t})\exp(yt^m) \quad (4.2)$$

$$\sum_{n=0}^{\infty} \frac{g_{2n}^{2m+1}(x,y)\,t^n}{(2n)!} = \cosh\left(x\sqrt{t} + y\,t^{m+\frac{1}{2}}\right) \quad (4.3)$$

$$\sum_{n=0}^{\infty} \frac{g_{2n+1}^{2m+1}(x,y)\,t^n}{(2n+1)!} = t^{-1/2}\sinh\left(x\sqrt{t} + y\,t^{m+\frac{1}{2}}\right) \quad (4.4)$$

$$\sum_{n=0}^{\infty} \frac{g_n^m(x,y)\cos(nt)}{n!} = \cos\{x\sin t + y\sin(mt)\}\exp\{x\cos t + y\cos(mt)\} \quad (4.5)$$

$$\sum_{n=0}^{\infty} \frac{g_n^m(x,y)\sin(nt)}{n!} = \sin\{x\sin t + y\sin(mt)\}\exp\{x\cos t + y\cos(mt)\} \quad (4.6)$$

Proofs:

Consider the well known generating function

$$\exp(xt + yt^m) = \sum_{n=0}^{\infty} g_n^m(x,y)\frac{t^n}{n!} \quad (4.7)$$

If we replace m by $2m$ in (4.7) and use well known series identity

$$\sum_{n=0}^{\infty}\Phi(n)=\sum_{n=0}^{\infty}\Phi(2n)+\sum_{n=0}^{\infty}\Phi(2n+1) \tag{4.8}$$

in the right hand side of (4.7), put $t=iT$ and equate real and imaginary parts; after that in each of the results obtained, replace T by $i\sqrt{t}$, on simplification we get (4.1) and (4.2).

Similarly if we replace m by $2m+1$ in (4.7), we can find (4.3) and (4.4).

If we replace t by $\cos t + i\sin t$ in (4.7), use Demoivre's theorem, equate real and imaginary parts, we can obtain (4.5) and (4.6).

For further study of similar types of generating relations, we refer the papers[1;4;5;6;8;9].

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