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A common fixed point theorem for weakly compatible mappings satisfying a new contractive condition of integral type in 2-metric space

M. E. Hassan* and Elhadi Elnour Elniel

Department of Mathematics, College of Science and Arts, Taif University, Saudi Arabia.

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ABSTRACT

In this paper we prove a unique common fixed point theorem in 2-metric space .the existence of fixed point for two weakly compatible maps into 2-metric space is established under new contractive condition of integral type by using another functions ϕ and ψ.

Introduction

As a generalization of the area function for Euclidean triangles the concept of 2-metric introduced by Menger [1] was investigated by Gahler in series of papers [2],[3],[4].

Fixed point theorems in 2-metric spaces have been established by several authors (see e.g.[5],[6],[7],[8],[9]).

In(1986)Jungck [10] introduced the concept of compatible mappings and used to obtain results which generalize a theorem by Park and Bae [11], a theorem by Hadzic [12], and others. Fixed point theorems for compatible mappings and weakly compatible mappings have been established by several authors (see e.g.[13],[14],[15],[16]).

In(2002)Branciari [17] obtained a fixed point result for a single mapping satisfying an analogue of a Banach contraction principle for integral type in the following theorem.

In (2011)V.Gupta and Vareen Manic[18]study the existence and uniqueness of common fixed point theorem for two weakly compatible maps under contractive condition of integral type.

Theorem 1. (Branciari). Let (X,d) be a complete metric space, $c\in[0,1)$ and let $T:X\to X$ be a mapping such that for each $x,y\in X$,

$$\int_0^{d(Tx.Ty)}\varphi(t)dt\le c\int_0^{d(x,y)}\varphi(t)dt$$

where $\varphi:[0,\infty)\to[0,\infty)$ is a Lebesgue- integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,\infty)$, and such that for each $\varepsilon>0,\int_0^{\varepsilon}\varphi(\mathrm{t})\mathrm{dt}>0,$ then T has a unique fixed point $a\in X$ such that for each $x\in X$, $\lim_{n\to\infty}T^nx=a.$

After the paper of V.Gupta, a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various known properties, in metric space and 2-metric space.

The aim of this paper is to translate the works of V.Gupta and Mani[18] from metric space into 2-metric space.

Definitions and Preliminaries

Definition 2. Let X be a non empty set. A real valued function d on $X\times X\times X$ is said to be 2-metric in X if

(i) To each pair of distinct points $x,y\in X$, there exists a point $z\in X$ such that $d(x,y,z)\ne 0$

(ii) $d(x,y,z)=0,$ if at least two of x,y,z are equal

(iii) $d(x,y,z)=d(y,z,x)=d(x,z,y)$

(iv) $d(x,y,z)\le d(x,y,w)+d(x,w,z)+d(w,y,z)$ for all $x,y,z,w\in X$. When d is 2-metric on X, then the pair (X,d) is called 2-metric space.

Definition 3. A sequence $\{x_n\}$ in 2-metric space (X,d) is said to be convergent to an element $x\in X$ if $\lim_{n\to\infty}d(x_n,x,a)=0$ for all $a\in X$. It follows that if the sequence $\{x_n\}$ converges to x then $\lim_{n\to n}d(x_n,a,b)=d(x,a,b)$ for all $a,b\in X$.

Tele:
E-mail addresses: elhadielniel_2003@hotmail.com

Definition 4. A sequence $\{x_n\}$ in 2-metric space (X,d) is said to be Cauchy sequence if $d(x_m,x_n,a)=0$ as $m,n\to\infty$ for all $a\in X.$

Definition 5. A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Proposition 6. If a sequence $\{x_n\}$ in a 2-metric space converges to x then every subsequence of $\{x_n\}$ also converges to the same limit $x.$

Proposition 7. Limit of a sequence in a 2-metric space, if exists, is unique.

Definition 8. Let f and g be two self maps on a set $X.$ Maps f and g are said to be commuting if $fgx=gfx$ for all $x\in X.$

Definition 9. Let (X,d) is a 2-metric space and $f,g:(X,d)\to(X,d).$ The mappings f and g are said to be compatible if whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=t$ for some $t\in X$ then $d(fgx_n,gfx_n,a)\to 0$ as $n\to\infty$ for all $a\in X.$

Definition 10. Let f and g be two self maps in a 2-metric space (X,d) then f and g are said to be weakly compatible if they commute at their coincidence points.

Lemma 11. Let f and g be weakly compatible self mapping of a set $X.$ If f and g have a unique point of coincidence $z,$ then z is the unique common fixed point of f and $g.$

Main Result

Theorem12. Let S and T be self compatible maps of a complete 2-metric space (X,d) satisfying the following conditions

(i) $S(X)\subset T(X)$ (1)

(ii) $$\psi\int_0^{d(Sx,Sy,a)}\varphi(t)dt\le\psi\int_0^{d(Tx,Ty,a)}\varphi(t)dt-\phi\int_0^{d(Tx,Ty,a)}\varphi(t)dt \quad (2)$$

for each $x,y,a\in X$ where $\psi:[0,\infty)\to[0,\infty)$ is a continuous and non decreasing function and $\phi:[0,\infty)\to[0,\infty)$ is a lower semi continuous and non decreasing function such that $\psi(t)=\phi(t)=0$ if and only if $t=0$ also $\varphi:[0,\infty)\to[0,\infty)$ is a "Lebesgue-integrable function" which is summable on each compact subset of R^+, non negative, and such that for each $\varepsilon>0,\int_0^{\varepsilon}\varphi(t)dt>0$. Then S and T have a unique common fixed point.

Proof : let x_0 be an arbitrary point of X . Since $S(X)\subset T(X)$. Choose a point x_1 in X such that $Sx_0=Tx_1$. Continuing this process, in general, choose x_{n+1} such that $y_n=Tx_{n+1}=Sx_n$, $n=0,1,2,\ldots\ldots$ For each integer $n\ge 1,$ and for all $a\in X$, we have from (2)

$$\psi\int_0^{d(y_n,y_{n+1},a)}\varphi(t)dt\le\psi\int_0^{d(y_{n-1},y_n,a)}\varphi(t)dt-\phi\int_0^{d(y_{n-1},y_n,a)}\varphi(t)dt$$

$$\le\psi\int_0^{d(y_{n-1},y_n,a)}\varphi(t)dt \quad (3)$$

Since ψ is continuous and has a monotone property, Therefore

$$\int_0^{d(y_n,y_{n+1},a)}\varphi(t)dt\le\int_0^{d(y_{n-1},y_n,a)}\varphi(t)dt$$

Let us take $z_n = \int_0^{d(y_n, y_{n+1}, a)} \varphi(t)dt$ then it follows that z_n is monotone decreasing and lower bounded sequence of numbers. Therefore there exist $k \geq 0$ such that $z_n \to k$ as $n \to \infty$. Suppose that $k > 0$. Taking limit as $n \to \infty$ on both sides of (3) and using that ϕ is lower semi continuous, we get,

$$\psi(k) \leq \psi(k) - \varphi(k) < \psi(k) \tag{4}$$

This is a contradiction. Therefore $k = 0$. This implies $z_n \to 0$ as $n \to \infty$

$$\int_0^{d(y_n, y_{n+1}, a)} \varphi(t)dt \to 0 \quad \text{as } n \to \infty \tag{5}$$

Now we prove that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Therefore there exists an $\varepsilon > 0$ and subsequence $\{y_{m(p)}\}$ and $\{y_{n(p)}\}$ such that for each positive integer $p(m)$, p(n) such that $m(p) < n(p) < m(p+1)$ with

$$d(y_{n(p)}, y_{m(p)}, a) \geq \varepsilon, \; d(y_{n(p)-1}, y_{m(p)}, a) < \varepsilon \tag{6}$$

Now

$$\varepsilon \leq d(y_{n(p)}, y_{m(p)}, a) \leq d(y_{n(p)}, y_{m(p)}, y_{n(p)-1}) + d(y_{n(p)}, y_{n(p)-1}, a) + d(y_{n(p)-1}, y_{m(p)}, a)$$

$$< d(y_{n(P)}, y_{m(p)}, y_{n(p)-1}) + d(y_{n(p)}, y_{n(p)-1}, a) + \varepsilon \tag{7}$$

Now

$$0 < \ell = \int_0^{\varepsilon} \varphi(t)dt \leq \int_0^{d(y_{n(p)}, y_{m(p)}, a)} \varphi(t)dt \leq \int_0^{d(y_{n(p)}, y_{m(p)}, y_{n(p)-1}) + d(y_{n(p)}, y_{n(p)-1}, a) + \varepsilon} \varphi(t)dt$$

Letting $p \to \infty$ and from (5)

$$\lim_{p \to \infty} \int_0^{d(y_{n(p)}, y_{m(p)} a)} \varphi(t)dt = \ell \tag{8}$$

Now consider triangle inequality,

$$d(y_{n(p)}, y_{m(p)}, a) \leq d(y_{n(p)}, y_{m(p)}, y_{n(p)-1}) + d(y_{n(p)}, y_{n(p)-1} a) + d(y_{n(p)-1}, y_{m(p)}, a)$$

$$d(y_{n(p)}, y_{m(p)}, a) \leq d(y_{n(p)}, y_{m(p)}, y_{n(p)-1}) + d(y_{n(p)}, y_{n(p)-1} a) + d(y_{n(p)-1}, y_{m(p)}, y_{m(p)-1})$$

$$+ d(y_{n(p)-1}, y_{m(p)-1}, a) + d(y_{m(p)-1}, y_{m(p)}, a) = \alpha \tag{9}$$

$$d(y_{n(p)-1}, y_{m(p)-1}, a) \leq d(y_{n(p)-1}, y_{m(p)-1}, y_{n(p)}) + d(y_{n(p)-1}, y_{n(p)}, a) + d(y_{n(p)} y_{m(p)-1}, a)$$

$$d(y_{n(p)-1}, y_{m(p)-1}, a) \leq d(y_{n(p)-1}, y_{m(p)-1}, y_{n(p)}) + d(y_{n(p)-1}, y_{n(p)}, a) + d(y_{n(p)} y_{m(p)-1}, y_{m(p)})$$

$$+ d(y_{n(p)}, y_{m(p)} a) + d(y_{m(p)} + y_{m(p)-1}, a) = \beta \tag{10}$$

and therefore

$$\int_o^{d(y_{n(p)}, y_{m(p)}, a)} \varphi(t)dt \leq \int_0^{\alpha} \varphi(t)dt \tag{11}$$

$$\int_0^{d(y_{n(p)-1}, y_{m(p)-1}, a)} \varphi(t)dt \leq \int_0^{\beta} \varphi(t)dt \tag{12}$$

Taking $p\to\infty$ and using (5) and (8) in (11) and (12). we get

$$\int_0^{d(y_{n(p)-1},y_{m(p)-1},a)}\varphi(\mathrm{t})\mathrm{dt}\le\ell\le\int_0^{d(y_{n(p)-1},y_{m(p)-1},a)}\varphi(\mathrm{t})\mathrm{dt}$$

This implies,

$$\lim_{p\to\infty}\int_0^{d(y_{n(p)-1},y_{m(p)-1},a)}\varphi(\mathrm{t})\mathrm{dt}=\ell \qquad (13)$$

Now from (2), we have

$$\psi\int_0^{d(y_{n(p)},y_{m(p)},a)}\varphi(t)dt\le\psi\int_0^{d(y_{n(p)-1},y_{m(p)-1},a)}\varphi(\mathrm{t})\mathrm{dt}-\phi\int_0^{\mathrm{d}(y_{n(p)-1},y_{m(p)-1},a)}\varphi(\mathrm{t})\mathrm{dt} \qquad (14)$$

Taking limit as $p\to\infty$ and using (8) and (13) in (14) we get

$$\psi(\ell)\le\psi(\ell)-\phi(\ell)$$

This is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since (X,d) is complete 2-metric space, therefore there exists a point v such that

$Sx_n\to v\ \&\ \mathrm{Tx_n}\to v$ as $\mathrm{n}\to\infty$. Consequently, we can find h in X such that $T(h)=v.$

Now, for all $a\in X$ we have

$$\psi\int_0^{d(Sx_n,Sh,a)}\varphi(t)dt\le\psi\int_0^{d(Tx_n,Th,a)}\varphi(\mathrm{t})\mathrm{dt}-\phi\int_0^{d(Tx_n,Th,a)}\phi(t)dt$$

On taking Limit as $n\to\infty$ implies

$$\psi(\int_0^{d(v,Sh,a)}\phi(t)dt\ )\le\psi(0)-\varphi(0)$$

So $\psi(\int_0^{d(v,Sh,a)}\phi(t)dt\ )=0$ implies that $S(h)=v.$

Hence v is the point of coincidence of S and T.

Now we prove that v is the unique point of coincidence of S and T. Suppose not, therefore there exists $u,\ (u\ne v\ )$)and there exists $\mu\in X$ such that $S\mu=T\mu=u.$

Using (2) we have for all $a\in X$

$$\psi\int_0^{d(Tv,T\mu,a)}\phi(t)dt=\psi\int_0^{d(Sv,S\mu,a)}\phi(t)dt\le\psi\int_0^{\mathrm{d(Tv,T}\mu\mathrm{,a)}}\phi(\mathrm{t})\mathrm{dt}-\varphi\int_0^{d(Tv,T\mu,a)}\phi(t)dt$$

Therefore

$$\psi\int_0^{d(Tv,T\mu,a)}\phi(t)dt<\psi\int_0^{d(Tv,T\mu,a)}\phi(t)dt$$

This is a contradiction which implies $u=v.$ This proves uniqueness of point of coincidence of S and $T.$ Therefore by using lemma (11), the theorem is proved.

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