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Generalized Hardy type transformation of certain spaces of distributions

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ABSTRACT

In the present paper, the classical generalized Hardy type transformation depending on three real parameters $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v})$ defined by

$$F_1(y) = \left(C_{1,\alpha,\beta,a,b,v} f\right)(y) = v\, b\, y^{-1-2a+2v} \int_0^\infty (xy)^a\, C_{\alpha-\beta}\left[b(xy)^v\right] f(x)\, dx,$$

$\left(\alpha - \beta \geq -\frac{1}{2}\right)$, where $C_{\alpha-\beta}(z) = \cos(p\pi)\, J_{\alpha-\beta}(z) + \sin(p\pi)\, Y_{\alpha-\beta}(z)$ with $J_{\alpha-\beta}(z)$ and $Y_{\alpha-\beta}(z)$ denote the Bessel type function of first and second kind of order $(\alpha - \beta)$ respectively is extended to certain spaces of generalized functions by the kernel method in such a way that the theory of Pathak and Pandey [11] in relation with Hardy type transformation

$$F(y) = \left(C_{\alpha,\beta} f\right)(y) = \int_0^\infty x\, C_{\alpha-\beta}(xy)\, f(x)\, dx, \qquad \left((\alpha - \beta) \geq -\frac{1}{2}\right)$$

appears then as a particular case for $v = b = a = 1$. By interpreting the convergence in the weak distributional sense, an inversion theorem is established. The theory thus developed is applied to solve certain boundary value problems.

1. Introduction

The study of integral transforms in spaces of generalized functions has been an active area of work in the last few years. The methods of the theory of generalized functions have permitted a generalization of the classical results with the emergence of the theory of generalized functions, many aspects of integral transformation theory have acquired new more general treatment. Some generalizations of the classical and the distributional Hankel type transformations.

$$\left(h_{\alpha,\beta} f\right)(y) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, f(x)\, dx, \ \ (\alpha - \beta) \geq -\frac{1}{2} \tag{1.1}$$

and

$$\left(h_{\alpha,\beta} f\right)(y) = \int_0^\infty x\, J_{\alpha-\beta}(xy)\, f(x)\, dx, \ \ (\alpha - \beta) \geq -\frac{1}{2}. \tag{1.2}$$

were given by many authors from time to time.

The generalized Hankel type transformation depending on three real parameters $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v})$ defined by

$$\left(F_{1,\alpha,\beta,a,b,v} f\right)(y) = vby^{-1-2a+2v} \int_0^\infty (xy)^a J_{\alpha-\beta}\left[b(xy)^v\right] f(x)\, dx, \ \ (\alpha - \beta) \geq -\frac{1}{2} \tag{1.3}$$

and $$\left(F_{2,\alpha,\beta,a,b,v} f\right)(y) = vb \int_0^\infty x^{-1-2a+2v} (xy)^a J_{\alpha-\beta}\left[b(xy)^v\right] f(x)\, dx, (\alpha - \beta) \geq -\frac{1}{2} \tag{1.4}$$

which encompass (1.1) and (1.2) and a number of Hankel type transforms both known as well as unknown as special cases, has been extended to certain class of generalized functions. (see [7,8]).

Following [2] we define the classical Hardy type transformation as

$$F(y) = \left(C_{\alpha,\beta} f\right)(y) = \int_0^\infty x\, C_{\alpha-\beta}(xy)\, f(x)\, dx \tag{1.5}$$

with the inversion formula

$$f(x) = \left(C_{\alpha,\beta}^{-1} F\right)(x) = \int_0^\infty x\, F_{\alpha-\beta}(xy)\, F(y)\, dy \tag{1.6}$$

where

$$C_{\alpha-\beta}(z) = \cos(p\pi)\, J_{\alpha-\beta}(z) + \sin(p\pi)\, Y_{\alpha-\beta}(z) \tag{1.7}$$

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and

$$F_{\alpha-\beta}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\alpha-\beta+2p+2m}}{\Gamma(p+m+1)\Gamma(p+m+3\alpha+\beta)} = 2^{2-(\alpha-\beta)-2p} S_{\alpha-\beta+2p-1,\alpha-\beta}(z), \quad (1.8)$$

$S_{1,\alpha-\beta}$ being Lommel's function (see Watson; p.345 [12]) has been extended to a certain class of distributions. (see Pathak and Pandey [11]).

The Hankel type transformations (1.2) and the famous Y and its reciprocal H –transformations are particular cases of the Hardy type transformations (1.5). However the transformation of the type (1.1) and related Y and its reciprocal H-transformations cannot be reduced as a particular case of (1.5). This and the work of Malgonde [8] initiated us to work on the generalized Hardy type transformation defined by

$$F_1(y) = \left(C_{1,\alpha,\beta,a,b,v} f\right)(y) = vb\, y^{-1-2a+2v} \int_0^{\infty} (xy)^a\, C_{\alpha-\beta}\left[b\,(xy)^v\right] f(x)\, dx \quad (1.9)$$

and

$$F_2(y) = \left(C_{2,\alpha,\beta,a,b,v} f\right)(y) = vb \int_0^{\infty} x^{-1-2a+2v} (xy)^a\, F_{\alpha-\beta}\left[b(xy)^v\right] f(x)\, dx \quad (1.10)$$

where $C_{\alpha-\beta}(z)$ and $F_{\alpha-\beta}(z)$ are as defined in (1.7) and (1.8) respectively, in the classical as well as in the distributional sense of which (1.5) is a particular case.

Following Erdelyi [4, p.22, 74, 40, 41, 85], Babister [1, p.83], Watson [12, p.457], we have following well-known results.

In our analysis p and $(\alpha-\beta)$ will be assumed to be real. If $(\alpha-\beta)$ is an integer, we assume that the expression on the right of (1.11) is defined by its limiting values of $(\alpha-\beta)$ tends towards an integer.

$$C_{\alpha-\beta}(z) = \frac{\sin(p+\alpha-\beta)\,\pi}{\sin(\alpha-\beta)\,\pi} J_{\alpha-\beta}(z) - \frac{\sin(p\pi)}{\sin(\alpha-\beta)\,\pi} J_{\beta-\alpha}(z). \quad (1.11)$$

$$S_{1,\alpha-\beta} = S_{1,\alpha-\beta}(z) - 2^{1-1}\Gamma\big((1+\alpha+3\beta)/2\big)\,\Gamma(1+3\alpha+\beta)/2]\ \mathrm{cosec}\ v\pi \quad (1.12)$$

$$\times \left[\mathrm{cosec}\ \pi\,(\alpha+3\beta)/2\ J_{\beta-\alpha}(z) - \mathrm{cosec}\ \pi(3\alpha+\beta)/2\ J_{\alpha-\beta}(z)\right]$$

$$S_{1,\alpha-\beta}(z) \sim z^{1-1}\{(1-[(1-1)^2-(\alpha-\beta)^2]\, z^{-2} + [(1-1)^2-(\alpha-\beta)^2]\,[(1-3)^2-(\alpha-\beta)^2]\, z^{-4} \ldots\ldots\} \quad (1.13)$$

$$J_{\alpha-\beta}[b\,x^v] \cong [\Gamma(1+v)]^{-1} (b\,x^v/2)^{\alpha-\beta}, as\ x \to 0 \quad (1.14)$$

$$J_{\alpha-\beta}[b\,x^v] \cong \left(\frac{2v}{b\pi}\right)^{\frac{1}{2}} x^{-v/2} \cos[b\,x^v - (\pi/2)(2\alpha)] + O(bx^v)^{-3/2},\ \ as\ x \to \infty \quad (1.15)$$

$$\frac{d}{dz}\left[C_{\alpha-\beta}(bz)\right] = \frac{\alpha-\beta}{z} C_{\alpha-\beta}(bz) - b\, C_{3\alpha+\beta}(bz) \quad (1.16)$$

$$\frac{d}{dz}\left[S_{1,\alpha-\beta}(bz)\right] = \frac{\alpha-\beta}{z} S_{1,\alpha-\beta}(bz) - b(1-3\alpha-\beta)\, S_{1-1,3\alpha+\beta}(bz) \quad (1.17)$$

$$\int_0^t t^v J_{\alpha-\beta}(t)\, dt = (v-\alpha-3\beta)\, r\, J_{\alpha-\beta}(r) S_{v-1,-\alpha-3\beta}(r) - r\, J_{-\alpha-3\beta}(r) S_{v,\alpha-\beta}(r) \quad (1.18)$$

$$\int_0^t t^v J_{\beta-\alpha}(t)\, dt = (v-\alpha-3\beta)\, r\, J_{\beta-\alpha}(r) S_{v-1,-\alpha-3\beta}(r) + r\, J_{\alpha+3\beta}(r) S_{v,\alpha-\beta}(r).$$

Following Watson [12], Cook [2] and Pathak and Pandey [11], we have following well known classical theorems:

Theorem 1.1 (Convergence Theorem):

For any arbitrary real parameters a, b, v and $(\alpha-\beta) \geq -\frac{1}{2}$, if $f(x)$ is a locally integrable function on $0 < x < \infty$ such that

$f(x) = O(x^{\eta}),\ x \to 0$ and

$$f(x) = O\left(x^{\xi}\right), \qquad x \to \infty,$$

then the integral

$$\int_0^{\infty} x^a\, C_{\alpha-\beta}(bx^v)\, f(x)\, dx$$

defining the transformation (1.9) is absolutely convergent according to asymptotic expansions of $C_{\alpha-\beta}(z)$ when $\eta > (-a-(\alpha-\beta)v-1)$ and $\xi < (-a+v/2-1)$.

Theorem 1.2: (Inversion formula) :

If (i) $p > -1,\ p+\alpha-\beta > -1, |\alpha-\beta+2p| < 3/2,$

(ii) $x^{\sigma} f(x)$ is integrable over $(0,\delta),\ \sigma = min\,(\alpha+3\beta-2p, 1-(\alpha-\beta),\ 1/2),\ \delta > 0,$

(iii) $x^{\sigma - v/2} f(x)$ is integrable over $(0,\delta)$; and

(iv) $f(x)$ is of bounded variation into a neighborhood of the point $x = x_0 > 0$, then

$$\lim_{R\to\infty} v\, b\, x_0^{-1-2a+2v} \int_0^{\infty} f(x) \int_0^{R} y^{-1-2a+2v}\,(xy)^a\, C_{\alpha-\beta}\,[b(xy)^v]\,[(x_0 y)^a F_{\alpha-\beta}\,[b(x_0 y)^v]dydx$$

$$= \frac{1}{2}\{f\,(x_0+0) + f\,(x_0-0)\} \qquad (1.20)$$

or

$$\lim_{R\to\infty} v\, b\, x_0^{-1-2a+2v} \int_0^{R} (x_0 y)^a\, F_{\alpha-\beta}\,[b(x_0 y)^v]\ F_1(y)\, dy\, dx$$

$$= \frac{1}{2}\{f(x_0+0) + f(x_0-0)\}$$

where $F_1(y)$ is defined by (1.9).

Theorem 1.3: (Operational calculus):

(i) If $y = x^a\, C_{\alpha-\beta}\,(b\, x^v)$ then it satisfies the differential equation

$$x^2 y'' + (1-2a)\, xy' + [b^2 v^2 x^{2v} + (a^2 - (\alpha-\beta)^2\, v^2)]\, y = 0$$

(ii) If $y = x^{-1-a+2v} F_{\alpha-\beta}\,(b\, x^v)$ then it satisfies the differential equation

$$x^2 y'' - (4v-2a-3)\, xy' - \{(\alpha-\beta)^2 v^2 - (a+1-2v)^2 + b^2 v^2 x^{2v}\}\, y = b^{\eta+1} v^2 x^{v+v\eta+2v-a-1}$$ where $\eta = \alpha-\beta+2p-1$.

(iii) $\frac{d}{dx}\left[x^{(\alpha-\beta)v}\, C_{\alpha,\beta}\,(bx^v)\right] = vbx^{v(\alpha-\beta)+v-1}\, C_{-\alpha-3\beta}\,(bx^v)$

(iv) $\frac{d}{dx}\left[x^{-(\alpha-\beta)v} C_{\alpha-\beta}\,(bx^v)\right] = -vbx^{-v(\alpha-\beta)+v-1}\, C_{3\alpha+\beta}\,(bx^v)$

(v) If $\Delta_{a,\alpha,\beta,v} = x^{-(\alpha-\beta)v+a+1-2v}\, D_x\, x^{2(\alpha-\beta)v+1}\, D_x\, x^{-a-(\alpha-\beta)v}$

$$= x^{2-2v} D_x^2 + (1-2a) x^{1-2v}\, D_x + (a^2 - (\alpha-\beta)^2 v^2)\, x^{-2v}, \qquad and$$

$$\Delta^*_{a,\alpha,\beta,v} = x^{-a-(\alpha-\beta)v}\, D_x\, x^{2(\alpha-\beta)v+1}\, D_x\, x^{-(\alpha-\beta)v+a+1-2v}$$

$$= x^{2-2v} D_x^2 - (4v-2a-3)\, x^{1-2v}\, D_x - [(\alpha-\beta)^2\, v^2 - (a+1-2v)^2]\, x^{-2v}$$

denote the Bessel type differential operators then for $(\alpha-\beta) \geq -1/2$ and for any arbitrary real numbers a, b, v

$$\Delta^k_{a,\alpha,\beta,v}\,[K_1(x,y)] = (-v^2 b^2)^k\, y^{2vk}\, K_1\,(x,y). \qquad (1.21)$$

and

$$\Delta^{*k}_{a,\alpha,\beta,v}\,[K_2\,(x,y)] = (-v^2 b^2)^k\, y^{vk}\, K_2\,(x,y) - P(x,y). \qquad (1.22)$$

where

$$P(x,y) = y^{(\alpha-\beta)v+2pv-a+2v-1} \sum_{i=1}^{k} a_i\, x^{(\alpha-\beta)v+2pv-a+1-2i}\, y^{2vk-2vi},$$

$$K_1\,(x,y) = vby^{-1-2a+2v}\,(xy)^a\, C_{\alpha-\beta}\,[b\,(xy)^v]$$

and

$$K_2\,(x,y) = vbx^{-1-2a+2v}\,(xy)^a\, F_{\alpha-\beta}\,[b\,(xy)^v]$$

are the Kernels of the transformations (1.9) and (1.10) respectively.

(vi) The transformation (1.9) satisfies the operational rule

$$F_{1,\alpha,\beta,a,b,v}\{\Delta^*_{a,\alpha,\beta,v}\, f(x)\}\,(y) = \,(-v^2 b^2)\, y^{2v}\, F_{1,\alpha,\beta,a,b,v}\,[f(x)]\,(y)$$

where $f(x)$ is a suitable function.

Theorem 1.4: If $(\alpha-\beta) \geq -1/2$, $p > -1$, $\alpha-\beta+p > -1$, $1+|\alpha-\beta|+1 > 0$, $1 = \alpha-\beta+2p-1$, then

$$b \int_0^{R^v} y^{-1+2v}\, C_{\alpha-\beta}\,[b\,(ty)^v]\;\; F_{\alpha-\beta}\,[b(xy)^v]\, dy$$

$$= \frac{R^v}{x^{2v}-t^{2v}} \left\{ \begin{array}{c} (x/t)^{v(1+1)}\, \dfrac{t^v}{sin(\alpha-\beta)\pi}\,(1-\alpha-3\beta)\left[\sin(p+\alpha-\beta)\pi\, J_{\alpha-\beta}[b(tR)^v\,\right] \\ -\sin p\pi\; J_{\beta-\alpha}\,[b(tR)^v]\, S_{1-1,-\alpha-3\beta}\,[b\,(tR)^v] - \,(\sin(p+\alpha-\beta)\pi)\, J_{-\alpha-3\beta}\,[b\,(tR)^v] \\ +\sin p\pi\, J_{\alpha+3\beta}\;[b(tR)^v]\; S_{1,\alpha-\beta}\,[b(tR)^v] - A\,[x^v\,(\alpha+3\beta-1) \\ \times\; C_{\alpha-\beta}\,[b(tR)^v]\, S_{1-1,3\alpha+\beta}\,[b\,(xR)^v] + t^v\, C_{3\alpha+\beta}\,[b(tR)^v] S_{1,\alpha-\beta}\,[b(xR)^v] \end{array} \right\}$$

where $1 = \alpha-\beta+2\beta-1$ and $A = \frac{2^{3\alpha+5\beta-2p}}{\Gamma(p)\,\Gamma(\alpha-\beta+p)}$. (1.23)

Proof : We know that $u = F_{\alpha-\beta}\,(ax)$ *and* $v = C_{\alpha-\beta}\,(bx)$ are solutions of the differential equations

$\frac{d^2u}{dy^2} + \frac{1}{y}\frac{du}{dy}\,(a^2 - 1^2/y^2)\, u = \; y^{1+1}\, a^{1+1}$ and

$\frac{d^2v}{dy^2} + \frac{1}{y}\frac{dv}{dy} + (a^2 - 1^2/y^2)\, v = 0$ respectively.

From above two equations, we get

$$(a^2-b^2)\int_0^{R^v} y\, uvdy = \int_0^{R^v} y^1\, a^{1+1} v\, dy - \left[y\left(v\,\frac{du}{dy} - u\frac{dv}{dy}\right)\right]_0^R$$

where $p > -1$ and $< \alpha - \;\beta + p > -1$. Using change of variables and the results (1.11), (1.15) –(1.19) this can easily be shown to be equivalent to

$$b\int_0^{R^v} y^{-1+2v}\, C_{\alpha-\beta}\,[b(ty)^v]\, F_{\alpha-\beta}\,[b(xy)^v]\, dy$$

$$= \frac{R^v}{x^{2v}-t^{2v}} \left\{ \begin{array}{c} (x/t)^{v(1+1)}\, \frac{t^v}{\sin(\alpha-\beta)\pi}\,(1-\alpha-3\beta)\left[\sin(p+\alpha-\beta)\pi\, J_{\alpha-\beta}\,[b(tR)^v]\right] \\ -\sin p\pi\, J_{\beta-\alpha}\,[b(tR)^v])\, S_{1-1,-\alpha-3\beta}\,[b(tR)^v] - (sin(p+\alpha-\beta)\pi) \\ \times\; J_{-\alpha-3\beta}\,[b(tR)^v] + \sin p\pi\, J_{\alpha+3\beta}\;[b(tR)^v])\, S_{1,\alpha-\beta}\,[b(tR)^v]] \\ -A\,\left[x^v\,(1-\alpha-3\beta)\, C_{\alpha-\beta}\,[b(tR)^v]\;\, S_{1-1,3\alpha-\beta}\,[b(xR)^v]\right] \\ +\, t^v\, C_{3\alpha+\beta}\,[b(tR)^v]\, S_{1,\alpha-\beta}\,[b(tR)^v]] \end{array} \right\}.$$

where A is the same as given in (1.23), $p > -1$ and $\alpha - \beta + p > -1.$ Thus proof is completed.

In this paper we extend the F_1 –transformation (1.9) to other spaces of generalized functions following a different procedure called the kernel method. For any real numbers $a, \alpha-\beta, v,$ and $c > 0$, we construct a testing function space $H_{a,\alpha,\beta,v,c}$ which contains the kernel $K_1\,(x,y)$, as function on $0 < x < \infty$, for each fixed positive real y. The F'_1 –transformation is now defined on the dual space $H'_{a,\alpha,\beta,v,c}$ as follows: For

$f \in H'_{a,\alpha,\beta,v,c}$,

$$F'_1\{f\}\,(y) = \langle f(x),\;\, K_1\,(x,y)\rangle,\; for\; y > 0. \tag{1.24}$$

This definition is more convenient for specific computation, e.g.

$$F'_1\{\delta(t-b)\} = \langle\delta(t-b),\, K_1\,(t,y)\rangle = vby^{-1-2a+2v}\,(by)^a\, C_{\alpha-\beta}\,[b(by)^v]$$

$$= v\, b\, y^{-1-a+2v}\, d^a\, C_{\alpha-\beta}\,[b(dy)^v]\, for\; d\; > 0.$$

Theorems on smoothness, boundedness, inversion and uniqueness, together with an operation-transform formula for a Bessel-type differential operator are given and the theory is illustrated solving certain distributional differential equation by our distributional generalized Hardy type transformation. The arbitrariness of the parameters (a, b, v) facilitates this work and allow us to obtain particularly the results of Pathak and Pandey [11], Koh and Zemanian [5] and Malgonde [8] in relation with the transformations (1.5), (1.1) and (1.3), respectively, and other known and unknown Hankel type integral transformations (see [3] and [10]).

We follow the notations and terminology of [13] and [8].

2. The testing function spaces $H_{a,\alpha,\beta,v,c}$ and $H_{a,\alpha,\beta,v}^{(\sigma)}$ and their duals:

Let a be a positive real number and $a, \alpha-\beta, v$ be any arbitrary real parameters. Then we define $H_{a,\alpha,\beta,v,c}$ as the space of testing functions $\phi(x)$ which are defined on $0<x<\infty$ and for which

$$\rho_k^{a,\alpha,\beta,v,c}\,\phi(x) = \sup_{0<x<\infty}\left|e^{-cx}\,x^{-a-(\alpha-\beta)v}\,\Delta_{a,\alpha,\beta,v}^{k}\,\phi(x)\right| < \infty, \qquad (2.1)$$

for $k = 0,1,2,3,\ldots..$

Now we list some properties of these spaces:

(a) Let $(\alpha-\beta) \geq -1/2,\ c>0\ and\ a, v$ be any arbitrary real parameters. For a fixed positive real number y,

$$\frac{\partial^m}{\partial y^m}\,[K_1(x,y)] \in H_{a,\alpha,\beta,v,c}\,, m = 0,1,2,3,\ldots..$$

It can be easily verified that

$$\frac{\partial^m}{\partial y^m}\,[K_1(x,y)] = \frac{\partial^m}{\partial y^m}\left[vby^{-1-2a+2v}\,(xy)^a\,C_{\alpha-\beta}\,[b(xy)^v]\right]$$

$$= vb\sum_{j=0}^{m} a_j\,(a,\alpha-\beta,v)\,(bv)^j\,y^{-1-a+2v-m+vj}\,C_{\alpha-\beta-j}\,[b(xy)^v]$$

$$= \sum_{j=0}^{m} a_j\,(a,\alpha-\beta,b,v)\,\,y^{-1-a+2v-m+vj}\,x^{a+vj}\,C_{\alpha-\beta-j}\,[b(xy)^v]$$

where the $a_j\,(a,\alpha-\beta,b,v)$ are constants depending upon $(a,\alpha-\beta,v)\ and\ b.$ By the series and asymptotic expansions of $C_{\alpha-\beta-j}\,[b\,(xy)^v]$ it follows that the quantities

$$\eta_k^{a,\alpha-\beta,v,c}\left[y^{-1-a+2v}\,x^{a+vj}\,C_{\alpha-\beta-j}\,[b(xy)^v]\right]$$

exists for all $k = 0,1,2,3,\ldots.,$ and $(\alpha-\beta) \geq -1/2$. Hence

$$\rho_k^{a,\alpha,\beta,v,c}\left\{\frac{\partial^m}{\partial y^m}\,[K_1\,(x,y)]\right\}$$

$$\leq \sum_{j=0}^{m} a_j\,(a,\alpha-\beta,b,v)\,y^{vj-m}\,\rho_k^{a,\alpha,\beta,v,c}\left\{y^{-1-a+2v}x^{a+vj}\,C_{\alpha-\beta-j}\,[b(xy)^v]\right\} < \infty$$

for any fixed $y>0$.

(b) $H_{a,\alpha,\beta,v,c}$ is sequentially complete and therefore a Frechet space. Hence $H'_{a,\alpha,\beta,v,c}$ is also sequentially complete.

(c) If $c>d>0$, then $H_{a,\alpha,\beta,v,d} \subset H_{a,\alpha,\beta,v,c}$, and the topology of $H_{a,\alpha,\beta,v,d}$ is tronger than that induced on it by $H_{a,\alpha,\beta,v,c}$.

(d) $D(I) \subset H_{a,\alpha,\beta,v,c}$, and the topology of $D(I)$ is stronger than that induced on it by $H_{a,\alpha,\beta,v,c}$. Hence, the restriction of any $f \in H'_{a,\alpha,\beta,v,c}$ to $D(I)$ is in $D'(I)$, and convergence in $H'_{a,\alpha,\beta,v,c}$ implies weak convergence in $D'(I)$.

(e) For every choice of $a,\alpha,\beta,v\ and\ c$, $H_{a,\alpha,\beta,v,c} \subset E(I)$. Moreover, it is dense in $E(I)$ because $D\,(I) \subset H_{a,\alpha,\beta,v,a}$ and $D\,(I)$ is dense in $E(I)$. The topology of $H_{a,\alpha,\beta,v,c}$ is stronger than that induced on it by $E(I)$. Hence $E'(I)$ can be identified with a subspace of $H'_{a,\alpha,\beta,v,c}$.

(f) The operation $\phi \rightarrow \Delta_{a,\alpha,\beta,v}\,\phi$ is a continuous linear mapping of $H_{a,\alpha,\beta,v,c}$ into itself because

$$\rho_k^{a,\alpha,\beta,v,c}\left(\Delta_{a,\alpha,\beta,v}\phi\right) = \rho_{k+1}^{a,\alpha,\beta,v,c}(\phi), \qquad for\ k = 0,1,2,\ldots..$$

(g) For each $f \in H'_{a,\alpha,\beta,v,c}$, there exist a non-negative integer r and a positive constant c such that, for all $\phi \in H_{a,\alpha,\beta,v,c}$,

$$|\langle f,\phi\rangle| \leq \subset \max_{0\leq k\leq r} \rho_k^{a,\alpha,\beta,v,c}(\phi) .$$

(h) Let $f(x)$ be a locally integrable on $0 < x < \infty$ and such that

$$\int_0^\infty \left|f(x)\, x^{a+(\alpha-\beta)v}\, e^{cx}\right| dx < \infty.$$

Then $f(x)$ generates a regular generalized function in $H'_{a,\alpha,\beta,v,c}$ defined by

$$\langle f,\phi\rangle = \int_0^\infty f(x)\,\phi(x)\,dx, \ \ \phi \in H_{a,\alpha,\beta,v,c} .$$

Next, we give a structure formula for the restriction of element in $H'_{a,\alpha,\beta,v,c}$ to $D(I)$.

(i) Let f be an arbitrary element of $H'_{a,\alpha,\beta,v,c}$. Then there exist bounded measurable functions $g_i(x)$ defined for $x > 0$ for $i = 0,1,2,\ldots.r$ where r is some non-negative integer depending upon f such that for an arbitrary $\phi \in D(I)$ we have

$$\langle f,\phi\rangle = \langle \sum_{i=0}^{r} \left(\Delta^*_{a,\alpha,\beta,v}\right)^i \left\{e^{-cx}\, x^{-a-(\alpha-\beta)v}\,(-D)\, g_i(x)\right\},\ \ \phi(x)\rangle.$$

By note (g) for every $f \in H'_{a,\alpha,\beta,v,c}$ there exist a non-negative r and a positive constant C such that for all $\phi \in D(I) \subset H_{a,\alpha,\beta,v,c}$ one has

$$|\langle f,\phi\rangle| \ \leq C\ \max_{0\leq k\leq r} \eta_k^{a,\alpha,\beta,v,c}(\phi)$$

$$\leq C \max_{0\leq k\leq r} \left\|D\left\{e^{-cx}\, x^{-a-(\alpha-\beta)v}\, \Delta^k_{a,\alpha,\beta,v}\, \phi(x)\right\}\right\|_{L_1(0,\infty)}$$

where $L_1(0,\infty)$ is the space of equivalence classes of Lebesgue integrable functions on $(0,\infty)$ whose topology is defined by the norm $\|f\|_{L_1(0,\infty)} = \int_0^\infty |f(x)|\,dx,\ f \in L_1(0,\infty)$. Hence in view of the Hahn-Banach theorem and since the dual of $L_1(0,\infty)$ is isomorphic with $L_\infty(0,\infty)$, the space of all equivalence classes of complex-valued integrable functions on $(0,\infty)$ such that for every $f \in L_\infty(0,\infty)$ there exist a constant M such that $|f(x)| < M\ a\ e.$, there are functions $g_i \in L_\infty(0,\infty)$,
$0 \leq i \leq r$ such that

$$\langle f,\phi\rangle = \langle \sum_{i=0}^{r} \left(\Delta^*_{a,\alpha,\beta,v}\right)^i \left\{e^{-cx}\, x^{-c-(\alpha-\beta)v}\,(-D_x)\, g_i(x)\right\},\ \ \phi(x)\rangle$$

where $\Delta^*_{a,\alpha,\beta,v}$ is the adjoint operator of $\Delta_{a,\alpha,\beta,v}$ as defined in (v) of Theorem 1.3. This proves our assertion.

Our subsequent discussion takes on a simpler form when the $H_{a,\alpha,\beta,v}(\sigma)$ spaces are used in place of the $H_{a,\alpha,\beta,v,c}$ spaces (see [5]). A generalized function f is $F'_{1,\alpha,\beta,a,b,v}$ - transformable if $f \in H'_{a,\alpha,\beta,v}(\sigma)$ for some $\sigma > 0$ where $H'_{a,\alpha,\beta,v}(\sigma)$ is the dual of $H_{a,\alpha,\beta,v}(\sigma)$.

In view of definitions of $H_{a,\alpha,\beta,v}(\sigma)$ and its dual, the following lemmas are immediate.

Lemma 2.1: For any fixed $y > 0,\ \frac{\partial^m}{\partial y^m}\,[K_1(x,y)] \in H_{a,\alpha,\beta,v}(\sigma),\ m = 0,1,2,3,\ldots.,$ where $\sigma > 0$.

Lemma 2.2: The operation $\phi \rightarrow \Delta_{a,\alpha,\beta,v}\,\phi$ is a continuous linear mapping of $H_{a,\alpha,\beta,v}(\sigma)$ into $H_{a,\alpha,\beta,v}(\sigma)$. Hence the operation $f \rightarrow \Delta^*_{a,\alpha,\beta,v}\, f$ is a continuous linear mapping of $H'_{a,\alpha,\beta,v}(\sigma)$ into itself (see [13]).

3. The distributional generalized Hardy type transformation:

Let $(\alpha-\beta) \geq -1/2$ and a, v be any arbitrary real numbers. In view of note (c) of §2, to every $f \in H'_{a,\alpha,\beta,vc}$ there exists an unique real number σ_f (possibly, $\sigma_f = +\infty$) such that $f \in H'_{a,\alpha,\beta,v,d}$ if $d < \sigma_f$ and $f \notin H'_{a,\alpha,\beta,v,d}$ if $d < \sigma_f$. Therefore, $f \in$

$H'_{a,\alpha,\beta,v}\left(\sigma_f\right)$. We define the $(\alpha-\beta)^{th}$ order distributional Hardy transform $F'_{1,\alpha,\beta,a,b,v}\,f$ of f as the application of f to the kernel of $K_1(x,y)$; i.e.

$$F_1\,(y) = \left(F'_{1,\alpha,\beta,a,b,v}\,f\,\right)(y) = \langle f(x), K_1\,(x,y)\rangle \tag{3.1}$$

where $0 < y < \infty$ and $\sigma_f > 0$. The right hand side of (3.1) has a sense because, by Lemma 2.1, $K_1(x,y) \in H_{a,\alpha,\beta,v}(\sigma_f)$ for each $y > 0$ and $\sigma_f > 0$.

Lemma 3.1: Let c and σ_f be fixed real numbers such that $0 < c < \sigma_f$. For all fixed $y > 0$, for $(\alpha-\beta) \geq -1/2$ and for $0 < x < \infty$

$$\left|e^{-cx}\,[b(xy)^v]^{-(\alpha-\beta)} C_{\alpha-\beta}\,[b(xy)^v]\right| < A_{\alpha,\beta,b,v}, \tag{3.2}$$

where $A_{\alpha,\beta,b,v}$ is a constant with respect to x and y.

Proof: Following Koh and Zemanian [5], proof can be given.

Now we will show that $F_1(y) = \left(F'_{1,\alpha,\beta,a,b,v}\,f\right)(y)$ is analytic.

Theorem 3.1: For $y > 0$, let $F_1(y)$ be defined by (3.1). Then

$$\frac{d}{dy}[F_1\,(y)] = \langle f(x), \frac{\partial}{\partial y}\,K_1\,(x,y)\rangle.$$

Proof: Proof is not much difficult and can be completed just by referring to Koh and Zemanian [5] again.

Theorem 3.2: Let $F_1(y)$ be defined by (3.1). Then $F_1(y)$ is bounded according to

$$|F_1(y)| \leq \begin{cases} c\,y^{-1-a+2v+(\alpha-\beta)v} & as\ y \to 0^+ \\ c\,y^{2vr-1-a+2v+(\alpha-\beta)v} & as\ y \to \infty \end{cases} \tag{3.3}$$

where c is positive constant and r is a non-negative integer depending on f.

Proof: As $f \in H'_{a,\alpha,\beta,v,d}$, where $0 < c < d < \sigma_f$, we see from note (g) of section 2 that there exist a constant c and a non-negative integer r such that

$$|F_1(y)| \leq c \max_{0\leq k\leq r} \sup_{0<x<\infty} \left|e^{-cx}\,x^{-a-(\alpha-\beta)v}\,\Delta^k_{a,\alpha,\beta,v}\,[k_1\,(x_1 y)]\,\right|.$$

By (1.21), the right-hand side is equal to

$$c \max_{0\leq k\leq r} \sup_{0<x<\infty} \left|e^{-cx}\,y^{-1-a+2v+(\alpha-\beta)v+2vk}[b(xy)^v]^{-(\alpha-\beta)} C_{\alpha-\beta}\,[b\,(xy)^v]\,\right|.$$

Theorem follows from Lemma 3.1.

Now we state an inversion theorem for our distributional generalized Hankel type integral transformation $\left(F'_{1,\alpha,\beta,a,b,v} - transformation\right)$.

Theorem 3.3: Let $F_1(y) = \left(F'_{1,\alpha,\beta,a,b,v} f\right)(y)$, $f \in H'_{a,\alpha,\beta,v}\,\left(\sigma_f\right)$ as in (3.1) where $(\alpha-\beta)$ is as defined in section 1 and $y > 0$. Then in the sense of convergence in $D'(I)$,

$$f(x) = \lim_{R\to\infty} \int_0^R F_1(y)\,K_2\,(x,y)\,dy\,, \tag{3.4}$$

where

$$K_2\,(x,y) = vbx^{-1-2a+2v}\,(xy)^a\;\;F_{\alpha-\beta}\,[b(xy)^v].$$

Proof: Let $\phi \in D(I)$. We want to show that

$$\langle \int_0^R F_1(y)\,K_2\,(x_1 y) dy,\;\; \phi(x)\rangle \to \langle f(x), \phi(x)\rangle. \tag{3.5}$$

as $R \to \infty$. From the smoothness of $F_1(y)$ and the fact that support of $f(x)$ is a compact subset of I, we may write (3.5) as a repeated integral on (x,y) having a continuous integrand and a finite domain of integration. Hence we can change the order of integration and obtain

$$\int_0^\infty \phi(x)\,\int_0^R F_1(y)\,K_2\,(x,y)\,dy\,dx = \int_0^R \langle f(t), K_1(t,y)\rangle \int_0^\infty \phi(x)\,K_2\,(x,y)\,dx\,dy. \tag{3.6}$$

By using an argument of Riemann sums for the integral

$\int_0^R \ldots.dy$, the right hand side of (3.6) can be written as

$$\langle f(t), \int_0^R K_1\,(t,y)\ \int_0^\infty \phi(x)\,K_2\,(x,y)\,dx\,dy\rangle. \tag{3.7}$$

By using the Theorem 1.4 and the asymptotic representations of the Bessel functions and Lommel's functions enable us to show that for any $a > 0$, the testing function in (3.7) converges in $H_{a,\alpha,\beta,v,c}$ to $\phi(t)$ as $R \to \infty$. Since $f \in H'_{a,\alpha,\beta,v,c}$ where $0 < c < \sigma_f$, it follows that (3.7) converges to $\langle f(t),\ \phi(t)\rangle$ as $R \to \infty$. This proves the theorem.

Theorem 3.4: (Uniqueness Theorem) :

Let $F_1(y) = \left(F'_{1,\alpha,\beta,a,b,v}f\right)(y)\ for\ y > 0\ and$

$G_1(y) = \left(F'_{1,\alpha,\beta,a,b,v}g\right)(y)\ for\ y > 0,\ fand\ g$ being in $H'_{a,\alpha,\beta,v}\,(\sigma)$. If $F_1(y) = G_1(y)$, for every $y > 0$, then $f = g$ in the sense of equality in $D'(I)$.

Proof: By Theorem 3.3

$$f - g = \lim_{R\to\infty} \int_0^R [F_1(y) - G_1(y)]\,K_2\,(x,y)\,dy = 0.$$

4. An Operational Calculus:

In this section, we shall apply the theory so far developed in solving certain differential equations involving generalized functions. We define the operator

$$\Delta^*_{a,\alpha,\beta,v}\colon\ H'_{a,\alpha,\beta,v}(\sigma_f) \to H'_{a,\alpha,\beta,v}(\sigma_f)$$

by the relation

$$\langle \Delta^*_{a,\alpha,\beta,v}\,f(x), \quad \phi(x)\rangle = \langle f(x), \quad \Delta_{a,\alpha,\beta,v}\,\phi(x)\rangle$$

for all $f \in H'_{a,\alpha,\beta,v}\,(\sigma_f)$ and $\phi\,(x) \in H_{a,\alpha,\beta,v}\,(\sigma_f)$ for $(\alpha-\beta)$ is as defined in section 1 and for $a\ and\ v$ arbitrary real numbers. It can be readily seen that

$$\langle \left(\Delta^*_{a,\alpha,\beta,v}\right)^k f(x), \quad \phi(x)\rangle = \langle f(x),\ \Delta^k_{a,\alpha,\beta,v}\,\phi(x)\rangle$$

for each $k = 1,2,3, \ldots$. In case f is a regular distribution generated by an element of $D(I)$, then

$\Delta^*_{a,\alpha,\beta,v} = x^{-a-(\alpha-\beta)v} D\, x^{2(\alpha-\beta)v+1}\, D\, x^{-(\alpha-\beta)v+a+1-2v}$

$\quad = x^{2-2v}\, D_x^2 - (4v - 2a - 3)\, x^{1-2v}\, D_x - \left[(\alpha-\beta)^2 v^2 - (a+1-2v)^2\right] x^{-2v}.$

The distributional generalized Hardy type transformation is used in solving initial value problems. In fact, we now establish a theorem that enables us to transform a differential equation of the form

$$P\left[\Delta^*_{a,\alpha,\beta,v}\right]u = g. \tag{4.1}$$

where $u\ and\ g$ posses $F'_{1,\alpha,\beta,a,b,v}$ - transforms and P is any polynomial having no zeros on $-\infty < x \le 0$ into an algebraic equation of the form

$$P\left[-y^{2v}\right] U(y) = G(y),$$

where $U\,(y) = \left(F'_{1,\alpha,\beta,a,b,v}\,u(x)\right)(y)\ and\ G(y) = \left(F'_{1,\alpha,\beta,a,b,v}\,g(x)\right)(y).$

Theorem 4.1: For $k = 0,1,2,3, \ldots$

$$F'_{1,\alpha,\beta,a,b,v}\left[\Delta^*_{a,\alpha,\beta,v}f\right](y) = (-b^2v^2)^k\, y^{2vk}\, F'_{1,\alpha,\beta,a,b,v}f\,, \tag{4.2}$$

for every $f \in H'_{a.\alpha,\beta,v}\,(\sigma_f)$.

Proof: Form definition of the operator $\Delta^*_{a,\alpha,\beta,v}$ in note of (f) of section 2 and from equation (1.21), we have

$$\begin{aligned} F'_{1,\alpha,\beta,a,b,v}\left[\Delta^{*k}_{a,\alpha,\beta,v}f\right](y) &= \langle \Delta^{*k}_{a,\alpha,\beta,v}f(x), \quad K_1\,(x,y)\rangle \\ &= \langle f(x), \quad \Delta^k_{a,\alpha,\beta,v}\,K_1\,(x,y)\rangle \\ &= (-b^2v^2)^k\, y^{2vk} \langle f(x), \quad K_1\,(x,y)\,\rangle \\ &= (-b^2v^2)^k\, y^{2vk}\, F'_{1,\alpha,\beta,a,b,v}\,f. \end{aligned}$$

We can find a generalized function $\boldsymbol{u} \in \boldsymbol{H}'_{a,\alpha,\beta,v}(\boldsymbol{\sigma})$ for some $\boldsymbol{\sigma} > \mathbf{0}$ satisfying the operator equation (4.1) by invoking Theorem 4.1. The distributional generalized Hardy type transformation $\boldsymbol{F}'_{1,\alpha,\beta,a,b,v}$ can be used in solving the generalized Cauchy problem as an application of the preceding theory to generalized Cauchy problems having a generalized function like initial condition (see [7,8]).

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