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An n-dimensional pseudo-differential type operator involving Hankel type transformation

B.B.Waphare
MIT ACSC, Alandi, Haveli, Pune, Maharashtra, India.

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ABSTRACT

In this paper Bessel type differential operator $\Delta_{\alpha,\beta}$ and an n-dimensional pseudo-differential type operator involving the n-dimensional Hankel type transformation is defined. The symbol class H^m is introduced. It is shown that pseudo-differential type operators associated with symbols belonging to this class are continuous linear mappings of the n-dimensional Zemanian space $H_{\alpha,\beta}(I^n)$ into itself. An integral representation for the pseudo-differential operator is obtained. By using the Hankel type convolution, it is shown that the pseudo-differential type operator satisfies a certain $L^1 -$ norm inequality.

1. Introduction

In this paper, we shall use the following notations:

$\mathbb{R}^n$ and $\mathbb{C}^n$ are respectively the real and complex n-dimensional Euclidean spaces. An n-tuple is denoted by $x = (x_1, \ldots\ldots, x_n)$. We shall restrict x and y to the first orthant of $\mathbb{R}^n$which we denote by I^n. Thus $I^n = \{x \in \mathbb{R}^n : 0 < x_i < \infty, i = 1, \ldots\ldots, n\}$ and the Euclidean norm $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. A function on a subset of $\mathbb{R}^n$ is denoted by $\phi(x) = \phi(x_1, \ldots\ldots, x_n)$. By $[x]$ we mean the product $x_1 \ldots\ldots x_n$. Thus $[x^q] = x_1^{q_1} \ldots . x_n^{q_n}$, where $q = (q_1, \ldots., q_n)$. The notation $x \le y$ means $x_i \le y_i \ (i = 1,2, \ldots., n)$. In what follows, the letters k and q are denoted by non-negative integers in $\mathbb{R}^n$ i.e. k_i and $q_i \ (i = 1,2, \ldots. n)$ are non-negative integers. Letting $|k| = k_1 + k_2 + \cdots + k_n$,

We shall write

$$D_x^k = \frac{\partial^{|k|}}{\partial x_1^{k_i} \ldots\ldots \partial x_n^{k_n}} ,$$

and denote

$$(x^{-1}D_x)^k = \prod_{i=1}^{n} \left(x_i^{-1} \frac{\partial}{\partial x_i} \right)^{k_i} .$$

Here we use the notations and terminology of [5,3]. For $(\alpha - \beta) \in \mathbb{R}$, the differential operators $N_{\alpha,\beta}$, $M_{\alpha,\beta}$ and $\Delta_{\alpha,\beta}$ are defined by

$$N_{i\alpha,i\beta} = x_i^{2\alpha} \frac{\partial}{\partial x_i} x_i^{2\beta-1} ,$$

$$N_{\alpha,\beta} = N_{1\alpha,1\beta} \ldots\ldots N_{n\alpha,n\beta} = [x]^{2\alpha} \frac{\partial^n}{\partial x_1 \ldots . \partial x_n} [x]^{2\beta-1} ,$$

Tele:
E-mail addresses: balasahebwaphare@gmail.com

$$M_{i\alpha,i\beta} = x_i^{2\beta-1}\frac{\partial}{\partial x_i}\, x_i^{2\alpha} \quad ,$$

$$M_{\alpha,\beta} = M_{1\alpha,1\beta} \ldots\ldots M_{n\alpha,n\beta} = [x]^{2\beta-1}\,\frac{\partial^n}{\partial x_1 \ldots\ldots \partial x_n}\,[x]^{2\alpha}$$

and

$$\Delta_{\alpha,\beta} = M_{\alpha,\beta}\, N_{\alpha,\beta} = [x]^{2\beta-1}\,\frac{\partial^n}{\partial x_1 \ldots\ldots \partial x_n}\,[x]^{4\alpha}\,\frac{\partial^n}{\partial x_1 \ldots\ldots \partial x_n}\,[x]^{2\beta-1}$$

$$\Delta_{\alpha,\beta} = \prod_{i=1}^{n} x_i^{4\alpha+4\beta-2}\left[\frac{\partial^2}{\partial x_i^2} + \frac{4\alpha+4\beta-2}{x_i}\,\frac{\partial}{\partial x_i} + \frac{(2\beta-1)(4\alpha+2\beta-2)}{x_i^2}\right].$$

Now, we define the test function space $H_{\alpha,\beta}\,(I^n)$ to be the space of all smooth complex valued functions $\phi(x)$ which are defined on I^n such that for each pair of non-negative integers q and k in $\mathbb{N}_o^n$.

$$\rho_{q,k}^{\alpha,\beta}\,(\phi) = \underset{x\,\epsilon\, I^n}{Sup}\left|[x^q]\,(x^{-1}\,D_x)^k\,[x]^{2\beta-1}\,\phi(x)\right| < \infty \tag{1.1}$$

The space $H_{\alpha,\beta}\,(I^n)$ is topologized by the family of seminorms $\left\{\rho_{q,k}^{\alpha,\beta}\right\}_{q,k,\epsilon\,\mathbb{N}_0^n}$.

The n-dimensional classical $(\alpha-\beta)^{th}$ order Hankel type transformation $h_{\alpha,\beta}$ is defined by

$$\hat{\phi}\,(y) = \left(h_{\alpha,\beta}\,\phi\right)(y) = \int_0^\infty \ldots.\int_0^\infty \phi\,(x_1, x_2 \ldots., x_n)$$

$$\times\,\left(\prod_{i=1}^n (x_i y_i)^{\alpha+\beta}\, J_{\alpha-\beta}\,(x_i y_i)\right) dx_1 \ldots..dx_n,\ \phi \in H_{\alpha,\beta}\,(I^n)\,, \tag{1.2}$$

where $y \in I^n$ and $J_{\alpha-\beta}$ is the Bessel type function of the first kind and order $(\alpha-\beta)$can be extended by transposition to distributions belonging to $H'_{\alpha-\beta}\,(I^n)$ the dual of the test function space $H_{\alpha,\beta}\,(I^n)$, provided that $(\alpha-\beta) \geq -\frac{1}{2}$ [5]. The inversion formula for (1.2) is given by

$$\phi(x) = \int_0^\infty \ldots.\int_0^\infty \hat{\phi}\,(y_1 \ldots..y_n)\left(\prod_{i=1}^n (x_i y_i)^{\alpha+\beta}\, J_{\alpha-\beta}\,(x_i y_i)\right) dy_1 \ldots.dy_n\,,\ x \in I^n \tag{1.3}$$

From [5] we have the following relations for any $\phi \in H_{\alpha,\beta}\,(I^n)$ and $, y \in I^n$:

$$h_{\alpha,\beta,1}\,([-x]\phi) = N_{\alpha,\beta}\, h_{\alpha,\beta}\,\phi\ , \tag{1.4}$$

$$h_{\alpha,\beta,1}\left(N_{\alpha,\beta}\,\phi\right) = [-y]\, h_{\alpha,\beta}\,\phi\ , \tag{1.5}$$

$$h_{\alpha,\beta}\left(\Delta_{\alpha,\beta}\,\phi\right) = (-1)^n\,[y]^2\, h_{\alpha,\beta}\,\phi\ . \tag{1.6}$$

We have the Leibniz formula and Bessel type differential operator of the r^{th} –order in $\mathbb{R}^n$ respectively as follows:

$$\prod_{i=1}^{n}\left(\left(x_i^{-1}\,\frac{\partial}{\partial x_i}\right)^{k_i}\left(x_i^{2\beta-1}(\psi\phi)\,(x_1 \ldots..x_n)\right)\right)$$

$$= \prod_{i=1}^n\left(\sum_{\nu_i=0}^{k_i}\binom{k_i}{\nu_i}\left(x_i^{-1}\,\frac{\partial}{\partial x_i}\right)^{\nu_i}\psi(x_1 \ldots..x_n)\left(x_i^{-1}\,\frac{\partial}{\partial x_i}\right)^{k_i-\nu_i}\left(x_i^{2\beta-1}\phi(x_1 \ldots..x_n)\right)\right) \tag{1.7}$$

$$\Delta_{\alpha,\beta,x}^r\,\phi\,(x_1, \ldots\ldots x_n) = \prod_{i=1}^n\left(\sum_{j_i=0}^{r_i} bj_i\, x_i^{2j_i+2\alpha}\left(x_i^{-1}\,\frac{\partial}{\partial x_i}\right)^{r_i+j_i}\left(x_i^{2\beta-1}\,\phi\,(x_1 \ldots..x_n)\right)\right) \tag{1.8}$$

where bj_i are constants depending only on $\alpha-\beta$.

For $i = 1,2,\ldots.n$, let $\Delta\,(x_i, y_i, z_i)$ be the area of a triangle with sides x_i, y_i, z_i if such a triangle exists and zero otherwise

For fixed $a - b > -\frac{1}{2}$, set

$$D_{a,b}\,(x,y,z) = \prod_{i=1}^{n} D_{a,b}\,(x_i, y_i, z_i)$$

$$= 2^{n\left(-2a-8b+\frac{3}{2}\right)}\,\pi^{-\frac{n}{2}}\left(\Gamma(3a+b)\right)^{2n}\left(\Gamma\left(a-b+\frac{1}{2}\right)\right)^{-n}$$

$$\times \prod_{i=1}^{n}\left((x_i y_i z_i)^{4b-1}\{\Delta(x_i, y_i, z_i)\}^{-2a-6b+1}\right) .$$

We note that the n-dimensional Delsarte Kernel $D_{a-b}\,(x,y,z)$ is non-negative and symmetric in x, y, z. We have the formula

$$\int_0^\infty \ldots . \int_0^\infty \prod_{i=1}^{n}\left(j_{a-b}\,(z_i t_i)\, D_{a-b}\,(x_i, y_i, z_i) d\sigma\,(z_i)\right)$$

$$= \prod_{i=1}^{n}\left(j_{a-b}\,(x_i t_i)\, j_{a-b}\,(y_i t_i)\right), \tag{1.9}$$

where

$$\prod_{i=1}^{n} d\sigma\,(x_i) = \left(2^{a-b}\,\Gamma(3a+b)\right)^{-n} \prod_{i=1}^{n}\left(x_i^{2(a-b)+1}\,dx_i\right), \tag{1.10}$$

$$\prod_{i=1}^{n} j_{a-b}\,(x_i) = \left(2^{a-b}\Gamma(3a+b)\right)^{n} \prod_{i=1}^{n}\left(x_i^{-(a-b)} J_{a-b}\,(x_i)\right) . \tag{1.11}$$

Let $f \in L^1\,(I^n)$. Then the n-dimensional Hankel type translate $T_x f$ of f by $x = (x_1, \ldots\ldots, x_n)$ is defined by

$$(\tau_x f)\,(y) = \int_0^\infty \ldots . . \int_0^\infty f\,(z_1, \ldots\ldots, z_n) \prod_{i=1}^{n}\left(D_{a-b}\,(x_i, y_i, z_i)\, d\sigma(z_i)\right),\ \ x_1\, y \in I^n . \tag{1.12}$$

Let f and g be functions in $L^1\,(I^n)$ and let their n-dimensional Hankel type convolution $f \# g$ be defined by

$$(f\#g)\,(x) = \int_0^\infty \ldots . \int_0^\infty (\tau_x f)\,(y_1 \ldots . . y_n)\, g\,(y_1, \ldots \ldots, y_n)\ d\sigma\,(y_1) \ldots . . d\sigma\,(y_n),\ \ x \in I^n \tag{1.13}$$

The integral defining $(f\#g)\,(x)$ converges for almost all $x \in I^n$ and

$$\|f\#g\|_{L^1} \leq \|f\|_{L^1}\,\|g\|_{L^1} . \tag{1.14}$$

The above result is a generalization of the corresponding one-dimensional case investigated by Hirschman in pp. 309-311 of [1]. Schwartz's theory of the Fourier transform of distributions in $S'(\mathbb{R}^n)$ has been exploited by many authors in the study of pseudo-differential operators, see for instance, Zaidman [7,6]. In this paper we have used the Zemanian's theory of the Hankel type transform of distributions in $H'_{\alpha,\beta}\,(I^n)$ to develop a theory of n-dimensional pseudo-differential operators corresponding to [4]. Unless otherwise stated, we shall always assume $(\alpha - \beta) \geq -\frac{1}{2}$.

2. The pseudo differential type operator $h_{\alpha,\beta,a}$:

Definition 2.1:

Let $a\,(x_1, \ldots . x_n;\ y_1, \ldots . . y_n)$ be a complex-valued function belonging to the space $C^\infty(I^n \times I^n)$, where $I = (0, \infty)$ and let its derivatives satisfy certain growth conditions such as (2.1) below. Then the pseudo-differential type operator $h_{\alpha,\beta,a}$ associated with the symbol $a\left(x_1 \ldots ., x_n\,;\ y_{1,} \ldots . . y_n\right)$ is defined by

$$\left(h_{\alpha,\beta,a}\,\phi\right)(y_1,\ldots\ldots,y_n) = \int_0^\infty \ldots \int_0^\infty \left(\prod_{i=1}^n (x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}\,(x_i y_i)\right)$$

$$\times\, a(x_1 \ldots . x_n;\, y_1,\ldots\ldots,y_n)\,\left(\left(h_{\alpha,\beta}\,\phi\right)(x_1,\ldots\ldots x_n)\right) dx_1 \ldots\ldots dx_n \cdot$$

Definition 2.2: Let $m \in (-\infty,\infty)$. The function $a\,(x_1,\ldots\ldots x_n\,;\,y_1,\ldots\ldots,y_n) \in C^\infty\,(I^n \times I^n)$ is said to belong to the class H^m if and only if for all $q,\nu,\delta \in \mathbb{N}_0^n$, there exists $D > 0$ such that

$$\prod_{i=1}^n (1+x_i)^{q_i} \left|\left(x_i^{-1}\,\frac{\partial}{\partial x_i}\right)^{\nu_i} \left(y_i^{-1}\,\frac{\partial}{\partial y_i}\right)^{\delta_i} a\,(x_1,\ldots\ldots,x_n;\,y_1,\ldots\ldots,y_n)\right| \le D\,\left(1+\prod_{i=1}^n y_i\right)^{m-|\delta|} \quad (2.1)$$

Theorem 2.1: Let the symbol $a\,(x_1,\ldots\ldots,x_n\,;\,y_1,\ldots\ldots,y_n)$ belong to H^m. Then for $(\alpha-\beta) \ge -\frac{1}{2}$ the pseudo-differential operator $h_{\alpha,\beta,a}$ is a continuous linear mapping of $H_{\alpha,\beta}\,(I^n)$ into itself.

Proof: Let $\Phi\,(y_1,\ldots.,y_n) = \left(h_{\alpha,\beta,a}\,\phi\right)(y_1,\ldots.y_n),\ \phi \in H_{\alpha,\beta}\,(I^n)$. Then by using formulas (1.4) and (1.5) and using Zemanian's technique $(p.\,141\ of\ [9])$, for $i = 1,2,\ldots\ldots,n$, we have

$$N_{i\alpha,i\beta}\,\Phi\,(y_1,\ldots\ldots,y_n) = y_i^{2\alpha}\left(\frac{\partial}{\partial yi}\right) y_i^{2\beta-1}\,\Phi\,(y_1,\ldots.,y_n)$$

$$= y_i^{2\alpha+1}\left(y_i^{-1}\,\frac{\partial}{\partial yi}\right) y_i^{2\beta-1}\,\Phi\,(y_1,\ldots.,y_n)\,;$$

$$N_{i\alpha,i\beta,i}\,N_{i\alpha,i\beta}\,\Phi\,(y_1,\ldots.,y_n) = y_i^{\alpha-\beta+\frac{1}{2}}\left(\frac{\partial}{\partial y_i}\right) y_i^{-(3\alpha+\beta)-\frac{1}{2}}\,N_{i\alpha,\ i\beta}\,\Phi\,(y_1,\ldots.,y_n)$$

$$= y_i^{\alpha-\beta+2+\frac{1}{2}}\left(y_i^{-1}\,\frac{\partial}{\partial y_i}\right) y_i^{2\beta-2}$$

$$\times\left[y_i^{4\alpha+2\beta}\left(y_i^{-1}\,\frac{\partial}{\partial y_i}\right) y_i^{2\beta-1}\,\Phi\,(y_1,\ldots\ldots y_n)\right]$$

$$= y_i^{5\alpha+3\beta+\frac{1}{2}}\left(y_i^{-1}\,\frac{\partial}{\partial y_i}\right)^2 y_i^{2\beta-1}\,\Phi\,(y_1,\ldots\ldots,y_n)\,.$$

Similarly

$$N_{i\alpha,i\beta,\ i\,(k_i-1)\ldots\ldots}N_{i\alpha,i\beta}\,\Phi\,(y_1,\ldots\ldots,y_n)$$

$$= y_i^{\alpha-\beta+k_i+\frac{1}{2}}\left(y_i^{-1}\,\frac{\partial}{\partial y_i}\right)^{k_i} y_i^{2\beta-1}\,\Phi\,(y_1,\ldots\ldots,y_n).$$

Now, we have

$$\left(N_{1\alpha,1\beta,k_{1-1}}\ldots\ldots N_{1\alpha,1\beta}\right)\ldots\ldots\left(N_{n\alpha,n\beta,k_n-1}\ldots\ldots N_{n\alpha,n\beta}\right)\Phi\,(y_1,\ldots\ldots,y_n).$$

$$= \left(y_1^{\alpha-\beta+k_1+\frac{1}{2}}\ldots.\,y_n^{\alpha-\beta+k_n+\frac{1}{2}}\right)\left(\left(y_1^{-1}\frac{\partial}{\partial y_1}\right)^{k_1}\ldots\ldots\left(y_n^{-1}\,\frac{\partial}{\partial y_n}\right)^{k_n}\right)$$

$$\times\left(y_1^{2\beta-1}\ldots.\,y_n^{2\beta-1}\right)\Phi\,(y_1,\ldots\ldots,y_n).$$

Therefore, using (1.3) and (1.7), we get

$$\prod_{i=1}^n \left(N_{i\alpha,i\beta,i\,(k_i-1)}\ldots.\,N_{i\alpha,i\beta}\right)\Phi\,(y_1,\ldots.,y_n)$$

$$= \left[y^{\alpha-\beta+k+\frac{1}{2}}\right] \left(y^{-1} D_y\right)^k [y]^{2\beta-1} \Phi(y)$$

$$= \left[y^{\alpha-\beta+k+\frac{1}{2}}\right] \left(y^{-1} D_y\right)^k [y]^{2\beta-1} \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{n} (x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}(x_i y_i)$$

$$\times a(x_1, \dots, x_n ; y_1, \dots, y_n) \hat{\phi}(x_1, \dots, x_n) dx_1 \dots dx_n$$

$$= \left[y^{\alpha-\beta+k+\frac{1}{2}}\right] \left(y^{-1} D_y\right)^k \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{n} (x_i)^{\alpha+\beta} (y_i)^{-(\alpha-\beta)} J_{\alpha-\beta}(x_i y_i)$$

$$\times a(x_1, \dots, x_n ; y_1, \dots, y_n) \hat{\phi}(x_1, \dots, x_n) dx_1 \dots dx_n$$

$$= \left[y^{\alpha-\beta+k+\frac{1}{2}}\right] \int_0^\infty \dots \int_0^\infty \left(\prod_{i=1}^{n} (x_i)^{\alpha+\beta} \sum_{r_i=0}^{k_i} \binom{k_i}{r_i} \left(y_i^{-1} \frac{\partial}{\partial y_i}\right)^{k_i - r_i} (y_i)^{-(\alpha-\beta)} \right)$$

$$\times J_{\alpha-\beta}(x_i y_i) \left(y_i^{-1} \frac{\partial}{\partial y_i}\right)^{r_i} a(x_1, \dots x_n ; y_1, \dots, y_n) \hat{\phi}(x_1, \dots, x_n) dx_1 \dots dx_n$$

$$= \left[y^{\alpha-\beta+k+\frac{1}{2}}\right] \int_0^\infty \dots \int_0^\infty [x]^{\alpha+\beta} \left(\prod_{i=1}^{n} \sum_{r_i=0}^{k_i} \binom{k_i}{r_i} (-xi)^{k_i - r_i} (y_i)^{-(\alpha-\beta)-k_i-r_i} \right)$$

$$\times J_{\alpha-\beta+k_i-r_i}(x_i y_i) \left(y_i^{-1} \frac{\partial}{\partial y_i}\right)^{r_i} a(x_1, \dots x_n ; y_1, \dots, y_n) \hat{\phi}(x_1, \dots, x_n) dx_1 \dots dx_n. \quad (2.2)$$

Therefore

$$\prod_{i=1}^{n} N_{i\alpha, i\beta,\ i(ki-1)} \dots N_{i\alpha, i\beta} \Phi(y_1, \dots, y_n)$$

$$= \sum_{r_1=0}^{k_1} \dots \sum_{r_n=0}^{k_n} \binom{k_1}{r_1} \dots \binom{k_n}{r_n} \int_0^\infty \dots \int_0^\infty \left[y^{r+\frac{1}{2}}\right] [x]^{\alpha+\beta} \left(y^{-1} D_y\right)^r$$

$$\times a(x_1, \dots x_n ; y_1, \dots, y_n) \hat{\phi}(x_1, \dots x_n) [-x^{k-r}] \prod_{i=1}^{n} J_{\alpha-\beta+k_i-r_i}(x_i y_i) dx_1 \dots dx_n$$

$$= \sum_{r_1=0}^{k_i} \dots \sum_{r_n=0}^{k_n} \binom{k_1}{r_1} \dots \binom{k_n}{r_n} [y^r] \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{n} (x_i y_i)^{\alpha+\beta} J_{\alpha-\beta+k_i-r_i}(x_i y_i)$$

$$\times \left(y^{-1} D_y\right)^r a(x_1, \dots, x_n ; y_1, \dots, y_n) [-x^{k-r}] \hat{\phi}(x_1, \dots, x_n) dx_1 \dots dx_n$$

$$= \sum_{r_1=0}^{k_1} \dots \sum_{r_n=0}^{k_n} \binom{k_1}{r_1} \dots \binom{k_n}{r_n} h_{\alpha,\beta,k-r} \left([y^r] \left(y^{-1} D_y\right)^r\right)$$

$$\times a(x_1, \dots, x_n ; y_{1,} \dots, y_n) [-x^{k-r}] \hat{\phi}(x_1, \dots, x_n)(y_1, \dots, y_n).$$

From equation (2.2) for $i = 1$,

$$\left(N_{1\alpha, 1\beta, k_1-1} \dots N_{1\alpha, 1\beta}\right) \Phi(y_1, \dots, y_n)$$

$$= y_1^{\alpha-\beta+k_1+\frac{1}{2}} \left(y_1^{-1} \frac{\partial}{\partial y_1}\right)^{k_1} y_1^{2\beta-1} \Phi (y_1, \ldots\ldots, y_n)$$

$$= y_1^{\alpha-\beta+k_1+\frac{1}{2}} \left(y_1^{-1} \frac{\partial}{\partial y_1}\right)^{k_1} \int_0^\infty (x_1)^{\alpha+\beta} J_{\alpha-\beta} (x_1 y_1) a(x_1, \ldots\ldots, x_n; y_1, \ldots\ldots, y_n)$$

$$\times \hat{\phi} (x_1, \ldots., x_n) dx_1$$

$$= y_1^{\alpha-\beta+k_1+\frac{1}{2}} \left(y_1^{-1} \frac{\partial}{\partial y_1}\right)^{k_1} \int_0^\infty x_1 (y_1)^{-(\alpha+\beta)} J_{\alpha-\beta} (x_1 y_1) a(x_1, \ldots., x_n ; y_1, \ldots\ldots, y_n)$$

$$\times \hat{\phi} (x_1, \ldots., x_n) dx_1$$

$$= y_1^{\alpha-\beta+k_1+\frac{1}{2}} \int_0^\infty x_1^{\alpha+\beta} \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} \left(y_1^{-1} \frac{\partial}{\partial y_1}\right)^{k_1-n} (y_1)^{-(\alpha-\beta)} J_{\alpha-\beta} (x_1 y_1)$$

$$\times \left(y_1^{-1} \frac{\partial}{\partial y_1}\right)^{r_1} a (x_1, \ldots\ldots, x_n ; y_1, \ldots., y_n) \hat{\phi} (x_1, \ldots., x_n) dx_1$$

$$= y_1^{\alpha-\beta+k+\frac{1}{2}} \int_0^\infty x_1^{\alpha+\beta} \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-x_1)^{k_1-n} y_1^{-(\alpha-\beta)-k_1-n} J_{\alpha-\beta+k_1+r_1} (x_1 y_1)$$

$$\times \left(y_1^{-1} \frac{\partial}{\partial y_1}\right)^{r_1} a (x_1, \ldots\ldots, x_n ; y_1, \ldots., y_n) \hat{\phi} (x_1, \ldots\ldots, x_n) dx_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} \int_0^\infty y_1^{r_1+\frac{1}{2}} x_1^{\alpha+\beta} \left(y_1^{-1} \frac{\partial}{\partial y_1}\right)^{r_1} a(x_1 \ldots., x_n ; y_1, \ldots., y_n)$$

$$\times \hat{\phi} (x_1, \ldots. x_n) \left(-x_1\right)^{k_1-r_1} J_{\alpha-\beta+k_1-r_1}(x_1 y_1) dx_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} y_1{}^{r_1} \int_0^\infty (x_1 y_1)^{\alpha+\beta} J_{\alpha-\beta+k_i-r_1} (x_1 y_1) \left(y_1^{-1} \frac{\partial}{\partial y_1}\right)^{r_1}$$

$$\times a (x_1, \ldots., x_n ; y_1, \ldots., y_n) (-x_1)^{k_1-r_1} \hat{\phi} (x_1, \ldots\ldots, x_n) dx_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} h_{\alpha,\beta,k_1-r_1} \{(y_1^{r_1}) \left(y_1^{-1} \frac{\partial}{\partial y_1}\right)^{r_1} a (x_1, \ldots., x_n ; y_1 \ldots. y_n)$$

$$\times \left((-x_1)^{k_1-r_1} \hat{\phi} (x_1, \ldots\ldots, x_n)\right) (y_1, \ldots., y_n)\}$$

Using the formula $(-y_1) h_{\alpha,\beta} \Phi = h_{\alpha,\beta,1} \left(N_{\alpha,\beta} \Phi\right)$ in the above, we have

$$(-y_1) \left(N_{1\alpha,1\beta,k_1-1} \ldots\ldots N_{1\alpha,1\beta}\right) \Phi (y_1, \ldots., y_n)$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-y_1) h_{\alpha,\beta,k_1-r_1} \left(y_1^{r_1} (y_1^{-1})\right)^{r_1} a(x_1, \ldots. x_n ; y_1, \ldots., y_n)$$

$$\times (-x_1)^{k_1-r_1} \hat{\phi} \left((x_1, \ldots. x_n) y_1, \ldots., y_n \right)$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} h_{\alpha,\beta,k_1-r_1+1} N_{\alpha,\beta,k_1-r_1} \left(y_1^{r_1} \left(y_1^{-1} \frac{\partial}{\partial y_1} \right) \right)^{r_1}$$

$$\times\, a(x_1, \ldots . x_n \,;\, y_1, \ldots ., y_n)\, (-x_1)^{k_1-r_1}\, \hat{\phi}\left((x_1, \ldots . x_n)\, y_1, \ldots ., y_n\right)$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (y_1^{r_1}) \int_0^\infty (x_1 y_1)^{\alpha+\beta} J_{\alpha-\beta+k_1-r_1+1} N_{\alpha,\beta,k_1-r_1}$$

$$\times \left(\left(y_1^{-1} \frac{\partial}{\partial y_1} \right)^{r_1} a\,(x_1, \ldots x_n \,;\, y_1, \ldots . y_n)\, (-x_1)^{k_1-r_1}\, \hat{\phi}\,(x_1, \ldots x_n) dx_1 \right)$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} \int_0^\infty (y_1^{r_1})(x_1 y_1)^{\alpha+\beta} J_{\alpha-\beta+k_i-r_1}(x_1 y_1) x_1^{\alpha-\beta+k_1-r_1+\frac{1}{2}} \left(\frac{\partial}{\partial x_1}\right) x_1^{-(\alpha-\beta)-k_1+r_1-\frac{1}{2}}$$

$$\times \left(\left(y_1^{-1} \frac{\partial}{\partial y_1} \right)^{r_1} a(x_1, \ldots x_n \,;\, y_1, \ldots . y_n)\, (-x_1)^{k_1-r_1} \hat{\phi}\,(x_1, \ldots x_n) \right) dx_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-1)^{k_1-r_1} \int_0^\infty y_1^{r_1+\frac{1}{2}}\, x_1^{\alpha-\beta+k_1-r_1+2} \left(x_1^{-1} \frac{\partial}{\partial x_1} \right)$$

$$\times \left(x_1^{2\beta-1} \left(y_1^{-1} \frac{\partial}{\partial y_1} \right)^{r_1} a\,(x_1, \ldots x_n \,;\, y_1, \ldots . y_n)\, (-x_1)^{k_1-r_1}\, \hat{\phi}\,(x_1, \ldots x_n)\, J_{\alpha-\beta+k_1-r_1+1}(x_1 y_1) \right) dx_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-1)^{k_1-r_1}\, {y_1}^{r_1} \int_0^\infty (x_1 y_1)^{\alpha+\beta} J_{\alpha-\beta+k_1-r_1+1}(x_1 y_1)\, x_1^{\alpha-\beta+k_1-r_1+\frac{1}{2}} \left(x_1^{-1} \frac{\partial}{\partial x_1} \right)$$

$$\times \left(x_1^{2\beta-1} \left(y_1^{-1} \frac{\partial}{\partial x_1} \right)^{r_1} a\,(x_1, \ldots x_n \,;\, y_1, \ldots . y_n)(-x_1)^{k_1-r_1}\, \hat{\phi}\,\, (x_1, \ldots x_n) \right) dx_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-1)^{k_1-r_1}\, {y_1}^{r_1}\, h_{\alpha,\beta,k_1-r_1+1} \left\{ x_1^{\alpha-\beta+k_1-r_1+\frac{1}{2}} \left(x_1^{-1} \frac{\partial}{\partial x_1} \right) \right.$$

$$\left. \times \left(x_1^{2\beta-1} \left(y_1^{-1} \frac{\partial}{\partial x_1} \right)^{r_1} a\,(x_1, \ldots x_n \,;\, y_1, \ldots . y_n)(-x_1)^{k_1-r_1}\, \hat{\phi}\,(x_1, \ldots x_n) \right) \right\} .$$

Using the formula $(-y_1)\, h_{\alpha,\beta}\, \Phi = h_{\alpha,\beta,1}\, N_{\alpha,\beta}\, \Phi$ repeatedly, we obtain

$$(-y_1)^{t_1} \left(N_{1\alpha,1\beta,k_1-1} \ldots . . N_{1\alpha,1\beta} \right) \Phi\,(y_1, \ldots ., y_n)$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-1)^{k_1-r_1} \int_0^\infty y_1^{r_1+\frac{1}{2}}\, x_1^{\alpha-\beta+k_1-r_1+t_1+1} \left(x_1^{-1} \frac{\partial}{\partial x_1} \right)^{t_1}$$

$$\times \left(x_1^{2\beta-1} \left(y_1^{-1} \frac{\partial}{\partial y_1} \right)^{r_1} a\,(x_1, \ldots x_n \,;\, y_1, \ldots . y_n)(-x_1)^{k_1-r_1} \hat{\phi}\,(x_1, \ldots x_n)\, J_{\alpha-\beta+k_1-r_1+t_1}(x_1 y_1) \right) dx_1$$

Similarly, for $i = 2, \ldots . ., n,$

$$(-y_1)^{t_i}\left(N_{i\alpha,i\beta,i(k_i-1)}\ldots . N_{i\alpha,i\beta}\right)\Phi\,(y_1,\ldots\ldots,y_n)$$

$$= \sum_{r_i=0}^{k_i}\binom{k_1}{r_1}(-1)^{k_1-r_1}\int_0^\infty y_i^{r_i+\frac{1}{2}}\,x_i^{\alpha-\beta+k_i-r_i+t_i+1}\left(x_i^{-1}\,\frac{\partial}{\partial x_i}\right)^{t_i}$$

$$\times\left(x_i^{2\beta-1}\left(y_i^{-1}\frac{\partial}{\partial y_i}\right)^{r_i} a\,(x_1,\ldots x_n\,;\,y_1,\ldots . y_n)(-x_i)^{k_i-r_i}\,\hat{\phi}\,(x_1,\ldots x_n)\right)J_{\alpha-\beta+k_i-r_i+t_i}\,(x_iy_i)\,dx_i$$

Therefore

$$\left((-y_1)^{t_1}\,N_{1\alpha,1\beta,k_1-1}\ldots\ldots N_{1\alpha,1\beta}\right)\ldots\ldots\left((-y_n)^{t_n}\,N_{n\alpha,n\beta,n(k_n-1)}\ldots\ldots N_{n\alpha,n\beta}\right)\Phi\,(y_1,\ldots . ,y_n)$$

$$= \sum_{r_1=0}^{k_1}\binom{k_1}{r_1}\ldots\ldots\sum_{r_n=0}^{k_n}\binom{k_n}{r_n}\left((-1)^{k_1-r_1}\ldots . (-1)^{k_n-r_n}\right)$$

$$\times\int_0^\infty\ldots .\int_0^\infty\left(y_1^{r_1+\frac{1}{2}}\ldots\ldots y_n^{r_n+\frac{1}{2}}\right)\left(x_1^{\alpha-\beta+k_1-r_1+t_1+1}\ldots . x_n^{\alpha-\beta+k_n-r_n+t_1+1}\right)$$

$$\times\left(\left(x_1^{-1}\,\frac{\partial}{\partial x_1}\right)^{t_1}\ldots\ldots\left(x_n^{-1}\,\frac{\partial}{\partial x_n}\right)^{t_n}\right)$$

$$\times\left(\begin{array}{c}\left(x_1^{2\beta-1}\ldots\ldots x_n^{2\beta-1}\right)\left(y_1^{-1}\,\frac{\partial}{\partial y_1}\right)^{r_1}\ldots\ldots\left(y_n^{-1}\,\frac{\partial}{\partial y_n}\right)^{r_n} a\,(x_1,\ldots x_n\,;\,y_1,\ldots . y_n)\\ \times(-x_1)^{k_1-r_1}\ldots . (-x_n)^{k_n-r_n}\,\hat{\phi}\,(x_1,\ldots x_n)\end{array}\right)$$

$$\times\,J_{\alpha-\beta+k_1-r_1+t_1}\,(x_1y_1)\ldots\ldots J_{\alpha-\beta+k_n-r_n+t_n}\,(x_ny_n)\,dx_1\ldots\ldots dx_n\,.$$

Or

$$\prod_{i=1}^{n}\left((-y_i)^{t_i}\,N_{i\alpha,i\beta,i(k_i-1)}\ldots\ldots N_{i\alpha,i\beta}\right)\Phi\;(y_1,\ldots y_n)$$

$$= \prod_{i=1}^{n}\sum_{r_{i=0}}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty\ldots\ldots\int_0^\infty\left[y^{r+\frac{1}{2}}\right]\left[x^{\alpha-\beta+k-r+t+1}\right]$$

$$\times\;(x^{-1}D_x)^t\,[x]^{2\beta-1}\,\hat{\phi}\,(x_1,\ldots x_n)\left(y^{-1}D_y\right)^r a\,(x_1,\ldots x_n\,;\,y_1,\ldots . y_n)$$

$$\times\left(\prod_{i=1}^{n}J_{\alpha-\beta+k_i-r_i+t_i}\,(x_iy_i)\right)dx_1\ldots\ldots dx_n$$

$$= \prod_{i=1}^{n}\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty\ldots\ldots\int_0^\infty\left[y^{r+\frac{1}{2}}\right]\left[x^{\alpha-\beta+k-r+t+1}\right]$$

$$\times\left(\sum_{\nu_1=0}^{t_1}\ldots\ldots\sum_{\nu_n=0}^{t_n}\binom{t_1}{\nu_1}\ldots\ldots\binom{t_n}{\nu_n}\right)(x^{-1}D_x)^{\nu}\left(y^{-1}D_y\right)^r$$

$$\times\;a\,(x_1,\ldots x_n\,;\,y_1,\ldots . y_n)\,(x^{-1}D_x)^{t-\nu}\,[x]^{2\beta-1}\hat{\phi}\;(x_1,\ldots x_n)$$

$$\times\,\prod_{i=1}^{n}J_{\alpha-\beta+k_i-r_i+t_i}\,(x_iy_i)\,dx_1\ldots\ldots dx_n\,. \qquad (2.3)$$

Now, multiplying both sides of (2.2) by $\prod_{i=1}^{n}(-y_i)^{t_i}$, we get

$$\prod_{i=1}^{n}\left((-y_i)^{t_i}\, N_{i\alpha,i\beta,i(k_i-1)} \dots \dots N_{i\alpha,i\beta}\right) \Phi\ (y_1, \dots y_n)$$

$$= \ (-1)^{|t|}\left[y^{\alpha-\beta+k+t+\frac{1}{2}}\right]\left(y^{-1}D_y\right)^k [y]^{2\beta-1}\,\Phi\,(y)\ \cdot \tag{2.4}$$

Comparing (2.3) and (2.4), we have

$$(-1)^{|t|}\left[y^{\alpha-\beta+k+t+\frac{1}{2}}\right]\left(y^{-1}D_y\right)^k [y]^{2\beta-1}\,\Phi\,(y)$$

$$= \prod_{i=1}^{n}\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty \dots . \int_0^\infty \left[y^{r+\frac{1}{2}}\right]$$

$$\times\ \left[x^{\alpha-\beta+k-r+t+1}\right]\left(\sum_{\nu_1=0}^{t_1}\dots . \sum_{\nu_n=0}^{t_n}\binom{t_1}{\nu_1}\dots . \binom{t_n}{\nu_n}\right)$$

$$\times\ (x^{-1}D_x)^{\nu}\, a_r\,(x_1, \dots \dots, x_n\,;\ y_1, \dots y_n)\,(x^{-1}D_x)^{t-\nu}$$

$$\times\ [x]^{2\beta-1}\,\hat{\phi}\,(x_1, \dots \dots, x_n)\prod_{i=1}^{n}J_{\alpha-\beta+k_i-r_i+t_i}(x_iy_i)\,dx_1 \dots \dots dx_n\ \cdot$$

Therefore,

$$(-1)^{|t|}\,[y^t]\left(y^{-1}D_y\right)^k [y]^{2\beta-1}\,\Phi(y)$$

$$= \prod_{i=1}^{n}\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty \dots . \int_0^\infty \left[y^{-(\alpha-\beta+k-r)}\right]$$

$$\times\ \left[x^{\alpha-\beta+k-r+t+1}\right]\left(\sum_{\nu_1=0}^{t_1}\dots \dots \sum_{\nu_n=0}^{t_n}\binom{t_1}{\nu_1}\dots \dots \binom{t_n}{\nu_n}\right)$$

$$\times\ (x^{-1}D_x)^{\nu}\ a_r\,(x_1, \dots \dots, x_n\,;\ y_1, \dots y_n)\,(x^{-1}D_x)^{t-\nu}\,[x]^{2\beta-1}\,\hat{\phi}\,(x_1, \dots x_n)$$

$$\times\ \prod_{i=1}^{n}J_{\alpha-\beta+k_i-r_i+t_i}(x_iy_i)\,dx_1 \dots \dots dx_n$$

$$= \prod_{i=1}^{n}\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty \dots . \int_0^\infty \left[x^{2\lambda+t+1}\right]$$

$$\times\left(\sum_{\nu_1=0}^{t_1}\dots \dots \sum_{\nu_n=0}^{t_n}\binom{t_1}{\nu_1}\dots \dots \binom{t_n}{\nu_n}\right)(x^{-1}D_x)^{\nu}\, a_r\,(x_1, \dots \dots, x_n\,;\ y_1, \dots ., y_n)$$

$$\times\,(x^{-1}D_x)^{t-\nu}\left[x^{2\beta-1}\right]\hat{\phi}\,(x_1, \dots x_n)\ \prod_{i=1}^{n}(x_iy_i)^{-\lambda i}\,J_{\lambda_i+t_i}(x_iy_i)\ dx_1 \dots \dots dx_n\ ,$$

where $\lambda_i = \ \alpha - \beta + k_i - r_i\ ,\ i = 1,2, \dots . n\ \cdot$

Setting $t_i = p_i + s_i$, respectively $t_i = p_i,\ i = 1,2,\ldots\ldots,n,$ in the above expression and using the property (2.1) with $q = (0,\ldots\ldots,0)$ and taking into account that $(\alpha - \beta) \geq -\frac{1}{2}$, we arrive at the following estimate:

$$(1+[y^s])\left|[y^p]\,[y^1 D_y)^k\,[y]^{2\beta-1}\,\Phi\,(y)\right|$$

$$= \left|[y^p]\left(y^{-1}D_y\right)^k [y]^{2\beta-1}\,\Phi(y)\right| + \ |[y^{p+s}]\left(y^{-1}D_y\right)^k [y]^{2\beta-1}\,\Phi(y)$$

$$\leq D\sum_{r_1=0}^{k_1}\ldots\ldots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\ldots\ldots\binom{k_n}{r_n}\int_0^\infty\ldots.\int_0^\infty(1+[y])^{m-|r|}$$

$$\times\left(\left[x^{2\lambda+p+1}\right]\sum_{\nu_1=0}^{p_1}\ldots.\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1}\ldots.\binom{p_n}{\nu_n}(x^{-1}D_x)^{p-\nu}\,[x]^{2\beta-1}\,\hat{\phi}\,(x_1,\ldots x_n)\right.$$

$$\left.+\ \left[x^{2\lambda+p+s+1}\right]\sum_{\nu_1=0}^{p_1+s_1}\ldots\ldots\sum_{\nu_n=0}^{p_n+s_n}\binom{p_1+s_1}{\nu_1}\ldots\ldots\binom{p_n+s_n}{\nu_n}\,(x^{-1}D_x)^{p+s-\nu}\,[x]^{2\beta-1}\,\hat{\phi}\,(x_1,\ldots x_n)\right)$$

$$\times\, dx_1\ldots\ldots dx_n\ \cdot$$

Since $(1+[y])^{m-|r|} \leq (1+[y])^m$, we have

$$(1+[y^s])\,|[y^p]\left(y^{-1}D_y\right)^k\,[y]^{2\beta-1}\,\Phi(y)$$

$$\leq D\sum_{r_1=0}^{k_1}\ldots\ldots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\ldots\ldots\binom{k_n}{r_n}\int_0^\infty\ldots\ldots\int_0^\infty(1+[y])^m$$

$$\times\left(\left[x^{2\lambda+p+s+1}\right]\sum_{\nu_1=0}^{p_1+s_1}\ldots\ldots\sum_{\nu_n=0}^{p_n+s_n}\binom{p_1+s_1}{\nu_1}\ldots\ldots\binom{p_n+s_n}{\nu_n}\right.$$

$$\times\ (x^{-1}D_x)^{p+s-\nu}\,[x]^{2\beta-1}\,\hat{\phi}\,(x_1,\ldots x_n) + \left[x^{2\lambda+p+1}\right]\sum_{\nu_1=0}^{p_1}\ldots\ldots\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1}$$

$$\times\ldots\ldots\binom{p_n}{\nu_n}\ (x^{-1}D_x)^{p-\nu}\,[x]^{2\beta-1}\,\hat{\phi}\ (x_1,\ldots x_n)\,dx_1\ldots\ldots dx_n\ \cdot$$

Hence

$$\left|[y^p]\left(y^{-1}D_y\right)^k [y]^{2\beta-1}\,\Phi(y)\right|$$

$$\leq D\sum_{r_1=0}^{k_1}\ldots\ldots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\ldots.\binom{k_n}{r_n}\int_0^\infty\ldots\ldots\int_0^\infty\frac{(1+[y])^m}{(1+[s])}$$

$$\times\ \left(\left[x^{2\lambda+p+s+1}\right]\sum_{\nu_1=0}^{p_1+s_1}\ldots\ldots\sum_{\nu_n=0}^{p_n+s_n}\binom{p_1+s_1}{\nu_1}\ldots\ldots\binom{p_n+s_n}{\nu_n}\right.$$

$$\times\ (x^{-1}D_x)^{p+s-\nu}\,[x]^{2\beta-1}\,\hat{\phi}\,(x_1,\ldots x_n) + \ \left[x^{2\lambda+p+1}\right]\sum_{\nu_1=0}^{p_1}\ldots\ldots\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1}\ldots\ldots\binom{p_n}{\nu_n}\ \ (x^{-1}D_x)^{p-\nu}$$

$$\times [x]^{2\beta 1} \hat{\phi}\, (x_1, \dots x_n)\, dx_1 \dots\dots dx_n \tag{2.5}$$

We note that if $m \le 0$, we have $(1+[y])^m$, and then

$$\frac{(1+[y])^m}{(1+[y])} \le 1\ , \tag{2.6}$$

while if $m > 0$, then

$$\frac{(1+[y])^m}{(1+[y^s])} \le\ 2^m\ \frac{(1+[y])^m}{(1+[y^s])}\ \cdot \tag{2.7}$$

Since s is an arbitrary n-tuple of non-negative integers we can choose $s = (s_1, \dots\dots, s_n)$ such that

$$\frac{(1+[y])^m}{(1+[y^s])} \le 1\ \cdot \tag{2.8}$$

From equations (2.7) and (2.8), we have

$$\frac{(1+[y])^m}{(1+[y^s])} \le\ 2^m \quad , \tag{2.9}$$

so that from (2.6) and (2.9), we get

$$\frac{(1+[y])^m}{(1+[y^s])} \le \max(1, 2^m) =\ D_m\ \cdot \tag{2.10}$$

Using (2.10) in (2.5), we have

$$\left|[y^p]\left(y^{-1}D_y\right)^k [y]^{2\beta-1}\,\Phi(y)\right|$$

$$\le\ D_m\, D \sum_{r_1=0}^{k_1} \dots\dots \sum_{r_n=0}^{k_n} \binom{k_1}{r_1} \dots\dots \binom{k_n}{r_n}$$

$$\times \int_0^\infty \dots\dots \int_0^\infty \left(\left[x^{2\lambda+p+s+1}\right] \sum_{v_1=0}^{p_1+s_1} \dots\dots \sum_{v_n=0}^{p_n+s_n} \binom{p_1+s_1}{v_1} \dots\dots \binom{p_n+s_n}{v_n}\right.$$

$$\times\ (x^{-1}D_x)^{p+s-v}\, [x]^{2\beta-1}\, \hat{\phi}\ (x_1, \dots x_n) + \left[x^{2\lambda+p+1}\right] \textstyle\sum_{v_1=0}^{p_1} \dots\dots \sum_{v_n=0}^{p_n} \binom{p_1}{v_1} \dots\dots \binom{p_n}{v_n}$$

$$\times\ (x^{-1}D_x)^{p-v}\ [x]^{2\beta-1}\, \hat{\phi}\ (x_1, \dots x_n)\, dx_1 \dots\dots dx_n \ \cdot$$

Now, let N_i be a non-negative integer such that $N_i > 2(\alpha - \beta + k_i) + p_i + s_i + 3,\ i = 1,2, \dots . n$ · Then

$$|[y^p]\left(y^{-1}D_y\right)^k [y]^{2\beta-1}\,\Phi\,(y)$$

$$\le D' \sum_{r_1=0}^{k_1} \dots\dots \sum_{r_n=0}^{k_n} \binom{k_1}{r_1} \dots\dots \binom{k_n}{r_n} \int_0^\infty \dots\dots \int_0^\infty \prod_{i=1}^{n} (1+x_i)^{N_i-2}$$

$$\times \Big(\sum_{v_1=0}^{p_1+s_1} \dots\dots \sum_{v_n=0}^{p_n+s_n} \binom{p_1+s_1}{v_1} \dots\dots \binom{p_n+s_n}{v_n}$$

$$\times\ \left|(x^{-1}D_x)^{p+s-v}\, [x]^{2\beta-1}\, \hat{\phi}\ (x_1, \dots x_n)\right| + \sum_{v_1=0}^{p_1} \dots\dots \sum_{v_n=0}^{p_n} \binom{p_1}{v_1} \dots\dots \binom{p_n}{v_n}$$

$$\times\ \left|(x^{-1}D_x)^{p-v}\, [x]^{2\beta-1}\, \hat{\phi}\ (x_1, \dots x_n)\right|\Big)\, dx_1 \dots\dots dx_n \ \cdot$$

$$\leq D'' \sum_{r_1=0}^{k_1} \dots \sum_{r_n=0}^{k_n} \binom{k_1}{r_1} \dots \binom{k_n}{r_n} \Big(\sum_{\nu_1=0}^{p_1+s_1} \dots \sum_{\nu_n=0}^{p_n+s_n} \binom{p_1+s_1}{\nu_1} \dots \binom{p_n+s_n}{\nu_n}$$

$$\times \sum_{j_1=0}^{N_1} \dots \sum_{j_n=0}^{N_n} \binom{N_1}{j_1} \dots \binom{N_n}{j_n} \rho^{\alpha,\beta}_{j,p+s-\nu}\, \widehat{\phi}$$

$$+ \textstyle\sum_{\nu_1=0}^{p_1} \dots \sum_{\nu_n=0}^{p_n} \binom{p_1}{\nu_1} \dots \binom{p_n}{\nu_n} \sum_{j_1=0}^{N_1} \dots \sum_{j_n=0}^{N_n} \binom{N_1}{j_1} \dots \binom{N_n}{j_n} \rho^{\alpha,\beta}_{j,p-\nu}\, \widehat{\phi} \Big) \cdot$$

Therefore in view of equation (1.1),

$$\rho^{\alpha,\beta}_{p,k}(\Phi) \leq D'' \sum_{r_1=0}^{k_1} \dots \sum_{r_n=0}^{k_n} \binom{k_1}{r_1} \dots \binom{k_n}{r_n} \sum_{j_1=0}^{N_1} \dots \sum_{j_n=0}^{N_n} \binom{N_1}{j_1} \dots \binom{N_n}{j_n}$$

$$\times \left(\sum_{\nu_1=0}^{p_1+s_1} \dots \sum_{\nu_n=0}^{p_n+s_n} \binom{p_1+s_1}{\nu_1} \dots \binom{p_n+s_n}{\nu_n} \rho^{\alpha,\beta}_{j,p+s-\nu}\, \widehat{\phi} + \sum_{\nu_1=0}^{p_1} \dots \sum_{\nu_n=0}^{p_n} \binom{p_1}{\nu_1} \dots \binom{p_n}{\nu_n} \rho^{\alpha,\beta}_{j,p-\nu}\, \widehat{\phi} \right),$$

where D'' is a positive constant. From above, the continuity of $h_{\alpha,\beta,a}$ follows.

3. An integral representation:

The function $a_\eta\,(y_1, \dots y_n)$, where $\eta = (\eta_1, \dots \eta_n)$, associated with the symbol $a\,(x_1, \dots, x_n\,;\, y_1, \dots, y_n)$ are defined by

$$a_\eta(y_1, \dots, y_n) = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^n (x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}\,(x_i y_i)$$

$$\times \left\{(x_i\eta_i)^{\alpha+\beta} J_{\alpha-\beta}\,(x_i\eta_i)\;\; a\,(x_1, \dots, x_n\,;\, \eta_1, \dots, \eta_n)\right\} dx_1 \dots dx_n \cdot \tag{3.1}$$

will play a fundamental role in our investigation.

An estimate for $a_\eta(y_1, \dots y_n)$ is given by

Lemma 3.1: Let the symbol $a\,(x_1, \dots, x_n\,;\, y_1, \dots, y_n)$ belongs to H^m .

Then the function $a_\eta(y_1, \dots y_n)$ defined by (3.1) satisfies the inequality

$$\left|a_\eta(y_1, \dots y_n)\right| \leq A\,(1+[\eta])^{\alpha-\beta+m+4r+\frac{1}{2}}\,(1+[y])^{2\alpha}\,(1+[y^{2r}])^{-1}\;,$$

where A is a positive constant, $\eta = (\eta_1, \dots \eta_n)$, and $r \in \mathbb{N}_0^n$ with $r > (0,\, 0, \dots, 0) \cdot$

Proof: For $\in \mathbb{N}_0^n$, using formulas (1.6) and (1.8), we have

$$\left(\left(\prod_{i=1}^n (-y_i^2)^{ri}\right) a_\eta\;(y_1, \dots y_n)\right.$$

$$\left.= \int_0^\infty \dots \int_0^\infty \left(\prod_{i=1}^n (x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}\,(x_i y_i)\; \Delta^{r_i}_{\alpha,\beta} \left\{(x_i\eta_i)^{\alpha+\beta} J_{\alpha-\beta}\,(x_i\eta_i)\; a\,(x_1, \dots, x_n\,;\, \eta_1, \dots, \eta_n)\right\}\right)\right)$$

$$\times\; dx_1 \dots dx_n \cdot$$

$$= \int_0^\infty \ldots\ldots \int_0^\infty \prod_{i=1}^{n} \left\{ (x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}(x_i y_i) \right\} \left(\begin{array}{c} \displaystyle\sum_{j_i=0}^{r_i} bj_i \, (x_i)^{2j_i+2\alpha} \left(x_i^{-1} \frac{\partial}{\partial x_i} \right)^{r_i+j_i} \\ \left((x_i)^{2\beta-1} \left\{ (x_i \eta_i)^{\alpha+\beta} J_{\alpha-\beta}(x_i \eta_i) \right\} a\,(x_1, \ldots\ldots, x_n \,;\, \eta_1, \ldots., \eta_n) \right) \end{array} \right)$$

$$\times \; dx_1 \ldots\ldots dx_n \,.$$

Using (1.7) , we get,

$$\left(\prod_{i=1}^{n} (-y_i^2)^{r_i} \right) a_\eta \, (y_1, \ldots\ldots y_n)$$

$$= \int_0^\infty \ldots\ldots \int_0^\infty \prod_{i=1}^{n} \left\{ (x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}(x_i y_i) \right\}$$

$$\times \left(\begin{array}{c} \displaystyle\sum_{j_i=0}^{r_i} bj_i \, (x_i)^{2j_i+2\alpha} \sum_{e_i=0}^{r_i+j_i} \binom{r_i+j_i}{e_i} \left(x_i^{-1} \frac{\partial}{\partial x_i} \right)^{r_i+j_i-e_i} \\ \times \left\{ (x_i)^{2\beta-1} (x_i \eta_i)^{\alpha+\beta} J_{\alpha-\beta}(x_i \eta_i) \right\} \left(x_i^{-1} \frac{\partial}{\partial x_i} \right)^{e_i} a\,(x_1, \ldots\ldots, x_n \,;\, \eta_1, \ldots., \eta_n) \end{array} \right)$$

$$\times \; dx_1 \ldots\ldots dx_n$$

to which an application of the formula

$$\left(x_i^{-1} \frac{\partial}{\partial x_i} \right)^{q_i} x_i^{-(\alpha-\beta)} J_{\alpha-\beta}(x_i \eta_i) = (-\eta_i)^{q_i} \, x_i^{-(\alpha-\beta)-q_i} J_{\alpha-\beta+q_i}(x_i \eta_i), i = 1, 2, \ldots. n,$$

yields

$$\left| \left(\prod_{i=1}^{n} (y_i^2)^{r_i} \right) a_\eta \, (y_1, \ldots\ldots y_n) \right|$$

$$\leq \int_0^\infty \ldots\ldots \int_0^\infty \left| \prod_{i=1}^{n} \left\{ (x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}(x_i y_i) \right\} \right.$$

$$\times \left(\sum_{j_i=0}^{r_i} bj_i \, (x_i)^{2j_i+2\alpha} \sum_{e_i=0}^{r_i+j_i} \binom{r_i+j_i}{e_i} \left(x_i^{-1} \frac{\partial}{\partial x_i} \right)^{e_i} a_\eta \, (x_1, \ldots\ldots, x_n \,;\, \eta_1, \ldots., \eta_n) \right) \left[\eta^{\alpha-\beta+2r+2j-2i+\frac{1}{2}} \right]$$

$$\times \prod_{i=1}^{n} (x_i \eta_i)^{-(\alpha-\beta)-r_i-j_i+e_i} J_{\alpha-\beta+r_i+j_i-e_i}(x_i \eta_i) \left| dx_i \ldots\ldots dx_n \right.$$

$$\leq \; [y]^{2\alpha} \int_0^\infty \ldots\ldots \int_0^\infty [x]^{2\alpha} \left| \prod_{i=1}^{n} (x_i y_i)^{-(\alpha-\beta)} J_{\alpha-\beta}(x_i y_i) \right|$$

$$\times \left| \prod_{i=1}^{n} \sum_{j_i=0}^{r_i} bj_i \, (x_i)^{2j_i+2\alpha} \sum_{e_i=0}^{r_i+j_i} \binom{r_i+j_i}{e_i} \left(x_i^{-1} \frac{\partial}{\partial x_i} \right)^{e_i} a_\eta \, (x_1, \ldots\ldots, x_n \,;\, \eta_1, \ldots., \eta_n) \left[\eta^{\alpha-\beta+2r+2j-2i+\frac{1}{2}} \right] \right|$$

$$\times \left| \prod_{i=1}^{n} (x_i \eta_i)^{-(\alpha-\beta)-r_i-j_i+e_i} J_{\alpha-\beta+r_i+j_i-e_i}(x_i \eta_i) \right| dx_i \ldots\ldots dx_n$$

$$\leq B\,[y]^{2\alpha} \prod_{i=1}^{n}\left(\sum_{j_i=0}^{r_i}\sum_{e_i=0}^{r_i+j_i}\binom{r_i+j_i}{e_i}\,|bj_i|\right)\left[\eta^{\alpha-\beta+2r+2j-2i+\frac{1}{2}}\right]$$

$$\times\; D'\,(1+[\eta])^m \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{n}\left((1+x_i)^{-q_i}\, x_i^{2j+4\alpha}\right) dx_i \ldots\ldots dx_n$$

$$\leq\; B\,[y]^{2\alpha} \prod_{i=1}^{n}\left(\sum_{j_i=0}^{r_i}\sum_{e_i}^{r_i+j_i}\binom{r_i+\,j_i}{e_i}\,|bj_i|\right)\left[\eta^{\alpha-\beta+2r+2j-2i+\frac{1}{2}}\right]$$

$$\times D'(1+[\eta])^m \left(B(2(\alpha-\beta)+2j_i+2,\; q_i-2(\alpha-\beta)-2j_i-2)\right)$$

$$\leq B\,[y]^{2\alpha} \prod_{i=1}^{n}\left(\sum_{j_i=0}^{r_i}\sum_{e_i=0}^{r_i+j_i}\binom{r_i+\,j_i}{e_i}\,|bj_i|\right)\left[\eta^{\alpha-\beta+2r+2j-2i+\frac{1}{2}}\right]$$

$$\times D'(1+[\eta])^m \left(\frac{\Gamma(2(\alpha-\beta)+2j_i+2)\;\;\Gamma(q_i-2(\alpha-\beta)-2j_i-2)}{\Gamma(q_i)}\right).$$

Thus

$$\left|a_\eta\,(y_1,\ldots\ldots y_n)\right| \;\leq A\,(1+[y])^{2\alpha}\,(1+[y^{2r}])^{-1}\left(1+[\eta]^{\alpha-\beta+m+4r+\frac{1}{2}},\right.$$

for all $r\; >\; (0,\ldots\ldots,0)\;\in\;\mathbb{N}_0^n$.

Thus proof is completed.

Theorem 3.1: For any symbol $a\,(x_1,\ldots\ldots,x_n\,;\;y_1,\ldots.,y_n)\;\in\;H^m$ the associated operator $h_{\alpha,\beta,a}$ can be represented by

$$\left(h_{\alpha,\beta,a}\;\phi\right)(x_1,\ldots\ldots,x_n)$$

$$= \int_0^\infty \ldots\ldots \int_0^\infty \left(\prod_{i=1}^{n}(x_iy_i)^{\alpha+\beta}\,J_{\alpha-\beta}\,(x_iy_i)\right)\left(\int_0^\infty \ldots\ldots \int_0^\infty a_\eta(y_1,\ldots.,y_n)\;\hat{\phi}\;(\eta_1,\ldots.,\eta_n)\right)$$

$$\times\; dy_1\ldots\ldots dy_n \quad,\quad \phi\;\in\;H_{\alpha,\beta}\,(I^n)\;, \tag{3.2}$$

where $\hat{\phi}\;(\eta_1,\ldots.,\eta_n) = \left(h_{\alpha,\beta}\;\phi\right)(\eta_1,\ldots.,\eta_n)$, all integrals are convergent.

Proof: Since

$$a_\eta\;(y_1,\ldots.,y_n) = \int_0^\infty \ldots\ldots \int_0^\infty \prod_{i=1}^{n}\left((x_iy_i)^{\alpha+\beta}\,J_{\alpha-\beta}\,(x_iy_i)\right)$$

$$\times\left(\prod_{i=1}^{n}(x_i\eta_i)^{\alpha+\beta}\,J_{\alpha-\beta}\,(x_i\eta_i)\;a\,(x_1,\ldots\ldots,x_n\,;\;\eta_1,\ldots.,\eta_n)\right) dx_1\ldots\ldots dx_n\,,$$

by inversion, we have

$$\int_0^\infty \ldots\ldots \int_0^\infty a_\eta\;\;(y_1,\ldots.,y_n)\;\prod_{i=1}^{n}(x_iy_i)^{\alpha+\beta}\;\;dy_1\ldots\ldots dy_n$$

$$= \prod_{i=1}^{n}(x_i\eta_i)^{\alpha+\beta}\;\;J_{\alpha-\beta}\,(x_i\eta_i)\;a\,(x_1,\ldots\ldots,x_n\,;\;\eta_1,\ldots.,\eta_n)$$

Therefore,

$$\left(h_{\alpha,\beta}\,\phi\right)(x_1,\ldots,x_n)$$

$$= \int_0^\infty \ldots \int_0^\infty \left(\prod_{i=1}^n (x_i\eta_i)^{\alpha+\beta} J_{\alpha-\beta}\ (x_i\eta_i)\right) a\,(x_1,\ldots\ldots,x_n\,;\ \eta_1,\ldots,\eta_n)$$

$$\times\ \hat{\phi}\,(\eta_1,\ldots,\eta_n)\,d\eta_1\ldots\ldots d\eta_n$$

$$= \int_0^\infty \ldots\ldots \int_0^\infty \hat{\phi}\ (\eta_1,\ldots,\eta_n)\,d\eta_1\ldots\ldots d\eta_n \int_0^\infty \ldots\ldots \int_0^\infty a_\eta\ (y_1,\ldots,y_n)$$

$$\times \left(\prod_{i=1}^n (x_i y_i)^{\alpha+\beta}\ J_{\alpha-\beta}(x_i y_i)\right) dy_1\ldots\ldots dy_n$$

$$= \int_0^\infty \ldots \int_0^\infty \left(\prod_{i=1}^n (x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}\ (x_i y_i)\right) dy_1\ldots\ldots dy_n \int_0^\infty \ldots\ldots \int_0^\infty \hat{\phi}\ (\eta_1,\ldots,\eta_n)$$

$$\times\ a_\eta\,(y_1,\ldots,y_n)\,d\eta_1\ldots\ldots d\eta_n\ \cdot$$

Now, using the estimate for $a_\eta\,(y_1,\ldots,y_n)$ given in Lemma 3.1, the above change in the order of integration can be justified and the existence of the last integral can be proved. Since $\hat{\phi}\,(\eta_1,\ldots,\eta_n)\ \in H_{\alpha,\beta}\,(I^n)$

we have

$$\left|\hat{\phi}\,(\eta_1,\ldots,\eta_n)\right| \le\ C\,[\eta]^{2\alpha}\,(1+[\eta])^{-l}\ \ ,\qquad for\ all\ l > 0\,\cdot$$

Hence

$$\left|\left(h_{\alpha,\beta,a}\,\phi\right)(x_1,\ldots,x_n)\right|$$

$$\le \int_0^\infty \ldots \int_0^\infty \left(\int_0^\infty \ldots \int_0^\infty [xy]^{2\alpha}\right) \left|\prod_{i=1}^n (x_i y_i)^{-(\alpha-\beta)} J_{\alpha-\beta}\,(x_i y_i)\right|$$

$$\times A\,C\,(1+[y])^{2\alpha}\,(1+[y^{2r}])^{-1}\left(1+\ [\eta]^{\alpha-\beta+m+4r+\frac{1}{2}}\,[\eta]^{2\alpha}\,(1+[\eta])^{-l}\,d\eta_1\ldots.d\eta_n\right) dy_1\ldots.dy_n\ \cdot$$

$$\le L\,[x]^{2\alpha} \int_0^\infty \ldots \int_0^\infty (1+[y])^{4\alpha}\,(1+[y^{2r}])^{-1}\ dy_1\ldots.dy_n$$

$$\times\ \int_0^\infty \ldots \int_0^\infty \left(1+\ [\eta]^{2(\alpha-\beta)+m+4r+1-l}\ d\eta_1\ldots.d\eta_n\right. \qquad (3.3)$$

The above integrals are convergent since $(\alpha-\beta)\ \ge\ -\frac{1}{2}$, and $l\,(> r)$ can be chosen sufficiently large. Indeed, one can show that

$$\int_0^\infty \ldots \int_0^\infty \frac{\left(1+(y_1,\ldots,y_n)\right)^{4\alpha}}{1+y_1^{2r_1}\ldots\ldots y_n^{2r_n}}\ dy_1\ldots\ldots dy_n$$

$$\le \int_0^1 \ldots\ldots \int_0^1 \frac{(1+1)^{4\alpha}}{1}\,dy_1\ldots\ldots dy_n + \int_1^\infty \ldots\ldots \int_1^\infty \frac{(2y_1\ldots\ldots y_n)^{4\alpha}}{y_1^{2r_1}\ldots\ldots y_n^{2r_n}}\ dy_1\ldots\ldots dy_n$$

$$= 2^{4\alpha} + 2^{4\alpha} \int_1^\infty \dots\dots \int_1^\infty y_1^{4\alpha-2r_1} \dots\dots . y_n^{4\alpha-2r_n}\, dy_1 \dots . . dy_n < \infty ,$$

for all $r_i > 3\alpha + \beta$, $i = 1,2, \dots . . , n$.

Similarly, the second integral in (3.3) can also be shown to converge. Thus proof is completed.

4. An L^1 norm inequality:

In the proof Theorem 4.1, we shall need the following estimate for the Hankel type transform of $[x]^{2\alpha}\, a\,(x_1, \dots\dots, x_n\, ;\, \eta_1, \dots . , \eta_n)$. We write

$$A_\eta\,(y_1, \dots . , y_n) = h_{\alpha,\beta} \{[x]^{2\alpha} a\,(x_1, \dots\dots, x_n\, ;\, \eta_1, \dots . , \eta_n)\}\; y_1, \dots . , y_n \; .$$

Lemma 4.1: For $(\alpha - \beta) \geq -\frac{1}{2}$ and $r \in \mathbb{N}_0^n$, $r > (0, \dots\dots, 0),$ there exists a constant C > 0 such that

$$\left| A_\eta\,(y_1, \dots . , y_n) \right| \leq C\,(1 + [\eta])^m\, [y]^{2\alpha}\,(1 + [y^{2r}])^{-1} \qquad (4.1)$$

Proof: As in the proof of Lemma 3.1, we have

$$\left(\prod_{i=1}^n (y_i^2)^{r_i}\right) A_\eta\,(y_1, \dots . , y_n)$$

$$= \int_0^\infty \dots . . \int_0^\infty \left(\prod_{i=1}^n (x_i y_i)^{\alpha+\beta}\, J_{\alpha-\beta}\,(x_i y_i) \Delta_{\alpha,\beta}^{r_i} \left([x]^{2\alpha}\, a\,(x_1, \dots\dots, x_n\, ;\, \eta_1, \dots . , \eta_n)\right)\right)$$

$$\times\, dx_1 \dots\dots . dx_n$$

$$= \int_0^\infty \dots\dots \int_0^\infty \prod_{i=1}^n \{(x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}\,(x_i y_i)\} \left(\sum_{j_i=0}^{r_i} bj_i\,(x_i)^{2j_i+2\alpha} \left(x_1^{-1}\, \frac{\partial}{\partial x_i}\right)^{r_i+j_i} a\,(x_1, \dots\dots, x_n\, ;\, \eta_1, \dots . , \eta_n)\right)$$

$$\times\, dx_1 \dots\dots . dx_n ,$$

so that

$$\left| \left(\prod_{i=1}^n (y_i^2)^{r_i}\right) A_\eta\,(y_1, \dots . , y_n) \right| \leq \int_0^\infty \dots . . \int_0^\infty \left| \prod_{i=1}^n \{(x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}\,(x_i y_i)\} \right|$$

$$\times \left(\prod_{i=1}^n \sum_{j_i=0}^{r_i} |bj_i|\,(x_i)^{2j_i+2\alpha}\right) \left| \left(x_1^{-1}\, \frac{\partial}{\partial x_i}\right)^{r_i+j_i} a\,(x_1, \dots\dots, x_n\, ;\, \eta_1, \dots . , \eta_n) \right| dx_1 \dots\dots . dx_n$$

$$\leq \int_0^\infty \dots\dots \int_0^\infty \left| \prod_{i=1}^n \{(x_i y_i)^{\alpha+\beta} J_{\alpha-\beta}\,(x_i y_i)\} \right| \left(\prod_{i=1}^n \sum_{j_i=0}^{r_i} |bj_i|\right)$$

$$\times\, [x^{2j+2\alpha}]\, D\,(1 + [\eta])^m \prod_{i=1}^n (1 + x_i)^{-q_i}\, dx_1 \dots\dots . dx_n$$

$$\leq \int_0^\infty \dots . . \int_0^\infty [y]^{2\alpha}\, [x]^{2\alpha} \left| \prod_{i=1}^n \{(x_i y_i)^{-(\alpha-\beta)} J_{\alpha-\beta}\,(x_i y_i)\} \right|$$

$$\times \left(\prod_{i=1}^{n}\sum_{j_i=0}^{r_i}|bj_i|\right)\left[x^{2j+2\alpha}\right] D\,(1+[\eta])^m$$

$$\times \prod_{i=1}^{n}(1+x_i)^{-q_i}\,dx_1\,\ldots\ldots dx_n$$

$$\leq \prod_{i=1}^{n}\sum_{j_i=0}^{r_i}[y]^{2\alpha}\,B\,(1+[\eta])^m\int_0^\infty\ldots\ldots\int_0^\infty\prod_{i=1}^{n}\left((1+x_i)^{-q_i}\,x_i^{2j+4\alpha}\right)\,dx_1\,\ldots\ldots dx_n$$

$$\leq \prod_{i=1}^{n}\sum_{j_i=0}^{r_i}[y]^{2\alpha}\,B\,\,(1+[\eta])^m\,\left(B\,(2(\alpha-\beta)+2j_i+2, q_i-2(\alpha-\beta)-2j_i-2)\right)$$

$$\leq \prod_{i=1}^{n}\sum_{j_i=0}^{r_i}[y]^{2\alpha}\,B\,(1+[\eta])^m\,\left(\frac{\Gamma(2(\alpha-\beta)+2j_i+2)\,\Gamma(q_i-2(\alpha-\beta)-2j_i-2)}{\Gamma(q_i)}\right)$$

Therefore,

$$\left|A_\eta\,(y_1,\ldots.,y_n)\right|\,\leq\,C\,(1+[\eta])^m\,[y]^{2\alpha}\,(1+[y^{2r}])^{-1},$$

where C is a positive constant. Thus proof is completed.

We shall use the above inequality in obtaining a Sobolev norm inequality for a subspace of $H_{\alpha,\beta}\,(I^n)$.

Definition 4.1: (Sobolev type space) : The space $G^s_{\alpha,\beta}\,(\mathbb{R}^n),\ s\in\mathbb{R}$ is defined to be the set of all those elements $\phi\in H_{\alpha,\beta}\,(I^n)$ which satisfy

$$\|\phi\|_{G^s_{\alpha,\beta}} = \eta^{s+2\beta-1}\,h_{\alpha,\beta}\,\phi\,| < \infty\,\cdot \tag{4.2}$$

Theorem 4.1: Let $(\alpha-\beta)\,>\,-\frac{1}{2}$. Then for all $\nu\in\mathbb{N}_0^n$ there exists C >0 such that

$$\left\|h_{\alpha,\beta,a}\,\phi\right\|_{G^0_{\alpha,\beta}}\,\leq\,\prod_{i=1}^n\sum_{l_i=0}^{\nu_i}\binom{\nu_i}{l_i}\,\|\phi\|_{G^{l_i}_{\alpha-\beta}}\quad\phi\in H_{\alpha,\beta}\,(I^n)\,\cdot \tag{4.3}$$

Proof: Taking the Hankel type transform with respect to x of (3.2), we get

$$\int_0^\infty\ldots..\int_0^\infty\left(\prod_{i=1}^{n}(x_iy_i)^{\alpha+\beta}J_{\alpha-\beta}\,(x_iy_i)\right)\,\left(h_{\alpha,\beta,a}\,\phi\right)(x_1,\ldots\ldots,x_n)\,dx_1\,\ldots\ldots dx_n$$

$$=\int_0^\infty\ldots.\int_0^\infty a_\eta\,\,(y_1,\ldots\ldots,y_n)\,\hat{\phi}\,(\eta_1,\ldots\ldots,\eta_n)\,d\eta_1\,\ldots\ldots d\eta_n\,\cdot$$

Now, multiplying both sides by $[y]^{2\beta-1}$ and using (1.11), we get

$$[y]^{2\beta-1}\,h_{\alpha,\beta}\left(h_{\alpha,\beta,a}\,\phi\right)(y_1,\ldots\ldots,y_n)=\int_0^\infty\ldots\ldots\int_0^\infty\hat{\phi}\,\,(\eta_1,\ldots\ldots,\eta_n)\,[y]^{2\beta-1}$$

$$\times\left(\int_0^\infty\ldots..\int_0^\infty\left\{\prod_{i=1}^{n}(x_iy_i)^{\alpha+\beta}J_{\alpha-\beta}\,(x_iy_i)\right\}\left\{\prod_{\nu=1}^{n}(x_\nu y_\nu)^{\alpha+\beta}J_{\alpha-\beta}\,(x_\nu y_\nu)\right\}a\,(x_1,\ldots\ldots,x_n\,;\,\eta_1,\ldots.,\eta_n)\times dx_1\,\ldots\ldots dx_n\right)$$

$$\times \quad d\eta_1, \dots \dots, d\eta_n \cdot$$

$$= \frac{1}{\left(2^{\alpha-\beta}\, \Gamma(3\alpha+\beta)\right) 2^{2n}} \int_0^\infty \dots \dots \int_0^\infty \hat{\phi}\ (\eta_1, \dots \dots, \eta_n)\ [y]^{2\beta-1}$$

$$= \int_0^\infty \dots . . \int_0^\infty \left\{ \prod_{i=1}^{n} (x_i y_i)^{\alpha+\beta}\ (x_i y_i)^{\alpha+\beta}\ j_{\alpha-\beta}\ (x_i y_i) \right\}$$

$$\times \left\{ \prod_{\nu=1}^{n} (x_\nu y_\nu)^{\alpha+\beta}\ \ (x_\nu y_\nu)^{\alpha-\beta}\ j_{\alpha-\beta}\ (x_\nu y_\nu) \right\}$$

$$\times\ a\,(x_1, \dots \dots, x_n\ ;\ \eta_1, \dots ., \eta_n)\ dx_1 \dots . . dx_n)\ \ d\eta_1, \dots \dots, d\eta_n \cdot$$

Now, applying equations (1.9) and (1.10), we can write the right-hand side of the above expression in the form

$$R \int_0^\infty \dots \dots \int_0^\infty \hat{\phi}\ (\eta_1, \dots \dots, \eta_n)\ [y]^{2\beta-1}\ \left(\int_0^\infty \dots . . \int_0^\infty [x]^{4\alpha}\ [\eta]^{3\alpha+\beta}\ [y]^{2\alpha} \right)$$

$$\times \left(\int_0^\infty \dots . . \int_0^\infty [z]^{3\alpha+\beta} \left\{ \prod_{i=1}^{n} (x_i)^{-(\alpha-\beta)}\ J_{\alpha-\beta}\ (z_i x_i)\ D_{\alpha,\beta}\ (\eta_i, y_i, z_i) \right\} dz_1, \dots \dots, dz_n \right)$$

$$\times\ a\,(x_1, \dots \dots, x_n\ ;\ \eta_1, \dots ., \eta_n)\ \ d\eta_1, \dots \dots, d\eta_n$$

$$= R \int_0^\infty \dots . . \int_0^\infty [x]^{3\alpha+\beta}\ [\eta]^{2\alpha}\ \ \hat{\phi}\ (\eta_1, \dots \dots, \eta_n)\ d\eta_1, \dots \dots, d\eta_n$$

$$\times \int_0^\infty \dots \dots \int_0^\infty [z]^{2\alpha} \left\{ \prod_{i=1}^{n} D_{\alpha,\beta}\ (\eta_i, y_i, z_i) \right\}\ dz_1, \dots \dots, dz_n\ )$$

$$\times \int_0^\infty \dots . . \int_0^\infty a\,(x_1, \dots \dots, x_n\ ;\ \eta_1, \dots ., \eta_n)\ [z]^{\alpha+\beta} \left(\prod_{i=1}^{n} J_{\alpha-\beta}\ (z_i x_i) \right) dx_1, \dots \dots, dx_n$$

$$= R \int_0^\infty \dots . . \int_0^\infty [x]^{2\alpha}\ [\eta]^{2\alpha}\ \ \hat{\phi}\ (\eta_1, \dots \dots, \eta_n)\ d\eta_1, \dots \dots, d\eta_n$$

$$\times \left(\int_0^\infty \dots . . \int_0^\infty [z]^{2\alpha} \left\{ \prod_{i=1}^{n} D_{\alpha,\beta}\ (\eta_i, y_i, z_i) \right\} dz_1, \dots \dots, dz_n \right)$$

$$\times \int_0^\infty \dots . . \int_0^\infty a\,(x_1, \dots \dots, x_n\ ;\ \eta_1, \dots ., \eta_n) \left(\prod_{i=1}^{n} (z_i x_i)^{\alpha+\beta}\ J_{\alpha-\beta}(z_i x_i) \right)\ dx_1, \dots \dots, dx_n$$

$$= R \int_0^\infty \dots . \int_0^\infty [\eta]^{2\alpha}\ \ \hat{\phi}\ (\eta_1, \dots \dots, \eta_n)\ d\eta_1, \dots \dots, d\eta_n \int_0^\infty \dots . \int_0^\infty [z]^{2\alpha} \left(\prod_{i=1}^{n} D_{\alpha,\beta}\ \ (\eta_i, y_i, z_i) \right)$$

$$\times\ dz_1, \dots \dots, dz_n \int_0^\infty \dots . \int_0^\infty [x]^{2\alpha}\ \ a\,(x_1, \dots \dots, x_n\ ;\ \eta_1, \dots ., \eta_n)$$

$$\times \left(\prod_{i=1}^{n}(z_i x_i)^{\alpha+\beta} J_{\alpha-\beta}\ (z_i x_i)\right) dx_1, \dots\dots, dx_n ,$$

where

$$R = \frac{1}{\left(2^{\alpha-\beta}\,\Gamma\,(3\alpha+\beta)\right)^{2n}} \cdot$$

Therefore,

$$[y]^{2\beta-1}\, h_{\alpha,\beta}\left(h_{\alpha,\beta,a}\right)\ (y_1, \dots\dots, y_n)$$

$$\leq \int_0^\infty \dots\dots \int_0^\infty [\eta]^{2\alpha}\ \hat{\phi}\ (\eta_1, \dots\dots, \eta_n)\ d\eta_1, \dots\dots, d\eta_n \int_0^\infty \dots\dots \int_0^\infty [z]^{2\alpha}$$

$$\times \left(\prod_{i=1}^{n} D_{\alpha,\beta}\ (\eta_i, y_i, z_i)\right) dz_1, \dots\dots, dz_n \int_0^\infty \dots\dots \int_0^\infty [x]^{2\alpha}\ a\,(x_1, \dots\dots, x_n\ ;\ \eta_1, \dots\dots, \eta_n)$$

$$\times \left(\prod_{i=1}^{n}(z_i x_i)^{\alpha+\beta} J_{\alpha-\beta}\ (z_i x_i)\right) dx_1, \dots\dots, dx_n \cdot \qquad (4.4)$$

By an application of the estimates (4.1) to (4.4), we have

$$\left|[y]^{2\beta-1}\, h_{\alpha,\beta}\left(h_{\alpha,\beta,a}\phi\right)(y_1, \dots\dots, y_n)\ \right|$$

$$\leq\ C\ R \int_0^\infty \dots\dots \int_0^\infty (1+[\eta])^m\ [\eta]^{2\alpha}\ \hat{\phi}\ (\eta_1, \dots\dots, \eta_n)\ d\eta_1, \dots\dots, d\eta_n$$

$$\times \int_0^\infty \dots\dots \int_0^\infty [z]^{4\alpha}\ (1+[z^{2r}])^{-1} \left(\prod_{i=1}^{n} D_{\alpha,\beta}\ (\eta_i, y_i, z_i)\right) dz_1, \dots\dots, dz_n$$

$$\leq D \sum_{l_1=0}^{\nu_1} \dots\dots \sum_{l_n=0}^{\nu_n} \binom{\nu_1}{l_1} \dots\dots \binom{\nu_n}{l_n} \int_0^\infty \dots\dots \int_0^\infty [\eta^{l+2\alpha}]\ \hat{\phi}\ (\eta_1, \dots\dots, \eta_n)\ d\eta_1, \dots\dots, d\eta_n$$

$$\times \int_0^\infty \dots\dots \int_0^\infty [z]^{4\alpha}\ (1+[z^{2r}])^{-1} \left(\prod_{i=1}^{n} D_{\alpha,\beta}\ (\eta_i, y_i, z_i)\right) dz_1, \dots\dots, dz_n \cdot \qquad (4.5)$$

Now, we set

$f\ (z_1, \dots\dots . z_n) =\ (1+[z^{2r}])^{-1}\ \in L^1\ (I^n)$ for $r_i\ > 0, i = 1,2 \dots\dots, n$,

and

$g\ (\eta_1, \dots\dots, \eta_n) = \left(2^{\alpha-\beta}\ \Gamma\ (3\alpha+\beta)\right)^n\ \left[\eta^{l+2\beta-1}\right] \hat{\phi}\ (\eta_1, \dots\dots, \eta_n)\ \in L^1\ (I^n),$ for

all $l\ \in\ \mathbb{N}_0^n$ such that $l_i\ \leq\ \nu_i\ ,\ i = 1,2 \dots\dots, n\ \cdot$

Then according to (1.12) and (1.13), we have

$$\left(\tau_y\ f\right)(\eta_1, \dots\dots, \eta_n) = \int_0^\infty \dots\dots \int_0^\infty f\ (z_1, \dots\dots . z_n) \left(\prod_{i=1}^{n} D_{\alpha,\beta}\ (\eta_i, y_i, z_i)\right)$$

$$\times\ [z]^{4\alpha} \left(2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right)^{-n}\ dz_1 \dots\dots dz_n\ \cdot$$

and

$$(f \neq g)(y) = \int_0^\infty \ldots\ldots \int_0^\infty (\tau_y f)\ (\eta_1, \ldots\ldots, \eta_n)\ g\ (\eta_1, \ldots\ldots, \eta_n)\ [\eta]^{4\alpha}$$
$$\times \left(2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right)^{-n}\ d\eta_1, \ldots\ldots, d\eta_n\ .$$

Therefore applying (1.14) to (4.5), we get

$$\left\| [y]^{2\beta-1}\, h_{\alpha,\beta}\left(h_{\alpha,\beta,\alpha}\,\phi\right)(y_1, \ldots\ldots, y_n) \right\|_{L^1}$$
$$\leq D \sum_{l_1=0}^{\nu_1} \ldots.. \sum_{l_n=0}^{\nu_n} \binom{\nu_1}{l_1} \ldots\ldots \binom{\nu_n}{l_n}\ \left\| \left[\eta^{l+2\beta-1}\right] \hat{\phi}\ (\eta_1, \ldots\ldots, \eta_n) \right\|_{L^1}\ \|(1+[z^{2r}])^{-1}\|_{L^1}$$
$$\leq\ C \sum_{l_1=0}^{\nu_1} \ldots.. \sum_{l_n=0}^{\nu_n} \binom{\nu_1}{l_1} \ldots\ldots \binom{\nu_n}{l_n}\ \left\| \left[\eta^{l+2\beta-1}\right] \hat{\phi}\ (\eta_1, \ldots\ldots, \eta_n) \right\|_{L^1}\ .$$

From which inequality (4.3) follows. This completes the proof.

Conclusions:

1. If we take $\alpha = \frac{1}{4} + \frac{\mu}{2}\ ,\ \beta = \frac{1}{4} - \frac{\mu}{2}$ in the present paper then results reduce to n dimensional case in [2] .
2. For $n = 1,\ \alpha = \frac{1}{4} + \frac{\mu}{2}\ ,\ \beta = \frac{1}{4} - \frac{\mu}{2}$, the results in the present paper reduce to the dimensional case in Zemanian [8].
3. Author claims that the results developed in the present paper are stronger than that of [2].

Remark: It is proposed to obtain more results on n- dimensional pseudo-differential type operator involving Hankel type transformation.

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