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Mean waiting time assessment and analysis to address hidden failures using reversed hazard rate

$Min \sum_{i=1}^{n} w_i x_i$

$s.t \; A_{(n+w)\times n} X_{n\times 1} \geq b_{(n+w)\times 1}$

$Min \sum_{i=1}^{n} w_i . x_i$

$s.t \; y = A_{n\times n} X_{n\times 1} \geq b_{n\times 1}$

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ABSTRACT

Analysis of Reversed Hazard Rate (RHR) can provide insights in making it suitable for industrial applications. From the published literature it is learnt that Nature of reversed Hazard rate for standard continuous distributions is a decreasing function. Obviously this makes RHR suitable in the field of maintenance engineering to address hidden failures in a given system. One of its most useful applications lies in the assessment of waiting time of hidden failures. RHR is closely related to another important concept known as the mean waiting time. This concept is useful in casualty insurance, reliability, and medicine including forensic science to predict times of occurrences of events. For instance, the incubation times of diseases, are difficult to measure because the infection time is unobserved in general. Mean waiting time will offer its great help in such situations, which are analyzed and incorporated appropriately in this paper.

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Introduction

In today's technological world, the industry depends upon the continuous functioning of a wide array of complex machineries, equipments and systems. They are expected to operate at high level of reliability in order to remain in the competition and ensure customer satisfaction. In this scenario, maintenance has assumed a lot of importance. Reliability and Availability are the key performance parameters in effective maintenance function. Hazard rate is an important reliability measure and is widely used in modeling the life performance of variety of equipments. However, in case of hidden failures, the Hazard rate loses its relevance as it cannot capture such failures by the very definition of it insisting the condition on till time‘t’ the system functions. Therefore, there is a need to have another characteristic known as RHR. It is defined as the conditional probability of a failure of an object per unit time in (t-Δt, t) given that the failure occurred in [0, t]. Recently, the properties of RHR have attracted the attention of researchers.

Problem on Hand

In real life we come across number of situations where our belongings like Mobile Phone, Wallet, driving license etc. are lost and we come to know about the loss at a later time and soon report the loss. Thus there is always delay in reporting the loss from the time the item is lost. It is more so with hidden failures that take place when a system is in random failure region. The cost of not knowing the failures on their occurrence is rather more dangerous and will disrupt the proceedings of any facility and hence business.

Although hazard captures the dynamic changes in the probability of a state transition, yet the analysis is enhanced by using both hazard and RHR. These two functions offer complementary information and provide further constraints on the assessment of probabilistic systems. Sometimes hazard might be more revealing in nature and on other occasions reversed hazard might be more valuable tool. If somehow the mean waiting time of a hidden failure is known, things can be better controlled as commented by Veres-Ferrer and Pavia (2012). This motivated authors of this paper to work on mean waiting time using Hazard rate. Hence this paper deals with analysis of mean waiting time for exponential distribution that helps in modeling the above situations. Further, the same can be applied to maintenance and reliability issues. An attempt is made in this direction.

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Literature Review

The RHR function has been receiving increasing attention in the recent literature of reliability analysis and stochastic modeling. Being in a certain sense a dual function to an ordinary Hazard rate, it still bears some interesting features useful in reliability analysis. One of its most important properties is the connection with the mean waiting time.

Veres-Ferrer and Pavia (2012) addressed the usefulness of reversed hazard function by relating it to other well known concepts like elasticity used in economics. Authors supported their work with a case study. Kundu and Gupta (2011) introduced a bivariate proportional reversed hazard model and discussed its different properties. According to Chechile (2011) Reversed hazard corresponds to the conditional density of an immediate failure or state change conditioned by the fact that the state change occurred. RHR exhibits many symmetrical properties with hazard rate models. To this effect the author developed a set of theorems that explicate the properties of reversed hazard for both continuous and discrete probability distributions. Dessai *et al* (2011) showed that RHR has monotonically decreasing pattern for all the Statistical distributions of engineering importance. They felt that RHR could prove to be a better tool as compared to Hazard rate for the investigation of failure modes. Hence it could be used in the area of conditioned based maintenance. Ortega *et al* (2009) demonstrated that stochastic comparisons from inactivity times can be used in several reliability problems. The inactivity time represents the interval time elapsed after an event occurs until the time of its observation, and it is useful to predict the exact times of occurrence of events. The authors obtained a characterization of the RHR order, and a functional relationship involving the mean inactivity time order for non-negative random variables that model lifetimes. Finkelstein (2002) discussed some of well-known and new properties of RHR. The author demonstrated that considering only non-negative random variables has a certain impact on the shape of the RHR. The author mentions that the RHR is closely related to the waiting time, time elapsed since the failure of an object. The Mean waiting time (MWT) is an important characteristic in many reliability applications. One of the most important properties of RHR is connecting MWT and RHR, which was discussed and presented in the paper. The author presented and reported possible applications.

From the above literature reviewed, we can infer that very limited work is done in the area of MWT using RHR.

Behaviour of RHR

It has been proved that r(t) decreases with decrease in h(t) and RHR and r(t) is a monotonically decreasing function for important continuous distributions including Exponential distribution, Normal distribution, Lognormal distribution and Weibull distribution. The basic definition of RHR is given by

$$\mathbf{r(t)} = \frac{\mathbf{h(t)}}{\mathbf{e}^{\int_0^t \mathbf{h(t)dt}} - \mathbf{1}} \quad (1)$$

Differentiating r(t) w.r.t 't' gives,

$$\mathbf{r'(t)} = \frac{\left(\mathbf{e}^{\int_0^t \mathbf{h(t)dt}} - \mathbf{1}\right)\mathbf{h'(t)} - \mathbf{h(t)}\frac{\mathbf{d}}{\mathbf{dt}}\left(\mathbf{e}^{\int_0^t \mathbf{h(t)dt}} - \mathbf{1}\right)}{\left(\mathbf{e}^{\int_0^t \mathbf{h(t)dt}} - \mathbf{1}\right)^2} \quad (2)$$

$$= \frac{\left(\mathbf{e}^{\int_0^t \mathbf{h(t)dt}} - \mathbf{1}\right)\mathbf{h'(t)} - \mathbf{h^2(t)}}{\left(\mathbf{e}^{\int_0^t \mathbf{h(t)dt}} - \mathbf{1}\right)^2} \quad (3)$$

When h(t) is a decreasing function h'(t) becomes negative, in turn r'(t) < 0, which professes that when h(t) is a decreasing function r(t) also becomes a decreasing function.

RHR for Exponential Distribution

As missing of failures just happen in random failure region of life cycle of any equipment, and they being rare events, it is rather required to know the RHR for exponential distribution. Using Equation (1), the RHR for exponential distribution is,

$$r(t) = \frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda t}} \quad (4)$$

$$= \frac{d(1-e^{-\lambda t})}{1-e^{-\lambda t}}$$

$$= \frac{d}{dt} ln(1 - e^{-\lambda t}) \quad (4a)$$

When t = 0, r(0) = ∞ and when t = ∞, r(∞) = 0. Differentiating Equation (4) w.r.t. 't' in order to investigate its nature

$$r'(t) = \frac{-\lambda^2 e^{-\lambda t}(1-e^{-\lambda t}) - \lambda^2 e^{-2\lambda t}}{(1-e^{-\lambda t})^2}$$

$$= \frac{-\lambda^2 e^{-\lambda t}[(1-e^{-\lambda t}) + e^{-\lambda t}]}{(1-e^{-\lambda t})^2}$$

$$= \frac{-\lambda^2 e^{-\lambda t}}{(1-e^{-\lambda t})^2} \quad (5)$$

From Equation (5), it is clear that RHR, r(t) is a monotonically decreasing function.

Waiting Time Assessment

The waiting time is the time elapsed since the failure of an item on condition that this failure had occurred in [0, t]. It turns out that RHR is closely related to another important random variable-the waiting time. Indeed, as a condition of a failure in [0, t] is already imposed while defining the RHR, it is of interest in different applications (reliability, actuarial science, and survival analysis) to describe the time which has elapsed since the failure. The observations of waiting time for instance, can be used for prediction of the governing distribution function. The waiting time could be of interest while describing different maintenance strategies.

Let $T_{W,X}$ denotes the waiting time elapsed since the failure of an item on condition that this failure had occurred in [0, x]. Therefore for the fixed x, Distribution function will be,

$$F_{w,x}(t) = P[(X - T > t)/T \leq X]$$

$$= P[(T < X - t)/T \leq X]$$

$$= \frac{P\{(T<X-t) \cap (T \leq X)\}}{P(T \leq X)}$$

$$= \frac{\Pr(T<x-t)}{\Pr(T \leq x)} = \frac{F(x-t)}{F(x)}$$

$$= \frac{F(u)}{F(x)} \quad (6)$$

Mean Waiting Time (MWT)

Mean waiting time (MWT) for an item failed in an interval [0, x] is given by,

$$MWT = \mu(x)$$

$$= E(T_{w,x})$$

$$= \int_0^x F_{w,x}(u)du$$

$$\therefore \mu(x) = \frac{\int_0^x F(u)du}{F(x)} \quad (7)$$

Relation between RHR and MWT

Assuming that $\mu(x)$ is differentiable, differentiating equation (7)

$$\mu'(x) = \frac{\frac{d}{dx}\{\int_0^x F(u)du\}F(x) - \frac{d}{dx}\{F(x)\}\int_0^x F(u)du}{[F(x)]^2}$$

Applying Leibnitz Rule to the first term in numerator and

Replacing $\frac{d}{dx}\{F(x)\} = f(x)$

We get,

$$\mu'(x) = \frac{[F(x)]^2 - f(x)\int_0^x F(u)du}{[F(x)]^2}$$

$$= 1 - \left\{\frac{f(x)}{F(x)}\right\}\left\{\frac{\int_0^x F(u)du}{F(x)}\right\}$$

$$= \mathbf{1 - r(x)\mu(x)}$$

$$\therefore r(x) = \frac{1-\mu'(x)}{\mu(x} \qquad (8)$$

Relation between MWT and Distribution Function

We know that

$$\boldsymbol{F(t)} = \mathbf{e}^{-\int_t^{\infty} \mathbf{r(x)dx}} \qquad (9)$$

$$\therefore \boldsymbol{F(t)} = \mathbf{e}^{-\int_t^{\infty} \frac{1-\mu'(u)du}{\mu(u)}}$$

As shown by Finkelstein (2002)

$$\boldsymbol{F(t)} = \mathbf{lim}_{x\to\infty} \frac{\mu(x)}{\mu(t)} e^{-\int_t^x \frac{1}{\mu(u)} du} \qquad (10)$$

MWT for Exponential Distribution

Using Equation (8)

$$\boldsymbol{\mu'(x) = 1 - r(x)\mu(x)}$$

For the exponential distribution, the Equation (11) becomes

$$\boldsymbol{\mu'(x) = 1 - \left(\frac{\lambda e^{-\lambda x}}{1-e^{-\lambda x}}\right)\mu(x)} \qquad (11)$$

By rearranging,

$$\boldsymbol{\frac{d\mu(x)}{dx} + \left(\frac{\lambda e^{-\lambda x}}{1-e^{-\lambda x}}\right)\mu(x) = 1} \qquad (12)$$

Solving the above differential equation with variable coefficient gives, with the integrating factor (IF)

$$\boldsymbol{I.F.} = \boldsymbol{e^{ln|(1-e^{-\lambda x})|}}$$

$$= \boldsymbol{1 - e^{-\lambda x}}$$

The general solution is given by,

$$\boldsymbol{\mu(x)(1 - e^{-\lambda x}) = \int (1 - e^{-\lambda x})dx + C}$$

$$\boldsymbol{\mu(x)(1 - e^{-\lambda x}) = x + \frac{e^{-\lambda x}}{\lambda} + C}\ )$$

At initial condition i.e., when μ(0) =0 *x* = 0 yields $\boldsymbol{C = -\frac{1}{\lambda}}$

$$\therefore \boldsymbol{\mu(x)(1 - e^{-\lambda x}) = x + \frac{e^{-\lambda x}}{\lambda} - \frac{1}{\lambda}}$$

$$\boldsymbol{\mu(x) = \frac{x}{(1-e^{-\lambda x})} - \frac{1}{\lambda}} \qquad (13)$$

Results and Discussions

The parameter λ which appears in Equation 13 governing the MWT and the one that appears in exponential distribution is same. Therefore, it is important to investigate the effect of this parameter.

When $\boldsymbol{x = 0}$; $\boldsymbol{\mu(x) = \frac{0}{0}}$ which is an indeterminate form, evaluating which we get

$$\boldsymbol{\lim_{x\to 0} \mu(x) = \lim_{x\to 0} \frac{1}{\lambda e^{-\lambda x}} - \frac{1}{\lambda} = 0}$$

When $\boldsymbol{x = \infty}$; $\boldsymbol{\mu(x) = \infty}$.

Equation (13) can be written as

$$\boldsymbol{\mu(x) = \frac{\lambda x - (1-e^{-\lambda x})}{\lambda(1-e^{-\lambda x})}} \qquad (14)$$

When $\boldsymbol{\lambda = 0}$; $\boldsymbol{\mu(x) = \frac{0}{0}}$ which is an indeterminate form, evaluating which we get

$$\boldsymbol{\mu(x) = \frac{x^2}{x+x} = \frac{x}{2}} \quad (15)$$

When $\boldsymbol{\lambda} = \infty$;

$$\boldsymbol{\mu(x) = x} \quad (16)$$

Consider Equation (13) to investigate the nature of MWT. Differentiating this equation w.r.t x, we get

$$\boldsymbol{\mu'(x) = \frac{1}{(1-e^{-\lambda x})} + \frac{\lambda e^{-\lambda x}}{(1-e^{-\lambda x})^2} = \frac{(1-e^{-\lambda x})+\lambda e^{-\lambda x}}{(1-e^{-\lambda x})} > 0} \quad (17)$$

From Equation (17) it is clear that MWT is non-linearly increasing function. Also $\boldsymbol{\mu'(0)} = \infty$ and $\boldsymbol{\mu'}(\infty)= 1$. Thus revealing the shape of the μ(x) versus x is as shown in Figure 1. At the same time there is parameter λ which restricts the value of MWT. From Equation (15) and Equation (16), MWT is just half the incubation time if the failure rate λ is 0 and it becomes as long as the incubation time itself if λ is ∞. The failure rate approaching to 0 represents the rareness of the event and when it assumes infinity it represents the sure and instant event. The bounds of the MWT are depicted in Figure 2. Failures of well maintained systems and non-mechanical systems (electronics and others without moving parts) and rare diseases in health care discipline all will have the Poisson occurrence revealing the inter-arrival time or time to occur to follow exponential. In any case, the elapsed time or incubation time or inactive time will have its mean between half the time elapsed if the arrival rate is zero and the elapsed time itself, if the arrival rate is infinity. To reveal the failure, it is recommended to go for regular checkups and regular periodic inspections in cases of health care and engineering segments, respectively.

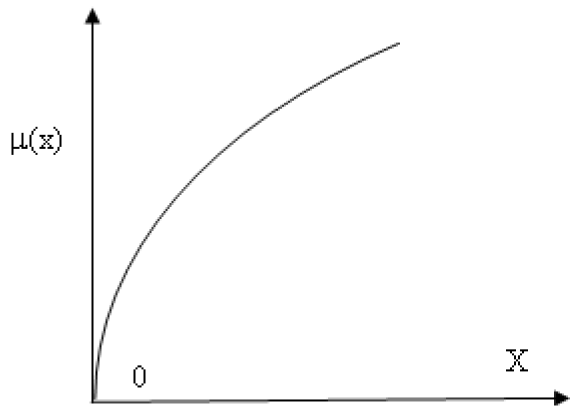

Figure 1: Nature of MWT on incubation time

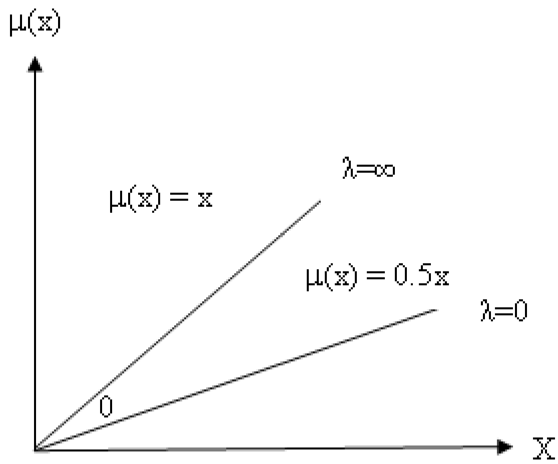

Figure 2: Bounds of MWT

To evaluate the value of ‘λ’ we apply the procedure of establishing attribute control chart, wherein we may have to take fixed no. of samples of size n (say 100). The central line of the control chart which is given by np = λ is then determined. In case of engineering segment, the failure times are collected and inferential statistics is used to fit exponential distribution, thereby obtaining the parameter ‘λ’. The inferential statistics may involve either goodness of fit test, probability plot or softwares such as Minitab, Stat graphics, SPSS etc.

Loss of ATM Card: A Case

Whenever any item like mobile phone, driving license etc. is lost, there is always a delay in reporting the loss from the time that item is lost. This delay in loss greatly impacts the investigation procedure, and in case of health care segment this delay may affect the medical treatment to cure a rare disease. In case of engineering field, the delay time may cause expensive maintenance.

Consider the case of ATM issuance office where customers arrive to report the loss of ATM card randomly. We assume that one reports the loss, after coming to know immediately i.e. zero time delay between knowing the loss and reporting the loss which actually had happened some time much before. Thus one reports the loss of ATM card at time‘t’, but the ATM card was already lost some time before i.e. time ‘t-dt’. Such a situation can be perfectly analyzed by using RHR. Thus the conditional probability that the ATM card is lost in an interval of width ‘dt’ preceding‘t’, given that it is lost before ‘t’ is modeled using RHR r(t). The time elapsed since the loss of the card is of interest in order to predict the actual time of loss. In this case, r(t)dt provides the probability of loss of card in (t − dt, t), when the card is found to be lost at time ‘t’. Number of lost cards reported in a randomly selected month is considered as rare event as it satisfies the following conditions:

i) At a small time interval, either one report is made or no report is made.

ii) At a given instant of time, the probability of this event is insignificantly small compared to the large population of the account holders that the bank has in a particular locality.

iii) Reporting times are independent.

The above realizations are reflecting the postulates of Poisson process and hence it can be rightly assumed that this event follows Poisson distribution and hence the inter-arrival times are following exponential distribution which is having ‘memory less’ property i.e. number of lost card reports made in a randomly selected month can be modeled by using Poisson distribution.

$$p(x) = \frac{e^{-\lambda t}(\lambda t)^{x}}{x!} \qquad (18)$$

where, x = 0, 1, 2, 3, 4, and λ rate of reporting loss. The pdf for exponential distribution is given by ,

$$f(t) = \lambda e^{-\lambda t}, t > 0$$

$$= 0, \text{ otherwise}$$

The distribution function for same is thus given by

$$F(t) = 1 - e^{-\lambda t}$$

We know that conditional probability that the ATM card is lost in an interval of width d*t* preceding *t*, given that it is lost before *t* is modeled by using RHR. RHR of exponential distribution using Equation (4) is given below

$$r(t) = \frac{\lambda}{e^{\lambda t}-1}$$

Thus probability of loss of card in (t − dt, t), when the card is found to be lost at time *t* is given by r(t)dt i.e. $\frac{\lambda}{e^{\lambda t}-1}\,\mathbf{dt}$. Using Equation (8) the MWT becomes for this case

$$\mu(t) = \frac{t}{(1-e^{-\lambda t})} - \frac{1}{\lambda}$$

and the bounds for the MWT become (0.5t, t). But it will tend to be 50% of the elapsed time due to true rareness prevailing. To make the average time truly to become 50% of the elapsed time, things like periodic checking of the ATM in the wallet, say every one hour has to be carried out.

Conclusions

The RHR is an important characteristic of a lifetime distribution function. It could be used as a better tool compared to Hazard rate for the investigation of failure modes, particularly the tricky hidden failures. It is known that RHR has monotonically decreasing pattern for all the continuous distributions of Engineering importance. RHR has already been used for modeling waiting times. But it is learnt that very little work is done and reported in MWT. The mean waiting time is the important characteristic in many Reliability applications in order to describe the time which has elapsed since failure.

In Maintenance Engineering, the primary concern is unrevealed and hidden failures. Though for revealed failures, there are methods like maintenance policies with Markov and without Markov modeling, for hidden and unrevealed failures, no much work has been done. In this paper, the investigation has been done on mean waiting time of exponential distribution, supported by a case. The bounds for MWT provide important information in Maintenance Engineering. With this investigation, scope is further widened for MWT analysis in various other statistical distributions, that in turn, will help in maintenance analysis and Reliability assessment. Also this MWT time analysis can be usefully extended to health care industry to treat incurable and rare diseases like cancer.

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