



Application modes in the algebra

Ebrahim Nazari¹, Mohsen Karami², Ezato Lah Aghahy², Adnan Shikhahmadi², Farogh Shari Fey² and Hamidreza Rostami²

¹Department of Mathematics, Farhangyan University, Iran.
²Department of Education, Bayangan, Ielam, Sanandaj Part Tow, Bayangan, Gilangharb, Iran.

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ABSTRACT

Modes are idempotent and entropic algebras. It is well known that each entropic groupoid with surjective operation embeds as a sub reduct into a semimodule over a commutative semiring. Surprisingly, this is no longer true for modes with operations of larger arity. As shown by M. Stronkowski [4] and [5], a mode embeds as a subreduct into a semimodule over a commutative semiring if and only if it satisfies the so-called Szendrei identities. A simpler proof was then given by D. Stanovsky [3]. Stronkowski also proved that free modes do not satisfy the Szendrei identities, while Stanovsky [3] provided a 3-element example of a mode with one ternary operation (Example 1.1). Medial modes, a natural generalization of normal bands, were investigated by P. Lonka. Rectangular algebras, a generalization of rectangular bands (diagonal modes) were investigated by P. Oschel and Reichel.

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Introduction

We paper properties of this variety, and show that it contains a broad class of modes not satisfying the Szendrei identities, i.e. not embeddable into semimodules over commutative semirings. To simplify notation, we consider only algebras with one ternary operation, but all our results may easily be extended to algebras with one basic operation of any arity $n > 3$. Note that the possibility of embedding given algebras as subreducts into other (richer) algebras provides an efficient method for investigating their structure. In particular, if these richer algebras are (semi)modules, such an embedding allows us to represent operations as linear combinations, providing so called linear representations for the algebras being embedded.

The algebras forming the main topic of this paper are *modes*, i.e. they are *idempotent*, in the sense that each singleton is a subalgebra, and they are *entropic*, i.e. each operation, as a mapping from a direct power of the algebra into the algebra, is actually a homomorphism. The two properties may be expressed algebraically by means of identities.

$$x_{11} \dots x_{1n} x_{m1} \dots \frac{x_{mn} w w}{=} x_{11} \dots \frac{x_{m1} w}{=} \dots x_{1n} \dots \frac{x_{mn} w}{=}$$

that are satisfied in each mode (A, ϵ) , for any n -ary operation w and m -ary operation w in ϵ . Such algebras are studied in detail in [7]. (See also [6]-[10].)

Given a mode (A, ϵ) , as a set A with a set ϵ of operations $w: A^{wx} \rightarrow A$ on it, one may form the set (A, ϵ) , S or AS of non-empty subalgebras of (A, ϵ) . This set AS carries an f -algebra structure under the complex products

$w: AS^{wx} \rightarrow AS; (X_1, \dots, X_{wx}) \rightarrow \{x_1, \dots, x_{wx} | x_i \in X_i\}$ and it turns out that the algebra (AS, ϵ) is again a mode, preserving many of the algebraic properties of (A, ϵ) [7, 146], [8], [9]. Affine spaces can be described as modes (E, R, P) with binary operations r for each r in R and one ternary Mal'cev (parallelogram-completion) operation P satisfying certain identities and it is possible to give much more simple description of affine spaces.

In [6] modes of subspaces of affine spaces over fields are investigated. The structure of certain reducts of such modes yields a direct, invariant passage from affine to projective geometry. Obviously each semilattice $(H, +)$ is a mode as well. We now recall two approaches to semilattices needed later in the paper. Let ϵ be a non-empty domain of operations with an arity mapping $\mu: \epsilon \rightarrow \{n \in \mathbb{N} | n \geq 3\}$. A semilattice $(H, +)$ may be considered as an Q -algebra (H, ϵ) , a so-called ϵ -semilattice, on defining

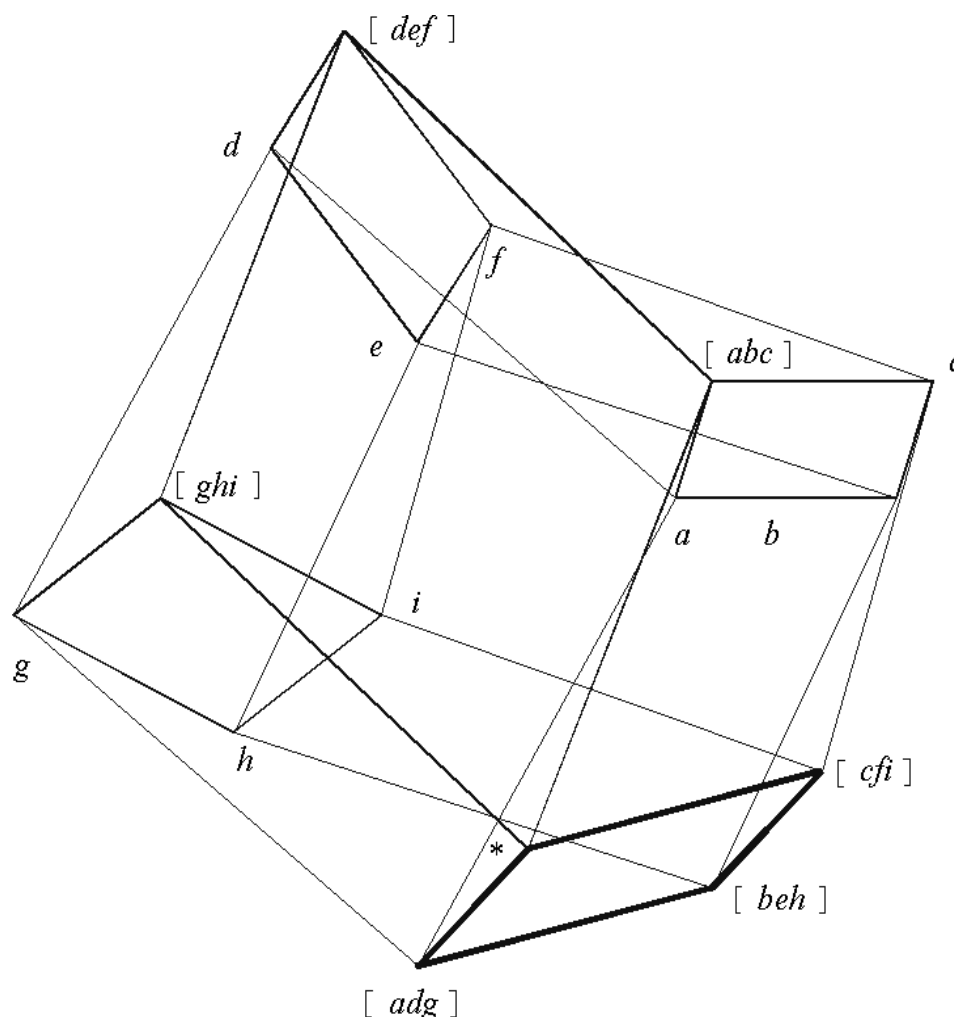
Tele:

E-mail addresses: nazari_ebrahim2008@yahoo.com

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$h_1 \dots h_{w\mu} w = h_2 + \dots + h_{w\mu} +$ for h_i in H . The semilattice operation $+$ is then recovered as $h + k = hk \dots kw$ for any $w \in E$. The semilattice $(H, +)$ may also be considered as a (small) category (H) with a set H of objects, and with a unique morphism $h \rightarrow k$ precisely when $h + k = k$, i.e. $h \leq k$.

Embedding one class of structures into a better understood one usually brings some new knowledge about the former class. We will focus on embeddings of algebras into reducts of semimodules over commutative semirings; hence we obtain linear representations for operations of the algebras. Modes are idempotent algebras where every pair of operations commute with one another [12]. Indeed, idempotent subreducts of semimodules over commutative semirings are modes and it had been an open problem [12] whether the converse statement is true. Quite recently, N. Dojer observed that such modes satisfy the so-called Szendrei identities (they appeared in the paper [15] by Ágnes Szendrei) and Michał Stronkowski found a syntactical proof that these identities do not follow from the axioms of modes [13]. Thus there exist modes that are not idempotent subreducts of semimodules over commutative semirings.



$$* \dots [abc][def][ghi] = [adg][beh][cfi]$$

Theorem :([19]). An algebra (A, f) with one n -ary basic operation is a medial mode if and only if it is a P lonka sum of algebras, each of them being the direct product of one $n - dimensional$ diagonal algebra and one $rn - algebra$. Diagonal algebras are characterized by the following proposition.

Proposition : ([20, Section 5.2]). Each $n -$ ary diagonal mode (A, d) is a direct product of n projection subalgebras (A_i, d) satisfying the identity

$$x_1 \dots x_i \dots x_n d = x_i$$

For $n = 2$ Proposition up reduces to well known fact that each rectangular band is a direct product of a left-zero semigroup and a right-zero semigroup. A further generalization of diagonal modes was considered by Pöschel and Reichel in [21] under the name of rectangular algebras. A mode (A, ϵ) of any finite type $\tau: \epsilon \rightarrow z^+$ is called a rectangular algebra if each operation ω in ϵ satisfies the diagonal identity. A projection algebra is an algebra (B, ϵ) for which every operation $\omega \in \epsilon$ is a projection.

Result:

- 1- The variety of semiprojection is modes.
- 2-if semiprojection with conduction property and idempotent is entropic algebras
- 3- Szendrei hemisemiprojection modes are left-zero algebras. This provides a new class of modes not embeddable into semimodules
- 4-every Szendrei mode is a semimodule over a commutative semiring and idempotent algebra, see (M. Stronkowski [16]).
- 5-every algebra (without constants) is a subreduct of a semimodule over a semiring
- 6-The following drawing illustrates the mediality of a ternary operation in (Q, ϵ_P)

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