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Clique Covering in Graphs

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ABSTRACT

In this paper, we introduce the concept of clique covering of a graph. Every vertex covering set, which contains all isolated vertices, is a clique covering set in a graph. We consider the effect of removing a vertex from the graphs on the clique covering number of the graph. We also define well clique covered graphs and prove some related result.

Introduction

Let G be a semigraph and S be a subset of $V(G)$ then S is said to be a vertex covering set of G, if $e \cap S \neq \phi$ for every edge e of G. For a given semigraph G there is uniquely associated graphs called adjacency graph $A(G)$. Note that S is a vertex covering set of G if every clique of $A(G)$ has non empty intersection with S.

We define the concept of a clique covering set in a graph and the clique covering number of a graph. In general, this number does not exceed the vertex covering number, if the graph has no isolated vertices. Here we consider the operations of removing a vertex from a graph. We prove that these operations will change the clique covering number of the graph. We also define upper clique covering number of a graph and prove some related results. Further, we consider well clique covered graphs.

For the concepts related to semigraphs, the reader can refer to E. Sampathkumar [2].

Preliminaries:

Definition 2.1: Clique

A maximal complete subgraph of a graph G is called a clique in the graph G.

Definition 2.2: Clique covering set

Let G be graph and S be a subset of $V(G)$. The set S is said to be a clique covering set if for every clique $K(G), K \cap S \neq \phi$.

Definition 2.3: Minimal clique covering set

A clique covering set S is said to be a minimal clique covering set, if for every vertex v in S, $S-\{v\}$ is not a clique covering set.

Definition 2.4: Minimum clique covering set

A clique covering set with minimum cardinality is called a minimum clique covering set. It is denoted as $\alpha_c - set$ of G.

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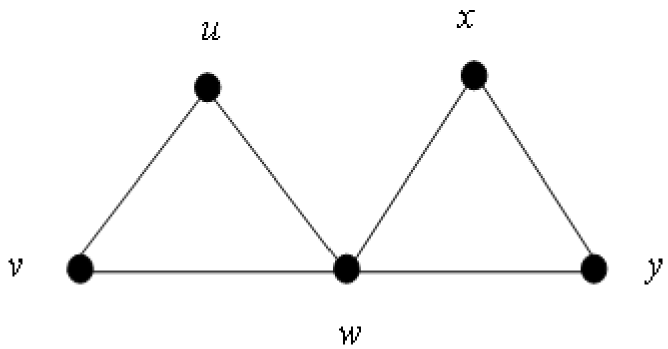
Definition 2.5: Clique covering number

The cardinality of a minimum clique covering set of K is called a clique covering number of the graph G and it is denoted as $\alpha_c(G)$.

Example 2.6:



Consider the graph whose vertex set $V(G)=\{u,v,w,x,y\}$ and edge set $E(G)=\{uv,uw,vw,xy,xw,yw\}$. In this graph $\{w\}$ is a minimum clique covering set. Also $\{w\}$ is not a vertex covering set of G.

If K is a clique in G then its vertex set will also be denoted as K. The symbol $K-v$ will denote the subgraph induced by the vertices of $V(K)-\{v\}$. Often $V(K)-\{v\}$ will also be denoted as $K-v$.

We make the following assumptions:

1) If v is an isolated vertex in G then $\{v\}$ is a clique in G.
2) If v is non isolated vertex in G and if K is a clique containing v then $K-v$ is a clique in $G-v$.

Minimum clique covering sets:

Now we consider the effect of removing a vertex v from the graph on the clique covering number of a graph.

Theorem 3.1: Let G be a graph and v is an isolate vertex of G then $\alpha_c(G-v)<\alpha_c(G)$.

Proof: Let S be a minimum clique covering set of G. Since v is and isolated vertex of G, $v \in S$.

Consider the set $S_1 = S-\{v\}$. Let K be a clique in $G-v$ then K is also clique in G. Hence $K \cap S \neq \phi$. Since $v \notin K$, $S_1 \cap K \neq \phi$. Thus, S_1 is a clique covering set of $G-v$. Therefore, $\alpha_c(G-v) \leq |S_1| < |S| = \alpha_c(G)$.

Hence, $\alpha_c(G-v)<\alpha_c(G)$. ∎

Now we state following theorem without proof.

Theorem 3.2: A clique covering set S of G is minimal if and only if for every vertex v in S there is a clique K of G such that $K \cap S = \{v\}$. ∎

Note that every vertex covering set of the graph G is clique covering set of G. However, converse is not true.

Theorem 3.3: Let G be a graph and $v \in V(G)$ such that v is not an isolated vertex in G then $\alpha_c(G-v) \geq \alpha_c(G)$.

Proof: Let S be a minimum clique in $G-v$. Let K be a clique of G. Let $K_1 = K-v$, then K_1 is a clique in $G-v$. Since S is a clique covering set in $G-v$, $K_1 \cap S \neq \phi$ and hence $K \cap S \neq \phi$.

Thus, S is clique covering set of G. Therefore, $\alpha_c(G) \leq |S| = \alpha_c(G-v)$.

Hence, $\alpha_c(G-v) \geq \alpha_c(G)$. ∎

Now we prove necessary and sufficient condition under which $\alpha_c(G-v)=\alpha_c(G)$.

Theorem 3.4: Let G be a graph and $v \in V(G)$, then $\alpha_c(G-v)=\alpha_c(G)$ if and only if there is a minimum clique covering set of G such that $v \notin S$.

Proof: First, suppose $\alpha_c(G-v)=\alpha_c(G)$.

Let S be a minimum clique covering set of $G-v$. Let K be a clique of G. If $v \notin K$ then K is a clique in $G-v$ and hence $K \cap S \neq \phi$.

If $v \in K$ then $K_1 = K - v$ is a clique in $G-v$ and hence $K_1 \cap S \neq \phi$ and therefore $K \cap S \neq \phi$.

Thus, S is a clique covering set of G. Since $\alpha_c(G-v)=\alpha_c(G)$, S is a minimum clique covering set G. Obviously $v \notin S$.

Conversely, suppose there is minimum clique covering set S of G such that $v \notin S$, then it can be easily prove that S is a clique covering set of $G-v$. This implies that $\alpha_c(G-v) \leq \alpha_c(G)$. It is already true that $\alpha_c(G) \leq \alpha_c(G-v)$.

Thus, $\alpha_c(G-v)=\alpha_c(G)$. ∎

Corollary 3.5: $\alpha_c(G) < \alpha_c(G-v)$ if and only if for every minimum clique covering set S of G, $v \in S$. ∎

Remark: If G is a graph and $S_1, S_2, \ldots, S_k$ are all the minimum clique covering set of G, then from the above corollary it is clear that $\bigcap S_i (i=1,2,\ldots,k) = \{v \in V(G) : \alpha_c(G-v) > \alpha_c(G)\}$.

Thus, $T = \{v \in V(G) : \alpha_c(G-v) > \alpha_c(G)\}$ then $|T| \leq \alpha_c(G)$.

Minimal clique covering sets with maximum cardinality:

Definition 4.1: Upper clique covering set

Let G be a graph. A minimal clique covering set S with maximum cardinality is called an upper clique covering set. It is also called $\alpha_B - set$ of G.

Definition 4.2: Upper clique covering number

The cardinality of a $\alpha_B - set$ is called the upper clique covering number of G and its denoted as $\alpha_B(G)$.

Note that $\alpha_c(G) \leq \alpha_B(G)$.

Next, we prove that the upper clique covering number of a graph does not increase when a vertex is removed from the graph.

Theorem 4.3: Let G be a graph and $v \in V(G)$ then $\alpha_B(G-v) \leq \alpha_B(G)$.

Proof: Let S be a $\alpha_B - set$ of $G-v$.

Case-I S is a clique covering set of G.

Let x be any vertex of S, then there is a clique K of $G-v$ such that $K \cap S = \{x\}$. Let K_1 be a clique of G such that $K \subset K_1$. Then $K_1 \cap S = \{x\}$ (because $v \notin S$). This proves that S is a minimal clique covering set of G. Therefore, $\alpha_B(G-v) = |S| \leq \alpha_B(G)$.

Hence, $\alpha_B(G-v) \leq \alpha_B(G)$.

<u>Case-II</u> S is not a clique covering set of $G-v$.

Then obviously $S \cup \{v\}$ is a minimal clique covering set of G. Hence $\alpha_B(G-v) < |S| \cup \{v\} \leq \alpha_B(G)$.

Hence, $\alpha_B(G-v) < \alpha_B(G)$. From cases above, $\alpha_B(G-v) \leq \alpha_B(G)$. ▌

Thus, we have this sequence of inequalities,

$\alpha_c(G) \leq \alpha_c(G-v) \leq \alpha_B(G-v) \leq \alpha_B(G)$.

Theorem 4.4: Let G be a graph and $v \in V(G)$, then $\alpha_B(G-v) = \alpha_B(G)$ if and only if every $\alpha_B - set$ of $G-v$ is an $\alpha_B - set$ in G.

Proof: Suppose $\alpha_B(G-v) = \alpha_B(G)$.

Let S be a $\alpha_B - set$ in $G-v$. If S is not $\alpha_B - set$ in G, then S cannot be a clique covering set of G. Hence $S \bigcup \{v\}$ is a minimal clique covering set of G. This implies that $\alpha_B(G-v) < \alpha_B(G)$, which contradicts the hypothesis. Thus, S must be $\alpha_B - set$ of G.

Converse is obvious.

Hence, the theorem is proved. ▌

Well clique covered graphs:

Here we consider those graphs for which all minimal clique covering sets have the same cardinality.

Definition 5.1: Well Clique covered graphs

Let G be a graph then G said to be a well clique covered graphs if $\alpha_c(G) = \alpha_B(G)$.

Equivalently a graph G is well clique covered if all minimal clique covering set have the same cardinality.

Theorem 5.2: Let G be a well clique covered graph and $v \in V(G)$ such that v is not an isolated vertex of G, then $G-v$ is also a well clique covered graph.

Proof: $\alpha_c(G) \leq \alpha_c(G-v) \leq \alpha_B(G-v) \leq \alpha_B(G)$.

Since $\alpha_c(G) = \alpha_B(G)$, $\alpha_c(G-v) = \alpha_B(G-v)$.

Hence, $G-v$ is well clique covered graph. ▌

Remark: If G is a graph which does not contain any triangle then obviously its vertex covering number is same as its clique covering number provided it has no isolated vertices. For such graphs, the vertex covering number may decrease when a vertex is removed from the graph [1]. Hence, the clique covering number may also decrease when a vertex is removed from the graph. Note that in such graphs edges are cliques and if $K = uv$ is an edge, then $K - v$ need not be a clique in $G-v$.

References:

[1] D. K. Thakkar and J. C. Bosamiya, *Vertex covering number of a graph,* Mathematics Today,Vol.27, June 2011, 30-35.

[2] E.Sampathkumar, *Semigraphs and their Applications,* Report on the DST (Department of Science and Technology) submitted to DST, India, May 2000.

[3] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Fundamentals of Domination in Graphs,* Marcel Dekker, Inc, (1998).

[4] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Domination in Graphs: Advanced Topics,* Marcel Dekker, Inc, (1998).