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On Unified Advection-Dispersion Problem and its Fourier Series Solution Involving Volterra Integral Equation

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ABSTRACT

In the present investigation, we introduce an unified space-time fractional advection-dispersion equation involving Caputo time fractional derivative of order β $(\beta > 0)$, Riesz-Feller space fractional derivatives of order γ $(0<\gamma<1)$ and asymmetry θ_1 $(|\theta_1| \le \min(\gamma, 1-\gamma))$ and of order α $(1<\alpha \le 2)$ and asymmetry θ_2 $(|\theta_2| \le \min(\alpha, 2-\alpha))$. Then, we consider a Fourier series to obtain its solution involving Volterra integral equation. We also evaluate its numerical approximation formula and discuss some of its particular cases.

1. Introduction

Regarding the linearity of the differential operators Kontecky [17] and Matsuda and Ayabe [22] studied the series solution of semi-differential equations (see also Oldham and Spanier [24, p.159]).

King et al. [15, p.123] described the Fourier series solution of ordinary one-dimensional diffusion equation for temperature distribution in the bar.

Özdemir et al. [25] obtained an analytic solution of fractional diffusion equation by applying Fourier series and also evaluated its numerical approximation formula.

Gorenflo, Luchko and Zabreiko [20] have solved the Cauchy problem and represented its series solution involving Mittag-Leffler function $E_{\alpha,\beta}(.)$defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \quad \left(z,\beta\in\mathbb{C};\Re(\alpha)>0\right) \tag{1.1}$$

where $\mathbb{C}$ is a set of complex numbers and $\Gamma(.)$ is the Gamma function (see Erdèlyi et al. [5] and Kilbas et al. [14]).

Many researchers such as Kilbas et al. [14], Oldham et al. [24], Podlubny [26], Samko et al. [27], and Mathai, Saxena and Haubold [19] presented a systematic study with analytical properties and applications of fractional derivatives, integrals and differential equations. Recently, Diethelm [3] has developed the theory and analysis of fractional differential equations involving Caputo type differential operators. Our work is concerning with the method developed by Diethelm [3] in the spaces of integrable, absolutely continuous and orthogonal functions.

Let $\Omega=[a,b],(-\infty\le a<b\le\infty)$ be a finite or infinite interval of the real axis $\mathbb{R}=(-\infty,\infty), L_p(a,b)(1\le p\le\infty)$ is the set of those Lebesgue complex-valued functions *f* on Ω for which $\|f\|_p<\infty$ where

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$$\|f\|_p = \left(\int_a^b |f(q)|^p\right)^{\frac{1}{p}} (1 \le p < \infty) \tag{1.2}$$

and $\|f\|_\infty = ess\sup_{a\le x\le b} |f(x)|$ = essential maximum of function $|f(x)|$. (1.3)

(see Nikol'ski [23, p.12-13]).

The weighted L^p-space with power of weight, denoted by $X_\beta^p(a,b)(\beta \in ¡ ,1 \le p < \infty)$, consists of those complex-valued Lebesgue measurable functions *f* on (a,b) for which

$$\|f\|_{X_\beta^p} < \infty \, with \|f\|_{X_\beta^p} = \left(\int_a^b \left|q^\beta f(q)\right|^p \frac{dq}{q}\right)^{\frac{1}{p}} (1 \le p < \infty) \tag{1.4}$$

and

$$\|f\|_{X_\beta^\infty} = ess\sup_{a\le x\le b} \left|x^\beta f(x)\right|, \text{ particularly } X_{\frac{1}{p}}^p(a,b) = L_p(a,b). \tag{1.5}$$

(See Kilbas et al. [14]).

For finite interval $[a,b](-\infty < a < b < \infty)$, $AC[a,b]$ be the space of functions f which are absolutely continuous on $[a,b]$ and $AC[a,b]$ coincides with space of primitives of Lebesgue summable functions

$$f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \phi(q)\,dq, \phi(q) \in L(a,b) \tag{1.6}$$

provided that $\phi(q) = f^{'}(q)\left(f^{'} = \frac{df}{dq}\right)$ almost everywhere on $[a,b]$ and on the triangle $(a \le q \le x \le b)$, and $c = f(a)$.

Again, for $n \in ¥, AC^n[a,b]$ be the space of complex-valued functions $f(x)$ which have continuous derivatives up to order *n*-1 on [*a*, *b*] such that $f^{n-1}(x) \in AC[a,b]$ almost everywhere on $[a,b]$ and defined by

$$AC^n[a,b] = \left\{f : [a,b] \to C \, and \left(D^{n-1}f\right)(x) \in AC[a,b]\right\}, D = \frac{d}{dx},$$

particularly, $AC^1[a,b] = AC[a,b]$. (1.7)

Lemma 1.1 The space $AC^n[a,b]$ consists of those and only these functions $f(x)$ which can be represented in the form

$$f(x) = \left(I_{a^+}^n \phi\right)(x) + \sum_{k=0}^{n-1} c_k (x-a)^k, \phi(q) = f^n(q), \phi(q) \in L(a,b), (a \le q \le x \le b) \tag{1.8}$$

where, $c_k = \frac{f^k(a)}{k!}, (k = 0,1,2,...,n-1)$ are arbitrary constants. (1.9)

Also $$\left(I_{a^+}^n \phi\right)(x) = \frac{1}{(n-1)} \int_a^x (x-q)^{n-1} \phi(q)\,dq \tag{1.10}$$

(see Samko et al. [27] and Diethelm [3]).

In our work we use the orthogonal property defined by

$$\int_{-\infty}^{\infty} e^{i(n-q)x} dx = 2\pi\delta(n-q), \delta(n-q) \, is \, a \, Dirac \, delta \;\; function \, \forall n,q \in ¡ \tag{1.11}$$

From Eqn. (1.11), particularly, we have

$$\int_{-\pi}^{\pi} e^{i(n-q)x}dx = \begin{cases} 2\pi, n = q, \forall n, q \in \mathbb{R} \\ 2\pi, n = q = 0 \\ 0, n \neq q \end{cases} \tag{1.12}$$

(see Bajpai [1] and Kumar [16]).

In our investigation, we also use following fractional derivatives:

The Caputo time fractional derivative ${}_0D_t^{\beta}$ *of order* $\beta, m-1 < \beta < m$ *and* $m = [\beta]+1$, $[\beta]$ is greatest integer not less than β, is defined by

$${}_0D_t^{\beta} f(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{d^m f(\xi)}{d\xi^m} (t-\xi)^{m-\beta-1} d\xi \tag{1.13}$$

(see Sousa [28]).

The Riesz-Feller space fractional derivative ${}_xD_{\theta}^{\alpha}$ of order α and asymmetry θ is defined by (see Feller [6], Gorenflo and Mainardi [10-11])

$${}_xD_{\theta}^{\alpha} = -{}_xI_{\theta}^{-\alpha} = -\left[c_+(\alpha,\theta)\, {}_xI_+^{\alpha} + c_-(\alpha,\theta)\, {}_xI_-^{\alpha}\right], \alpha \neq 1, 0 < \alpha \leq 2, |\theta| \leq \min(\alpha, 2-\alpha), x \in \mathbb{R} ;$$

$$c_+(\alpha,\theta) = \frac{\sin\left(\frac{(\alpha-\theta)\pi}{2}\right)}{\sin(\alpha\pi)}, \text{ and } c_-(\alpha,\theta) = \frac{\sin\left(\frac{(\alpha+\theta)\pi}{2}\right)}{\sin(\alpha\pi)} \tag{1.14}$$

${}_xI_+^{-\alpha}$ *and* ${}_xI_-^{-\alpha}$ are the inverse of the Riemann-Liouville integrals of ${}_xI_+^{\alpha}$ *and* ${}_xI_-^{\alpha}$ respectively given by (see Miller and Ross [22])

$${}_xI_+^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1} f(\xi) d\xi,$$

$$\text{and} \tag{1.15}$$

$${}_xI_-^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi-x)^{\alpha-1} f(\xi) d\xi, \alpha > 0, x \in \mathbb{R} ,$$

The Riemann-Liouville fractional derivatives ${}_xD_{\pm}^{\gamma}$ of order γ for $m = [\gamma]+1$ are defined as (see Samko et al. [27])

$${}_xD_+^{\gamma} f(x) = \frac{1}{\Gamma(m-\gamma)} \left(\frac{d}{dx}\right)^m \int_{-\infty}^{x} (x-\xi)^{m-\gamma-1} f(\xi) d\xi, x < \infty \tag{1.16}$$

and $${}_xD_-^{\gamma} f(x) = \frac{(-1)^m}{\Gamma(m-\gamma)} \left(\frac{d}{dx}\right)^m \int_{x}^{\infty} (\xi-x)^{m-\gamma-1} f(\xi) d\xi, x > -\infty \tag{1.17}$$

Gorenflo and Mainardi [10-11] presented a generalized diffusion equation and solved it through stochastic processes from probabilistic stand point. Further Liu et al. [17] made its extension through considering the Lévy-Feller advection-dispersion equation in the form

$$\frac{\partial U(x,t)}{\partial t} = a\, {}_xD_{\theta}^{\alpha} U(x,t) - b \frac{\partial U(x,t)}{\partial x} \tag{1.18}$$

with initial condition

$$U(x,0) = f(x), x \in \mathbb{R} , t > 0, a > 0, b \geq 0. \tag{1.19}$$

They [17] solved these Eqns. (1.18)-(1.19) through stochastic processes and obtained its numerical solution with the help of an explicit finite difference approximation formula.

Mathai, Saxena and Haubold [19] have derived an explicit solution of the fractional differential equation

$$ {}_0D_t^{\beta}U(x,t)=\eta\, {}_xD_{\theta}^{\alpha}U(x,t)+\phi(x,t), \eta, t>0, x\in\mathbb{R}; \alpha,\theta,\beta\ real\ parameters \tag{1.20}$$

with the constraints $0<\alpha\le 2,\ |\theta|\le\min(\alpha,2-\alpha),\ 0<\beta\le 2,$ (1.21)

also, with the initial conditions

$$U(x,0)=f(x), U_t(x,0)=g(x),\ \forall\ x\in\mathbb{R}, \left[\ U_t(x,0)=\frac{\partial U(x,t)}{\partial t}\ \text{at}\ t=0\right], \tag{1.22}$$
$$\lim_{x\to\pm\infty}U(x,t)=0,\ t>0.$$

Diethelm [3, Eqn. (7.9a, b), p.143], analyzed the following fractional differential equation

$$ {}_0D_x^{\beta}U(x)=f(x)U(x)+g(x) \tag{1.23}$$

subject to the initial condition

$$U^{(k)}(0)=U_0^{(k)}, \left[U^{(k)}(0)=\frac{d^kU(x)}{dx^k}\ \ \text{at}\ t=0\right](k=0,1,....,[\beta]-1) \tag{1.24}$$

and found its solution involving Volterra integral equation.

Motivated by above work, we present an unified advection-dispersion equation containing Caputo time fractional derivative of order $\beta,(\beta>0)$, Riesz-Feller space fractional derivatives of order $\alpha,(0<\alpha<1)$ and asymmetry $\theta_1, |\theta_1|\le\min(\alpha,\ 1-\alpha)$ and of order $\gamma,(1<\gamma\le 2)$ and asymmetry $\theta_2,\ |\theta_2|\le\min(\gamma,\ 2-\gamma)$ and then solve it by introducing a Fourier series to obtain its solution consisting Volterra integral equation. A numerical approximation formula is also derived and then analyzed it through computation by using certain hypergeometric approximation results.

2. An Unified Advection-Dispersion Equation and Analysis

In this section, we consider an unified advection-dispersion equation

$$ {}_0D_t^{\beta}U(x,t)=\mu\, {}_xD_{\theta_1}^{\alpha}U(x,t)+\nu\psi(t)\, {}_xD_{\theta_2}^{\gamma}U(x,t)+\phi(x,t) \tag{2.1}$$

Here $\nu,t\in\mathbb{R}_+$ and $\mu\in\mathbb{R}_+\cup\{0\}$ ($\mathbb{R}_+$ is the set of positive real numbers) $\alpha,\beta,\gamma\ and\ \theta$ are real parameters with the constraints; $0<\alpha<1, |\theta_1|\le\min(\alpha,\ 1-\alpha), \beta>0, 1<\gamma\le 2, |\theta_2|\le\min(\gamma,\ 2-\gamma)$, the initial conditions are:

$U^{(k)}(x,0)$ are continuous functions of $x,(\forall x\in\mathbb{R})$ only, where $(k=0,1,...,m-1),\ m=[\beta],$

and $\lim_{x\to\pm\infty}U(x,t)=0,\ t>0.$

Here, we define the sets: $G^*=\{(t):t\in\mathbb{R}_+\}$ and $G=\{(x,t):(x,t)\in\mathbb{R}\times\mathbb{R}_+,\ x\in\mathbb{R},\ t\in\mathbb{R}_+\}$, the functions $\psi\ and\ \phi$ are such that $\psi:G^*\to\mathbb{R}_+$ and $\phi:G\to\mathbb{R}$. The $\nu\psi(t)$ is the diffusion coefficient which has the limit

$$\lim_{t\to 0}\nu\psi(t)=\nu,\ \nu>0. \tag{2.2}$$

In order to solve the above problem (2.1)-(2.2), we first present the following theorems:

Theorem 2.1 Let $\beta>0$. $U^{(k)}(x,0)\ \left(x\in\mathbb{R}; k=0,1,...,m-1; m=[\beta]\right)$ are continuous functions and are $L_1(\mathbb{R})$. Again for the sets G^* and G, the functions ψ and ϕ defined in Eqn. (2.2), also under the conditions given in Eqn. (2.2), the advection-dispersion Eqn. (2.1) has the solution $U(x,t)=\sum_{n=-\infty}^{\infty}U_n(t)e^{inx}$ whose Fourier inversion is $U_n(t),\ (t>0)(n=-\infty,...,-2,-1,0,1,2,...,\infty)$ and consists the Volterra integral equation

$U_n(t)=\sum_{k=0}^{m-1}U_{n_0}^{(k)}\frac{t^k}{k!}+\frac{1}{\Gamma(\beta)}\int_0^t(t-q)^{\beta-1}H_n(q)dq+\frac{1}{\Gamma(\beta)}\int_0^t(t-q)^{\beta-1}(\nu_n^*\psi(q)+\mu_n^*)U_n(q)dq;\ H_n(q)$ be $L_1[0,T]$ on triangle $\{(t,q):0\le q<t\le T,\ T>0\},\ H_n(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-inx}\phi(x,t)dx,(t>0),\ U_{n_0}^{(k)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-inx}U^{(k)}(x,0)dx,$

$(U_{n_0}^{(k)}\in\mathbb{R})$ for all $(n=-\infty,....,-2,-1,0,1,2,....,\infty)$ and $(k=0,1,2,...m-1;\ m=[\beta])$. (2.3)

Here the constraints are $\mu_n^*=\mu n^{\alpha}\{c_+(\alpha,\theta_1)(i)^{\alpha}+c_-(\alpha,\theta_1)(-i)^{\alpha}\},0<\alpha<1,\quad |\theta_1|\le\min(\alpha,\ 1-\alpha)$, and

$\nu_n^*=\nu n^{\gamma}\{c_+(\gamma,\theta_2)(i)^{\gamma-2}+c_-(\gamma,\theta_2)(-i)^{\gamma-2}\},\ 1<\gamma\le 2,\ |\theta_2|\le\min(\gamma,\ 2-\gamma).$ (2.4)

Proof: Consider the solution of Eqns. (2.1)-(2.2) in the form $U(x,t)=\sum_{n=-\infty}^{\infty}U_n(t)e^{inx}$ (2.5)

Then, the Fourier inversion formula of $U(x,t)$ is given by

$$U_n(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-inx}U(x,t)dx,\ (\forall\ n=-\infty,....,-2,-1,0,1,2,....,\infty),\ t>0. \tag{2.6}$$

Now, make an appeal to Eqns. (2.1), (2.2), (2.4), (2.5) and (2.6) and then using orthogonal property (1.11), we get the fractional differential equation identical to Eqns. (1.23)-(1.24) due to Diethelm [3, p.143]

$${}_0D_t^{\beta}U_n(t)=(\nu_n^*\psi(t)+\mu_n^*)U_n(t)+H_n(t),\ \beta>0 \tag{2.7}$$

with the initial conditions:

$$U_n^{(k)}(0)=U_{n_0}^{(k)}\quad(\forall n=-\infty,....,-2,-1,0,1,2,....,\infty),\ (k=0,1,2,....,m-1;m=[\beta]). \tag{2.8}$$

Again, make an appeal to the theorem due to Diethelm [3, theorem 7.9, p. 143] in Eqns. (2.7)- (2.8), on the triangle $\{(t,q):0\le q<t\le T,\ T>0\}$, we get the solution in form of Volterra integral equation (2.3) given by

$$U_n(t)=\sum_{k=0}^{m-1}U_{n_0}^{(k)}\frac{t^k}{k!}+\frac{1}{\Gamma(\beta)}\int_0^t(t-q)^{\beta-1}H_n(q)dq+\frac{1}{\Gamma(\beta)}\int_0^t(t-q)^{\beta-1}(\nu_n^*\psi(q)+\mu_n^*)U_n(q)dq;$$

$$(\forall\ n=-\infty,....,-2,-1,0,1,2,....,\infty),\ t>0. \tag{2.9}$$

Lemma 2.1 Under the conditions of the Eqn. (2.2), the fractional differential Eqn. (2.1) has the solution in which on the triangle $\{(t,q):0\le q<t\le T,\ T>0\}$ and $(\forall n\in\{-\infty,....,-2,-1,0,1,2,....,\infty\},$

$$K_n(t,q)=\frac{1}{\Gamma(\beta)}\left[(t-q)^{\beta-1}(\mu_n^*\psi(t)+\nu_n^*)\right],\ \|\psi\|_{\infty}=ess\ \sup_{0\le q\le r\le t\le T}|\psi(t)|, \tag{2.10}$$

Also, here $\psi(q)$ and $H_n(q)\in L_1(0,T)$, then

(a) on above triangle the resolvent kernel $R_n(t,q)$ has the inequality

$$\left|R_n(t,q)\right| \le \left(\left|\nu_n^*\right|.\left\|\psi\right\|_\infty + \left|\mu_n^*\right|\right)(t-q)^{\beta-1} E_{\beta,\beta}\left(\left(\left|\nu_n^*\right|.\left\|\psi\right\|_\infty + \left|\mu_n^*\right|\right)(t-q)^{\beta-1}\right) \tag{2.11}$$

where, $E_{\beta,\beta}(.)$ is the generalized Mittag-Leffler function defined in Eqn. (1.1).

(b) The approximation function $\phi_{n,j}(t),\ \forall n \in \{-\infty,....,-2,-1,0,1,2,....,\infty\},\ j \in \{0,1,2,...\}, t>0$

has the equality $\phi_{n,j}(t) = \int_0^t K_n(t,q)\phi_{n,j-1}(q)\,dq \quad \forall\, j \in \{0,1,2,.....\}$ (2.12a)

and the inequality

$$\phi_{n,j}(t) \le \left(\left|\nu_n^*\right|.\left\|\psi\right\|_\infty + \left|\mu_n^*\right|\right)^j \left[\sum_{k=0}^{m-1} \frac{U_{n_0}^{(k)} t^{\beta j+k}}{\Gamma(\beta j+k+1)} + \frac{1}{\Gamma(\beta j+\beta)}\int_0^t (t-r)^{\beta j+\beta-1} H_n(r)\,dr\right] \tag{2.12 b}$$

Proof (a): On the set $\{(t,q): 0 \le q < t \le T,\ T>0\}$, we define the iterated kernel

$K_{n,1}(t,q) = K_n(t,q)\ \ (\forall\ n \in \{-\infty,....,-2,-1,0,1,2,....,\infty\})$

and $K_{n,j}(t,q) = \int_q^t K_n(t,r) K_{n,j-r}(r,q)\,dr, (j=2,3,..)$. (2.13)

Then on using Eqns. (2.10) and (2.13), particularly, we have

$$\left|K_{n,2}(t,q)\right| \le \frac{\left(\left|\nu_n^*\right|.\left\|\psi\right\|_\infty + \left|\mu_n^*\right|\right)^2}{\Gamma(2\beta)}(t-q)^{2\beta-1} \tag{2.14}$$

and $$\left|K_{n,3}(t,q)\right| \le \frac{\left(\left|\nu_n^*\right|.\left\|\psi\right\|_\infty + \left|\mu_n^*\right|\right)^3}{\Gamma(3\beta)}(t-q)^{3\beta-1} \tag{2.15}$$

By mathematical induction, Eqn. (2.15) gives us

$$\left|K_{n,j}(t,q)\right| \le \frac{\left(\left|\nu_n^*\right|.\left\|\psi\right\|_\infty + \left|\mu_n^*\right|\right)^j}{\Gamma(\beta j)}(t-q)^{\beta j-1}, (j=2,3,....)\,\bullet \tag{2.16}$$

Now use the resolvent kernel defined by $R_n(t,q) = \sum_{j=1}^{\infty} K_{n,j}(t,q)$ (c.f. Diethelm [p.144]) in Eqn. (2.16), we get the required inequality $\left|R_n(t,q)\right| \le \left(\left|\nu_n^*\right|.\left\|\psi\right\|_\infty + \left|\mu_n^*\right|\right)(t-q)^{\beta-1} E_{\beta,\beta}(\left(\left|\nu_n^*\right|.\left\|\psi\right\|_\infty + \left|\mu_n^*\right|\right)(t-q)^{\beta-1})$. (2.17)

Here in inequality (2.17), the resolvent kernel $R_n(t,q)$ has the entire function $E_{\beta,\beta}(.)\left(of\ order\,\frac{1}{\beta}\right)$ and hence, $R_n(t,q)$ is continuous on the set $\{(t,q): 0 \le q < t \le T,\ T>0\}$. Again $R_n(t,q)$ is the infinite series consisting the terms $K_{n,j}(t,q)$ $(\forall j = 1,2,....)$ and thus $K_{n,j}(t,q)$ is also continuous on the set $\{(t,q): 0 \le q < t \le T,\ T>0\}$.

Proof (b): From Eqn. (2.12b), we find that

$$\phi_{n,j-1}(t) \le \left(\left|\nu_n^*\right|.\left\|\psi\right\|_\infty + \left|\mu_n^*\right|\right)^{j-1} \left[\sum_{k=0}^{m-1} \frac{U_{n_0}^{(k)} t^{\beta j+k-\beta}}{\Gamma(\beta j+k-\beta+1)} + \frac{1}{\Gamma(\beta j)}\int_0^t (t-r)^{\beta j-1} H_n(r)\,dr\right], (j=1,2,....) \tag{2.18}$$

Again, suppose that $\phi_{n,0}(t) = M_n(t) = \sum_{k=0}^{m-1} \frac{U_{n_0}^{(k)} t^k}{\Gamma(k+1)} + \frac{1}{\Gamma(\beta)} \int_0^t (t-r)^{\beta j-1} H_n(r)\,dr$ (2.19)

Now, making an appeal to the Eqns. (2.10), (2.12a), (2.18) and (2.19), we find

$$\phi_{n,j}(t) \le \int_0^t \frac{1}{\Gamma(\beta)} \left\{ (t-q)^{\beta-1} \left(|\nu_n^*|.\|\psi\|_\infty + |\mu_n^*| \right) \right\} \left(|\nu_n^*|.\|\psi\|_\infty + |\mu_n^*| \right)^{j-1}$$
$$\times \left[\sum_{k=0}^{m-1} \frac{U_{n_0}^{(k)} q^{\beta j + k - \beta}}{\Gamma(\beta j + k - \beta + 1)} + \frac{1}{\Gamma(\beta j)} \int_0^q (q-r)^{\beta j-1} H_n(r)\,dr \right] dq \tag{2.20}$$

From (2.20), we get

$$\phi_{n,j}(t) \le \left(|\nu_n^*|.\|\psi\|_\infty + |\mu_n^*| \right)^{j}$$
$$\times \left[\sum_{k=0}^{m-1} \frac{U_{n_0}^{(k)}}{\Gamma(\beta)\Gamma(\beta j + k - \beta + 1)} \int_0^t (t-q)^{\beta-1} q^{\beta j + k - \beta} dq + \frac{1}{\Gamma(\beta)\Gamma(\beta j)} \int_0^t (t-q)^{\beta-1} \int_0^q (q-r)^{\beta j-1} H_n(r)\,dr\,dq \right] \tag{2.21}$$

On changing the order of integration in Eqn. (2.21), we get

$$\phi_{n,j}(t) \le \left(|\nu_n^*|.\|\psi\|_\infty + |\mu_n^*| \right)^{j}$$
$$\times \left[\sum_{k=0}^{m-1} \frac{U_{n_0}^{(k)}}{\Gamma(\beta)\Gamma(\beta j + k - \beta + 1)} \int_0^t (t-q)^{\beta-1} q^{\beta j + k - \beta} dq + \frac{1}{\Gamma(\beta)\Gamma(\beta j)} \int_0^t H_n(r) \int_r^t (t-q)^{\beta-1} (q-r)^{\beta j-1} dq\,dr \right] \tag{2.22}$$

which, in view of the definition of the beta integral $\int_r^t (t-q)^{\beta-1} (q-r)^{\beta j-1} dq = (t-r)^{\beta j + \beta - 1} \frac{\Gamma(\beta)\Gamma(\beta j)}{\Gamma(\beta + \beta j)}$ gives the required result (2.12).

Theorem 2.2: The fractional differential equation given in Eqn. (2.1), under the given conditions in Eqn. (2.2) has the solution $U(x,t)$ given in Eqn. (2.5) whose Fourier inversion formula is equivalent to $U_n(t),\ t>0,$ $(\forall n \in \{-\infty,....,-2,-1,0,1,2,....,\infty\})$ (see Eqn. (2.6)) and then for $m=[\beta],$ there exists

$$U_n(t) = \sum_{k=0}^{m-1} U_{n_0}^{(k)} t^k E_{\beta,k+1}\left(\left(\nu_n^* \psi(t) + \mu_n^* \right) t^\beta \right)$$
$$+ \int_0^t (t-r)^{\beta-1} E_{\beta,\beta}\left(\left(\nu_n^* \psi(t) + \mu_n^* \right)(t-r)^\beta \right) H_n(r)\,dr \tag{2.23}$$

Proof: From the theory of approximation method of Volterra integral equation and in consequence of lemma 2.1, we get the solution of fractional differential Eqn. (2.1) under the given condition (2.2) in the form of $U_n(t) = \sum_{j=0}^{\infty} \phi_{n,j}(t).$ (2.24)

Then, making an appeal to the Eqns. (2.22) and (2.24), we obtain the required Eqn. (2.23).

3. Special Cases

Set $\beta \to 1, \mu = -b, \alpha \to 1, \theta_1 = 0, \nu = a, \psi = 1, \phi = 0$ and $U(x,0) = f(x)$ in Eqns. (2.1)-(2.2). Then our unified advection-dispersion equation becomes Lévy -Feller advection-dispersion equation due to Liu et al. [17] given in Eqns. (1.18)-(1.19).

Again if we set $\mu=0,\ 0<\beta\le 2$ and $0<\gamma\le 2,\ \psi=1,\ U(x,0)=f(x)$ and $U_t(x,0)=g(x)$ in Eqns. (2.1)-(2.2), then our unified advection-dispersion equation becomes fractional differential equation of Mathai, Saxena and Haubold [19] given in the Eqns. (1.20)-(1.22).

Further set $\mu=0,\ \psi=1,\ \phi=0,\theta_2=\gamma;\gamma>0, and\, 1<\beta<l;\ l\in ¥$ in Eqn. (2.1).Then it becomes the equation due to Kilbas, Srivastava and Trujillo [14, p.380 Eqn. (6.4.1)] and its solution is equivalent to

$$U(x,t)=\sum_{k=0}^{m-1}\sum_{n=-\infty}^{\infty}U_{n_0}^{(k)}(t)^k E_{\beta,k+1}(\nu_n^* t^\beta)e^{inx} \tag{3.1}$$

Again, set $\psi=1,\ U(x,t)=\sum_{n=-\infty}^{\infty}U_n(t)e^{inx}$ in our unified Equation (2.1), then it becomes the equation of Kilbas, Srivastava and Trujillo [14, p. 323 Eq. (5.3.69)] whose solution is $U(x,t)=\sum_{n=-\infty}^{\infty}U_n(t)e^{inx}$ (3.2)

where $U_n(t)=\sum_{k=0}^{m-1}U_{n_0}^{(k)}(t)^k E_{\beta,k+1}\left(\left(\nu_n^*+\mu_n^*\right)t^\beta\right)+\int_0^t (t-r)^{\beta-1}E_{\beta,\beta}\left(\left(\nu_n^*+\mu_n^*\right)(t-r)^\beta\right)H_n(r)dr,$ and $H_n(r)$

(see also Eqn. (2.4)). (3.3)

Several other fractional differential equations may be found after making some manipulations in our Eqns. (2.1)-(2.6), see for example Gorenflo and Mainardi [9], Gorenflo and Rutman [13], Gorenflo, Mainardi and Srivastava [17], Luchko and Gorenflo [18], Diethelm [2], Gorenflo and Mainardi [8] and Diethelm and Ford [4].

4. G L Approximation for Numerical Solution

To obtain the approximation formula for numerical solution, we define mesh points

$t_j=t_0+jh,\ j=0,1,2,....,N,\ t_0\le t\le T$,where h denotes the uniform time steps. (4.1)

The shifted Grünwald-Letnikov formula is $D_{GL,S}^{\beta,h}U(t_j)=\frac{1}{h^\beta}\sum_{k=0}^{j}\omega_k^{(\beta)}U(t_{j+1-k})$ (4.2)

$$\text{where, } \omega_k^{(\beta)}=(-1)^k\binom{\beta}{k}=\frac{(-1)^k\beta(\beta-1)......(\beta-k+1)}{k!} \tag{4.3}$$

$$or\quad \omega_k^{(\beta)}=\frac{\Gamma(k-\beta)}{\Gamma(-\beta)\Gamma(k+1)} \tag{4.4}$$

Particularly, from eqns. (4.3) or (4.4) we find the recurrence relation

$$\omega_0^{(\beta)}=1;\ \omega_k^{(\beta)}=\left(1-\frac{\beta+1}{k}\right)\omega_{k-1}^{(\beta)},k=1,2,3,.... \tag{4.5}$$

Again, let the function $U(t),$ is $m-1$ times differentiable in $[t_0,T]$ and that the $m-th$ derivative of $U(t)$ is integrable in $[t_0,T]$. Then, for every $m-1<\beta<m,$ there exists $D_{GL,S}^{\beta,h}U(t_j)=D_{RL}^{\beta}U(t),\ t_0\le t\le T$ (4.6)

where $D_{RL}^{\beta}U(t)$ is the Riemann-Liouville fractional derivative of order β of in $[t_0,T]$. Further, for every $m-1<\beta<m$ and for $t_0\le t\le T,$ the relation between Caputo fractional derivative ${}_0D_t^{\beta}(of\ order\ \beta)$ and the Riemann-Liouville fractional derivative $D_{RL}^{\beta}(of\ order\ \beta)$ is given by ${}_0D_t^{\beta}U(t)=D_{RL}^{\beta}U(t)-\sum_{k=0}^{m-1}\frac{d^kU}{dt^k}(t_0)\frac{(t-t_0)^{-\beta+k}}{\Gamma(-\beta+k+1)}.$ (4.7)

Therefore, making an appeal to Eqns. (4.2)-(4.7) in Eqn. (2.7), for $t_0 = 0,$ and for the real value of $\mu_n^* \, and \, \nu_n^*$, we obtain the approximation formula

$$U_n(t) = \frac{1}{\left\{\frac{1}{h^\beta} - \left(\left|\mu_n^*\right| + \left|\nu_n^*\right| \psi(t)\right)\right\}} \times \left[H_n(t) + \sum_{k=0}^{m-1} \frac{d^k U_n(0)}{dt^k} \frac{t^{-\beta+k}}{\Gamma(-\beta+k+1)} - \frac{1}{h^\beta} \sum_{k=1}^{M} (-1)^k \binom{\beta}{k} U_n(hM - hk) \right] \quad (4.8)$$

where $t = \frac{M}{h}, \ \beta > 0, also \left|\mu_n^*\right| and \left|\nu_n^*\right|$ may be found by Eqn. (2.4).

Now, for $0 < \beta £ \ 2, \ 1 < \gamma £ \ 2, \ 0 < \alpha < 1,$ the above approximation formula (4.8) becomes

$$U_n(t) = \frac{1}{\left\{\frac{1}{h^\beta} - \left(\left|\mu_n^*\right| + \left|\nu_n^*\right| \psi(t)\right)\right\}} \times \left[H_n(t) + \frac{U_{n_0} t^{-\beta}}{\Gamma(1-\beta)} + U_{n_0}^{(1)} \frac{(t)^{-\beta+1}}{\Gamma(2-\beta)} - \frac{1}{h^\beta} \sum_{k=1}^{2} (-1)^k \binom{\beta}{k} U_n(hM - hk) \right] \quad (4.9)$$

where $M = \frac{t}{h}, \left|\mu_n^*\right| and \left|\nu_n^*\right|$ may be found by Eqn. (2.4).

Example: Consider

$$\phi(x,t) = \frac{2t}{t^2 + x^2}, \psi(t) = (1+t)^2, \ \alpha = 0.5, \theta_1 = 0.5, \gamma = 1.5, \beta = 1.5, h = \frac{1}{10},$$
$$\mu = \frac{1}{4}, \nu = \frac{1}{5}, U(x,0) = \frac{2}{1+x^2}, U_t(x,0) = \frac{4}{4+x^2} \quad (4.10)$$

Then, make an appeal to Eqns. (1.14), (2.4) and (4.10) to get

$$H_n(t) = e^{-t|n|}, U_{n_0} = e^{-|n|}, U_{n_0}^{(1)} = e^{-2|n|}, \ \left|\mu_n^*\right| = \frac{\sqrt{|n|}}{4}, \left|\nu_n^*\right| = \frac{|n|\sqrt{|n|}}{5}, \left|\theta_1\right| = 0.5 \, and \left|\theta_2\right| = 0.5. \quad (4.11)$$

Again, using the Eqns. (4.9), (4.10) and (4.11), we get following computational formulae (4.12) and (4.13) and their graphs (By Wolfram *Mathematica* 7)

$$U_n(t) = \frac{1}{\left\{(10)^{1.5} - \frac{\sqrt{|n|}}{4} - \frac{|n|\sqrt{|n|}}{5}(1+t)^2\right\}} \times \left[e^{-|n|t} + \frac{e^{-|n|} t^{-1.5}}{\Gamma(-0.5)} + \frac{e^{-2|n|} t^{-0.5}}{\Gamma(0.5)} + 1.5(10)^{0.5} e^{-2|n|} + 1.125(10)^{1.5} e^{-|n|} \right], \quad (4.12)$$

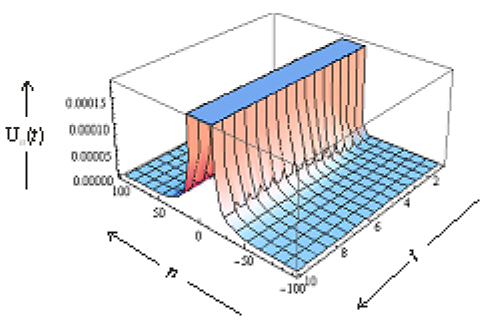
$t > 0, \ n \in I$ (the set of Integers).

$$U(x,t)=\sum_{n=-\infty}^{\infty}\cos nx\frac{1}{\left\{(10)^{1.5}-\frac{\sqrt{|n|}}{4}-\frac{|n|\sqrt{|n|}}{5}(1+t)^{2}\right\}} \tag{4.13}$$

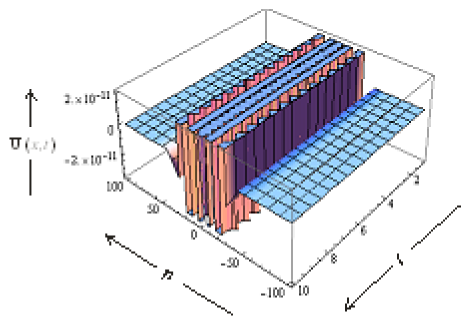
$$\times\left[e^{-|n|t}+\frac{e^{-|n|}t^{-1.5}}{\Gamma(-0.5)}+\frac{e^{-2|n|}t^{-0.5}}{\Gamma(0.5)}+1.5(10)^{0.5}e^{-2|n|}+1.125(10)^{1.5}e^{-|n|}\right],$$

$t>0,\ x\in\mathbb{R}$ (the set of real numbers).

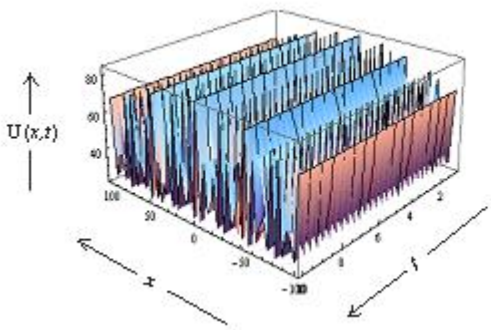
(I)



(II)



(III)



Further on using the Eqns. (2.5) and (2.23), we get the following results

$$U_n(t)=\left[\sum_{k=0}^{\infty}\left(\frac{|n|\sqrt{|n|}}{5}(1+t)^{2}+\frac{\sqrt{|n|}}{4}\right)^{k}\left\{\frac{e^{-|n|}t^{1.5k}}{\Gamma(1.5k+1)}+\frac{e^{-2|n|}t^{1.5k}}{\Gamma(1.5k+2)}+\frac{1}{\Gamma(1.5k+1.5)}\int_0^t(t-r)^{1.5k+0.5}e^{-|n|r}dr\right\}\right]$$

$$=\left[\sum_{k=0}^{\infty}\left(\frac{|n|\sqrt{|n|}}{5}(1+t)^{2}+\frac{\sqrt{|n|}}{4}\right)^{k}\left\{\frac{e^{-|n|}t^{1.5k}}{\Gamma(1.5k+1)}+\frac{e^{-2|n|}t^{1.5k}}{\Gamma(1.5k+2)}+\frac{\Gamma(1.5k+1.5)(t)^{1.5k+1.5}}{\Gamma(1.5k+2.5)}{}_1F_1(1;\ 1.5k+2.5;\ -|n|t)\right\}\right],\ t>0,\ n\in I \tag{4.14}$$

and

$$U(x,t)=\sum_{n=-\infty}^{\infty}\cos nx\sum_{k=0}^{\infty}\left(\frac{|n|\sqrt{|n|}}{5}(1+t)^{2}+\frac{\sqrt{|n|}}{4}\right)^{k}$$

$$\times\left\{\frac{e^{-|n|}t^{1.5k}}{\Gamma(1.5k+1)}+\frac{e^{-2|n|}t^{1.5k}}{\Gamma(1.5k+2)}+\frac{1}{\Gamma(1.5k+1.5)}\int_0^t(t-r)^{1.5k+0.5}e^{-|n|r}dr\right\}$$

$$=\left[\sum_{k=0}^{\infty}\left(\frac{|n|\sqrt{|n|}}{5}(1+t)^{2}+\frac{\sqrt{|n|}}{4}\right)^{k}\right.$$
$$\left.\times\left\{\frac{e^{-|n|}t^{1.5k}}{\Gamma(1.5k+1)}+\frac{e^{-2|n|}t^{1.5k}}{\Gamma(1.5k+2)}+\frac{\Gamma(1.5k+1.5)(t)^{1.5k+1.5}}{\Gamma(1.5k+2.5)}\,{}_{1}F_{1}\left(1;\ 1.5k+2.5;\ -|n|t\right)\right\}\right], \tag{4.15}$$

$t>0,\ x\in\mathbb{R}$.

.

Now, in Eqns. (4.14) and (4.15), the series involving integral is not valid for computation of structures as $t>0,$ therefore we convert it in the series consisting Kummer hypergeometric function and thus on using the approximate value of ${}_{1}F_{1}(a;c;-z)=\frac{\Gamma(c)}{\Gamma(c-a)}(z)^{-a}, z\to\infty,$ and c,a are bounded (see Srivastava and Manocha [29, p. 38]) at time t large (as $t\to\infty$), , we get the following approximation formulae (4.16) and (4.17) and their structures respectively

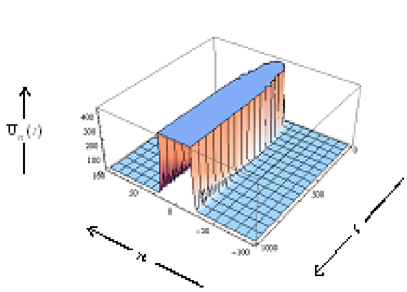
$$U_{n}(t)=\left[\sum_{k=0}^{\infty}\left(\frac{|n|\sqrt{|n|}}{5}(t)^{1.5}(1+t)^{2}+(t)^{1.5}\frac{\sqrt{|n|}}{4}\right)^{k}\left\{\frac{e^{-|n|}}{\Gamma(1.5k+1)}+\frac{e^{-2|n|}}{\Gamma(1.5k+2)}+\frac{\sqrt{t}}{|n|\Gamma(1.5k+1.5)}\right\}\right], \tag{4.16}$$

where $t>0,\ n\in I.$

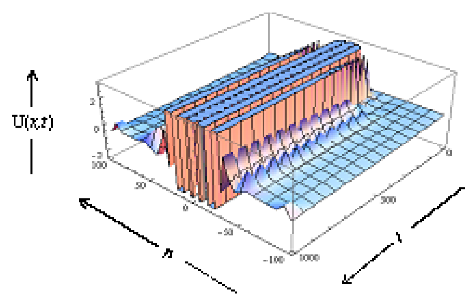
$$U(x,t)=\sum_{n=-\infty}^{\infty}\cos nx\left[\sum_{k=0}^{\infty}\left(\frac{|n|\sqrt{|n|}}{5}(t)^{1.5}(1+t)^{2}+(t)^{1.5}\frac{\sqrt{|n|}}{4}\right)^{k}\left\{\frac{e^{-|n|}}{\Gamma(1.5k+1)}+\frac{e^{-2|n|}}{\Gamma(1.5k+2)}+\frac{\sqrt{t}}{|n|\Gamma(1.5k+1.5)}\right\}\right],$$

$t>0,\ x\in\mathbb{R}$. (4.17)

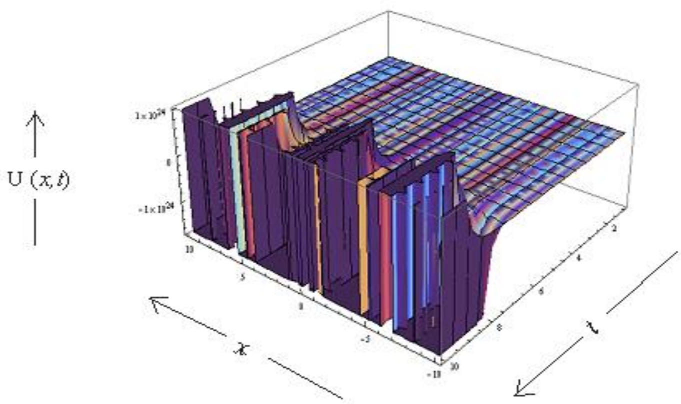
(IV)



(V)



(VI)



Again, on using the approximate value of ${}_1F_1(a;c;-z) = e^{-a}, c \to \infty$, and a, z are bounded (see Srivastava and Manocha [29, p. 38]) and large c (as $c \to \infty$), in Eqns. (4.14) and (4.15), we get the following approximation formulae (4.18) and (4.19) respectively for computational work

$$U_n(t) = \left[\sum_{k=0}^{\infty} \left(\frac{|n|\sqrt{|n|}}{5} (t)^{1.5} (1+t)^2 + (t)^{1.5} \frac{\sqrt{|n|}}{4} \right)^k \left\{ \frac{e^{-|n|}}{\Gamma(1.5k+1)} + \frac{e^{-2|n|}}{\Gamma(1.5k+2)} + \frac{t\sqrt{t}e^{-|n|t}}{\Gamma(1.5k+2.5)} \right\} \right],$$

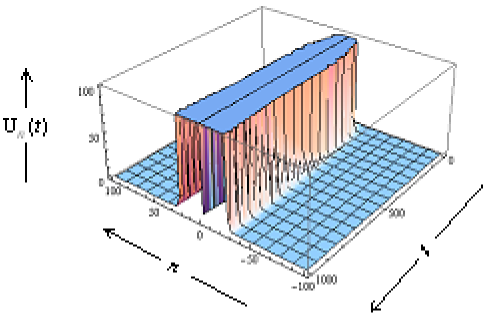
$$t > 0,\ n \in I. \tag{4.18}$$

and

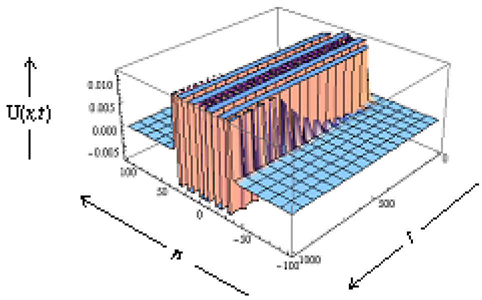
$$U(x,t) = \sum_{n=-\infty}^{\infty} \cos nx \left[\sum_{k=0}^{\infty} \left(\frac{|n|\sqrt{|n|}}{5} (t)^{1.5} (1+t)^2 + (t)^{1.5} \frac{\sqrt{|n|}}{4} \right)^k \left\{ \frac{e^{-|n|}}{\Gamma(1.5k+1)} + \frac{e^{-2|n|}}{\Gamma(1.5k+2)} + \frac{t\sqrt{t}e^{-|n|}t}{|n|\Gamma(1.5k+2.5)} \right\} \right],$$

$$t > 0,\ x \in \mathbb{R}. \tag{4.19}$$

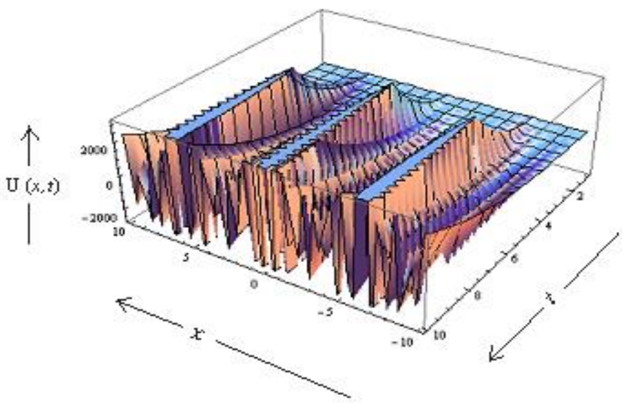
(VII)



(VIII)



(IX)



Conclusions:

(a) When we consider the dimensions $(n, t, U_n(t))$ and $(n, t, U(x,t))$ the structures (I) and (II) found on using numerical G-L numerical approximation formulae (4.12) and (4.13) are similar to the structures (IV) and (V) obtained due to our approximation formulae (4.16) and (4.17) respectively. Again to get these same structures of advection- dispersion for these dimensions, it take more time and greater values of $U_n(t)$ and $U(x,t)$ than above G-L numerical approximation formulae. This shows that in our results, advection-dispersion is slow and the values of $U_n(t)$ and $U(x,t)$ require larger to achieve the same action of advection-dispersion due to the G-L formulae.

(b) The structures (VII) and (VIII) found on using our approximation formulae (4.18) and (4.19) in above dimensions are similar to the structures (I) and (II) respectively and also identical to the structures (IV) and (V). But in these results the advection- dispersion is slow. The ratio of values of $U_n(t)$ and $U(x,t)$ due to our results to take the same action of advection-dispersion from the G-L formulae is given below:

The values of $U_n(t)$ and $U(x,t)$ to take similar action of advection-dispersion due to numerical G-L numerical approximation formulae {(4.12) and (4.13)} < the values of $U_n(t)$ and $U(x,t)$ on using our approximation formulae {(4.18) and (4.19)} < the values of $U_n(t)$ and $U(x,t)$ on using our approximation formulae {(4.16) and (4.17)}.

(c) The structures (III), (VI) and (IX), in dimension $(x,t,U(x,t))$, seem to be different to perform the action of advection-dispersion but most part of this action is lateral in these three structures and also the ratio of values of $U(x,t)$ due to our results and G-L numerical approximation formula is same as given above.

The value of $U(x,t)$ to take action of advection-dispersion due to numerical G-L numerical approximation formulae {(4.12) and (4.13)} < the value of $U(x,t)$ on using our approximation formulae {(4.18) and (4.19)} < the value of $U(x,t)$ on using our approximation formulae {(4.16) and (4.17)}.

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