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Review of Haar Wavelet Methods for Solution of Integral Equations

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ABSTRACT

Wavelet transform or wavelet analysis is a recently developed mathematical tool in applied mathematics. In this paper, A survey on the use of the Haar wavelet method to solve linear and nonlinear integral equations (Volterra, Fredholm, integro-differential equations) is presented. Also, the investigation of Haar wavelet method for solving singular integral equations is presented. Review shows that the Haar wavelet method is efficient and powerful in solving wide class of linear and nonlinear integral equations.

Introduction

Integral equations form one of the most useful techniques in many branches of pure analysis such as the theories of functional analysis and stochastic process. Integral equation is one of the most important branches of mathematical science. Also, Integral equations occurs in many fields of mechanics and mathematical physics. They are also connected with problems in mechanical vibrations, theory of analytic functions, orthogonal systems, theory of quadratic forms of infinitely many variables. In general, integral equation is an equation where an unknown function occurs under an integral sign. The name integral equation was first suggested by Paul du Bois-Reymond [143] in 1888, although the first appearance of integral equations is accredited to Abel for his work on the Tautochrone [144]. Integral equations arises in several problems of science and technology and may be obtained directly from physical problems e.g radiation transfer problem and neutron diffusion problem.

Fredholm and Volterra Integral Equations:

$$\int_a^b K(x,y)\varphi(y)dy = f(x) \qquad (1)$$

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,y)\varphi(y)dy \qquad (2)$$

where a, b are constants and parameter λ is unknown factor, which plays the same role as the eigenvalue in linear algebra. In these equations the function φ is the unknown, Kernel K and the right hand side f are given functions. The equations (1) and (2) carry the name of Ivar Fredholm and are called fredholm integral equations of the first and second kind, respectively. The equation (1) and (2) are linear equations since the unknown function φ appears in a linear fashion. If one of the limit is variable in (1) and (2), it is called volterra integral equations of first and second kind respectively.

Integro-differential equations:

A Integro-differential equation has the form

$$a\varphi'(x) + b\varphi(x) - \int_0^x K_1 dt = f(x) \qquad (3)$$

where $K_1 = K(x,t,\varphi(t),\varphi'(t))$ and $\varphi'(0) = \varphi_0,\ x,t \in [0,b]$.

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If the kernel K is a linear function in regard of $\varphi(t)$ and $\varphi'(t)$, we have a linear integral equation, otherwise the equation is nonlinear.

The integral equations play an important role in mathematical research on inverse problems and ill-posed problems. Some are discussed in [103, 104, 105, 121]. A wealth of inverse problems arises in the mathematical modeling of non - invasive evaluation and imaging methods in science, medicine and technology. The nonlinear integral equations also play an important role in mathematical theory. Many methods are developed for the solution of linear and nonlinear integral equations. A survey is presented in Atkinson [8]. Such methods are Spectral Galerkin method, Galerkin method, Petrov-Galerkin method, Wavelet-Galerkin method, Collocation Methods, Variational iterative method, Quadrature methods, sinc-collocation method, Adomian decomposition method, messless method, Block boundary value method, Homotopy perturbation method, Legendre wavelet method, Sequential approach method, Hybrid Legendre polynomial and Block-Pulse function approach, Legendre multiwavelet method, Method of moments, Harmonic wavelet method, Single-term walsh series method, Taylor-series expansion method, Haar wavelet method. Recently, Saha ray & Sahu [127] presented a review of various numerical methods to solve linear and nonlinear Fredholm integral equations of second kind. The first paper in which the wavelet method was applied for solving integral equations belong to Beylkin *et. al* [16]. After that important contributions to this subject are due to many authors.

Integral equations appears in the mathematical formulations of a variety of modeling procedures. These include jump diffusion and option pricing, fluid dynamics, biomedical areas, chemical kinetics, ecology, control theory of financial mathematics, aerospace systems, industrial mathematics, etc. Initial and boundary value problems in integro-differential equations have several applications in the study of chemical , physical and biological sciences. Yang & Shen [97] studied the periodic boundary-value problems for second order implusive integro-differential equations. Thieme [88] presented a model for the spatial spread of an epidemic. Cuminato *et al.* [27] presented a singular integro-differential equation model for dryout in LMFBR boiler tubes. Cont & Voltchkova [26] presented the integro-differential equations for option prices in exponential levy models. Cushing [28] presented integro- differential equations and delay models in population dynamics. Makroglou *et al.* [64] presented the mathematical models and software tools for the glucose-insulin regulatory system and diabetes. Choi & Lui [25] studied the integro-differential equation arising from an electrochemistry model. Schmidlen *et al.* [79] presented a rapid solution of first kind boundary integral equations in R^3. Sanchez-Avila [78] studied the wavelet domain signal de-convolution with singularity-preserving regularization.

Wavelet Transform

A wavelet is a mathematical function used to divide a given function or continuous-time signal into different scale components. The word wavelet is due to Morlet and Grossmann in the early 1980s. The study of wavelets has attained the present growth due to mathematical analysis of wavelets by Stromberg [86], Grossmann & Morlet [38], and Meyer [69]. Wavelets have been applied extensively in scientific and engineering fields. Many problems from physics and other disciplines lead to linear and nonlinear equations. Several methods have been proposed for numerical solution of these equations in Atkinson [8]. A wavelet transform is the representation of a function by wavelets. Wavelet transforms have advantages over traditional Fourier transforms for representing functions that have discontinuities and sharp peaks and for accurately deconstructing and reconstructing finite, non-periodic and/or non-stationary signals.

Types of Wavelet Transform

There are a large number of wavelet transforms each suitable for different applications. The common wavelet transform are continuous wavelet transform (CWT), discrete wavelet transform (DWT), multi-resolution discrete wavelet transform, fast wavelet transform (FWT), wavelet packet decomposition (WPD), stationary wavelet transform (SWT), fractional fourier transform (FRFT), fractional wavelet transform (FRWT), lifting scheme & generalized lifting scheme.

Survey on Methods to find solution of Integral equations

Spectral Galerkin method is used for the solution of a boundary integral equations in Mclean [108]. Also in Liang *et al.* [61], solution of second kind integral equations by Galerkin methods with continuous orthogonal wavelets is presented. Petrov-Galerkin method is used for the solution of second kind integral equation in Chen *et al.* [24]. In Maleknejad & Karimi [65], Petrov- Galerkin method is applied for the numerical solution of nonlinear fredholm integral equations using multiwavelets. Also, Petrov-Galerkin method is used for the solution of Hammerstein equations in Kaneko *et al.* [47]. Avudanayagam & Vano [9] presented Wavelet – Galerkin method for integro-differential equations. Wavelet - Galerkin method is used for the solution of boundary integral equations in Lage & Schwab [52]. Mahmoudi [63] used wavelet-Galerkin method to find the numerical solution of nonlinear integral equation. Also, Wavelet-Galerkin method is used for the solution of weakly singular fredholm integral equations in Micchelli & Xu [70]. Also, Galerkin wavelet methods is used for the solution of two-point boundary value problems in Xu & Shann [96]. In Fedotov [120], Justification of a Galerkin method for a regularized cauchy singular integro - differential equations is presented. Sidi & Israeli [111] presented the Quadrature methods for periodic singular and weakly singular fredholm integral equations. One of the favourite technique is the collocation method that is presented in Anioutline & Kyurkchan [5], Atkinson & Flores [7], Atkinson [8], Aziz *et al.* [11], Hsiao & Rathsfeld [43], Kaneko *et al.* [46], Okayama *et al.* [74], Pedas & Vainikko [75]. In Yan [112], the collocation method for first kind boundary integral equations on polygon domains is presented. In Mclean *et al.* [109], discrete trigonometric collocation method is used to solve integral equations. Avaji *et al.* [117] presented the solution of delay volterra integral equations using the variational iterative method. Also, Sweilam [87], presented the solution of fourth order integro-differential equations using variational iteration method. In [74], Okayama *et al.* presented the improved sinc-collocation method for the solution of fredholm integral equations of the second kind. Also, in Zarebnia [101], sinc numerical solution for the Volterra integro-differential equations is presented. Adomian decomposition method is applied to solve boundary value problems for fourth order integro-differential equations in Hasim [42]. Assari *et al.* [6] applied meshless method to solve nonlinear two-dimensional integral equations of the second kind on non-rectangular domains using radial basis functions with error analysis. Also, Dehghan & Salehi [30] presented the numerical solution of the nonlinear integro-differential equations based on the meshless method. Mirzaei & Dehghan [71] applied meshless based method for the solution of integral equations. In Chen & Zhang [21], block boundary value method is used to solve volterra integral and intrgro - differential equations. Logarithmic kernel, degenerate kernel, weakly singular kernal and arbitrary kernals are also used for the solution of integral equations. In Xu & Zhao [106], logarithmic kernel is used for the solution of boundary integral equations of first kind. In Atkinson & Sloan [107], the numerical solution of first kind logarithmic kernal integral equations on smooth open arcs is presented. In Saranen [110], the modified quadrature method for logarithmic kernel integral equations on closed curves is presented. Shamivand *et al.* [113] presented the solution of fredholm fuzzy integral equations with degenerate kernel. Miller and Feldstein [136] presented the solutions of volterra integral equations with weakly singular kernels. Brunner [138] presented the numerical solution of integral equations with weakly singular kernals. Friedmen *et al.* [114] presented the solutions of fuzzy integral equations with arbitrary kernals. Piessens & Branders [118] presented the numerical solution of linear fredholm integral equations of second kind. The solution involves a chebyshev series approximation. Arzhang [119] presented the numerical solution of weakly singular integral equations. Taylor series and Legendre functions of the second kind are used to remove the singularity of the weakly singular fredholm integral equations of the second kind. Yildirim [98], presented the solution of boundary value problems of fourth order integro-differential equations by using homotopy perturbation method. Yousefi & Razzaghi [99], presented the legendre wavelets methods to solve the nonlinear volterra-fredholm integral equations. Wanger & Chew [93] used wavelets to solve the electromagnetic integral equations. Berenguer *et al.* [15] applied a sequential approach to solve the fredholm integro-differential equations. Maleknejad *et al.* [68] applied hybrid legendre polynomials and block pulse functions approach to solve the nonlinear volterra – fredholm integro-differential equations. The solution of singular integral equations are presented by many authors. Singular integral equations are used in many problems related to science and engineering. Lakestani *et al.* [53] presented the numerical solution of the weakly singular fredholm integro-differential equations using legendre multiwavelets. Baratella & Orsi [137] presented new approach to find the numerical solution of weakly singular volterra integral equations. Pandey *et al.* [139] presented an efficient algorithms to solve singular integral equations of abel's type. Krenk [50], used the interpolation polynomial to find the solution of singular integral equations. Vainikko & Pedas [140] presented the numerical solution of singular integral equation. Splines are also used to find the solution of integral equations. Kumar & Sangal [51] presented the numerical solution of singular integral equations using cubic spline interpolations. Maleknejad & Lotfi [66], used expansion method to solve linear integral equations by cardinal B-Splines wavelet and Shanon wavelet bases to obtain Galerkin system. Feng & Pang [34], presented a class of three-point boundary-value problems for second –order impulsive integro-differential equations in banach spaces. Al – Mdallal [2] presented a monotone iterative sequences to solve the nonlinear integro-differential equations of second order. In Alipanah & Dehghan [3], numerical solution of the nonlinear fredholm integral equations by positive definite functions is presented. In Alpert [4], a class of bases in l^2 is used for the sparse representation of integral operators. Anioutline & Kyurkchan [5] used some application of wavelets technique to solve the integral equations. In Babolian *et al.* [13], numerical solution of nonlinear volterra – fredholm integro- differential equations by direct method using triangular functions is presented. Baker [14] also presented the numerical solution of integral equations. Cattani & Kudreyko [17] presented the harmonic wavelet method to solve the fredholm type integral equations of second kind. Chen & Lin [22] presented the Galerkin trigonometric wavelet methods to solve the natural boundary integral equations. Delves & Mohammad [31] introduced computational method to solve the integral equations. In Derili *et al.* [32] used two-dimensional wavelets for numerical solution of integral equations. Goswami *et al.* [37] solved the first – kind integral equations using wavelets on a bounded interval. Razani & Goodarzi [116] presented a solution of volterra-hamerstain integral equation in partially ordered sets. Zhu *et al.* [102] studied an application of fast and adaptive battle-lemarie wavelets to modeling lossy transmission lines. Yulan *et al.* [100] presented new algorithm for second order boundary value problems of integro-differential equation. In Sepehriana & Razzaghi [126], single-term walsh series method to solve volterra integro- differential equations is presented. Solution of integral equations of second kind are presented by many authors. Emamzadeh & Kajani [128] presented the solution of non-linear fredholm integral equation of the second kind with quadrature methods. Maleknejad *et al.* [129] presented a numerical solution of integral equations of the second kind by block-pulse functions. Maleknejad *et al.* [130] presented numerical solution of second kind fredholm integral equations system by using a Taylor-series expansion method. Ganji *et al.* [131] homotopy-perturbation method is used to find the solution of second kind nonlinear integral equations. Maleknejad & Sahlan [132] applied method of moments to solve second kind fredholm integral equations based on B-spline wavelets. Von Petersdorff *et al.* [92] used multiwavelet for the solution of second kind integral equations. Heydari *et al.* [142] presented wavelet methods to solve systems of nonlinear singular fractional volterra integro-differential equations.

Haar Wavelet

In mathematics, the Haar wavelet is a sequence of rescaled "square-shaped" functions which together form a wavelet family or basis. Haar wavelet methods are found to be giving high accuracy on low computational costs and are computationally attractive. Wavelet analysis is similar to Fourier analysis in that it allows a target function over an interval to be represented in terms of an orthonormal function basis. The Haar sequence is now recognised as the first known wavelet basis and extensively used as a teaching example. The Haar sequence was proposed in 1909 by Alfréd Haar [40]. Haar used these functions to give an example of an orthonormal system for the space of square-integrable functions on the unit interval [0, 1]. As a special case of the Daubechies wavelet, the Haar wavelet is also known as D2. The Haar wavelet is also the simplest possible wavelet. The technical disadvantage of the Haar wavelet is that it is not continuous and therefore not differentiable. This property can, however, be an advantage for the analysis of signals with sudden transitions, such as monitoring of tool failure in machines. Since the Haar wavelets are defined only for the interval [0,1], we must first normalize equations (1) - (3). This can be done by the change of the variables

$$\begin{cases} x_* = (x-a)/(b-a) \\ t_* = (t-a)/(b-a) \end{cases}$$

Haar wavelet method

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval [0,1].

$$h_i(x) = \begin{cases} 1 & for\ \alpha \le x < \beta \\ -1 & for\ \beta \le x < \gamma \\ 0 & otherwise\ in\ [0,1). \end{cases} \tag{4}$$

where $\alpha = \frac{k}{m},\ \beta = \frac{k+0.5}{m},\ \gamma = \frac{k+1}{m}$.

Integer $m = 2^j,\ (j = 0,1,2,3,\dots\dots\dots\dots,J)$ indicates the level of the wavelet.

$k = 0,1,2,\dots,m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m+k+1$; in the case of minimal values. $m = 1, k = 0$ we have $i = 2$. The maximal value of i is $i = 2M$. where $M=2^J$. It is assumed that the value $i = 1$, corresponds to the scaling function in [0,1].

$$h_1(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & otherwise \end{cases} \tag{5}$$

Let us define the collocation points $x_l = (l-0.5)/2M$, $(l = 1,2,3,\dots\dots,2M)$ and discredits the Haar function $h_i(x)$. In the collocation points, the first four Haar functions can be expressed as follow:

$h_1(x) = [1,\ 1,\ 1,\ 1]$, $h_2(x) = [1,\ 1,-1,-1]$, $h_3(x) = [1,-1,\ 0,\ 0]$, $h_4(x) = [0,0,1,-1]$.

We introduce the notation:

$H_4(x) = [h_1(x),h_2(x),h_3(x),h_4(x)]^T$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

where $H_4(x)$ is called Haar coefficient matrix . It is a square matrix of order 4.

In this way we get the coefficient matrix $H(i,l) = (h_i(x_l))$, which has the dimension 2M×2M.

Let us integrate equation (4),

$$q_i = \int_0^x h_i(x)dx, \tag{6}$$

In the collocation points equation (6) gets the form $Q(i,l) = q_i(x_l)$, where Q is a $2M \times 2M$ matrix.

Chen and Hsiao [20] presented this matrix in the form $Q_n = P_n H_n$, where $P_n H_n$ is interpreted as the product of the matrices P_n and H_n, called Haar integration and coefficient matrix, respectively.

The operational matrix of integration P, which is a 2M square matrix, is defined by the equation:

$$P_{i,1}(x) = \int_0^{x_l} h_i(x)dx \tag{7}$$

$$P_{i,v+1}(x) = \int_0^x P_{i,v}(x)dx, \tag{8}$$

where $v = 1,2,3,4,\ \dots\dots$

These integrals can be evaluated using equation (4) and first four of them are given:

$$P_{i,1}(x) = \begin{cases} x-\alpha & for\ x \in [\alpha,\beta) \\ \gamma - x & for\ x \in [\beta,\gamma) \\ 0 & elsewhere \end{cases}$$

$$P_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2 & for\ x \in [\alpha,\beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2 & for\ x \in [\beta,\gamma) \\ \frac{1}{4m^2} & for\ x \in [\gamma,1) \\ 0 & elsewhere \end{cases}$$

$$P_{i,3}(x) = \begin{cases} \frac{1}{6}(x-\alpha)^3 & for\ x \in [\alpha,\beta) \\ \frac{1}{4m^2}(x-\beta) - \frac{1}{6}(\gamma - x)^3 & for\ x \in [\beta,\gamma) \\ \frac{1}{4m^2}(x-\beta) & for\ x \in [\gamma,1) \\ 0 & elsewhere \end{cases}$$

$$P_{i,4}(x) = \begin{cases} \frac{1}{24}(x-\alpha)^4 & for\ x \in [\alpha,\beta) \\ \frac{(x-\beta)^2}{8m^2} - \frac{(\gamma - x)^4}{24} + \frac{1}{192m^4} & for\ x \in [\beta,\gamma) \\ \frac{(x-\alpha)^2}{8m^2} + \frac{1}{192m^4} & for\ x \in [\gamma,1) \\ 0 & elsewhere \end{cases}$$

Chen and Hsiao[20] showed that the following recursive formula for operational matrix of integration holds:

$$P_{\mu\times\mu} = \begin{bmatrix} P_{\mu/2} & -\frac{1}{2\mu}H_{\mu/2} \\ \frac{1}{2\mu}H_{\mu/2}^{-1} & 0 \end{bmatrix}$$

Function approximations

Any square integrable function $u(x)$ in the interval [0,1] can be expanded by a Haar series of infinite terms:

$$u(x) = \sum_{i=0}^{\infty} c_i h_i(x), \quad i \in \{0\} \cup N \qquad (9)$$

where the Haar coefficients c_i are determined as:

$$c_0 = \int_0^1 u(x)h_0(x)dx,$$

$$c_n = 2^j \int_0^1 u(x)h_i(x)dx,$$

$i = 2^j + k,\ j \geq 0,\ 0 \leq k < 2^j,\ x \in [0,1)$ such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 \left[u(x) - \sum_{i=0}^{m-1} c_i h_i(x)\right]^2 dx,$$

$$m = 2^j,\ j \in \{0\} \cup N.$$

Usually the series expansion of (9) contains infinite terms for smooth $u(x)$. If $u(x)$ is piecewise constant by itself , or may be approximation as piecewise constant during each subinterval, then $u(x)$ will be terminated at finite m terms, that is:

$$u(x) = \sum_{i=0}^{m-1} c_i h_i(x) \; = \; c_{(m)}^T h_{(m)}(x)$$

where the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as:

$c_{(m)}^T = [c_0, c_1, c_2, \dots \dots \dots \dots \dots \dots, c_{m-1}]$ and $h_{(m)}(x) = [h_0(x), h_1(x), \dots \dots \dots, h_{m-1}(x)]^T$

where T is the transpose.

In recent years interest to find the solution of integral equations by the Haar wavelet methods has greatly increased. Many research papers are published by many authors for this purpose. For numerical solution of linear integral equations, traditional quadrature formula methods and spline approximations are used. In the case of these methods systems of linear equations must be solved. For big matrices this requires a hugh number of arithmetic operations and a large storage capacity. A lot of computing time is saved if we succeed in replacing the fully populated transform matrix with a sparse matrix. Recently, Aziz & Siraj-ul-Islam [10] presented a new algorithms for numerical solution of nonlinear fredholm and volterra integral equations using Haar wavelets. Also, Siraj-ul-Islam *et al.* [83] presented an improved methods based on Haar wavelets for the numerical solution of nonlinear integral and integro-differential equations of first and higher orders. Siraj-ul-Islam *et al.* [84] also presented a new approach for numerical solution of integro-differential equations by Haar wavelet. Fayyaz & Azram [125] presented a new algorithms for numerical solution of nonlinear integro-differential equations of third order using Haar wavelets. Hariharan & Kannan [41] presented an overview of Haar wavelet methods for solving differential and integral equations. Also, Aziz *et al.* [11] presented the numerical solution of second order boundary-value problems by collocation method with the Haar wavelets. Babolian & Shahsavaran [12] presented the numerical solution of nonlinear fredholm integrals equations of second kind using Haar wavelets. Cattani [19] used Haar wavelet splines to solve the equations. Ghanbari *et al.* [36] presented the numerical solution of singular integral equation using Haar wavelet. Jun-li [45] presented numerical solution of fredholm integral-differential equations using Haar wavelet. Lepik & Tamme [54] used the application of Haar wavelets to solve linear integral equations. Lepik & Tamme [55] presented the solution of nonlinear integral equations by Haar wavelet method. In Lepik [56], application of Haar wavelet transform is used to solve integral and differential equations. Lepik [57] solved differential and integral equations by Haar wavelet method. Lepik [58] solved the integral and differential equations by the aid of non-uniform Haar wavelet method. In Lepik [59], Haar wavelet method is used for the numerical solution of nonlinear integro-differential equations. Lin *et al.* [62] presented a Haar wavelet methods to solve fredholm equations. Maleknejad & Mirzaee [67] used rationalized Haar wavelet to solve linear integral equations. Mishra *et al.* [72] presented Haar wavelet algorithm to solve certain differential, integral and integro-differential equations. Saeedi *et al.* [77] presented an operational Haar wavelet method to solve fractional volterra integral equations. Shamsi *et al.* [80] presented Haar wavelet method to solve pocklington's integral equation. In Shahsavaran [122], computational method to solve volterra integral equations of the second kind with weakly singular kernel which is based on the use of Haar wavelets and properties of block-pulse functions, is presented. Lepik [123] solved the fractional integral equations by the Haar wavelet method. Shihab & Mohammed [124] presented an efficient algorithm for nth order integro-differential equations using new Haar wavelets matrix designation. Bahmanpour [133] presented numerical solution of fredholm and volterra integral equations of the first kind using wavelets bases. Haar wavelet, Legendre wavelet and Chebyshev wavelets are used. Single-term Haar wavelet series (STHW) is used to solve some singular system problems. Sekar & Kumar [134] presented numerical investigation of nonlinear volterra-Hammerstein integral equations by single term Haar wavelet series. This method is more efficient than other methods. In Kumar & Pandit [135], historical development of wavelets and different numerical methods based on Haar and Daubechies wavelets for the numerical solution of integral equations and integro-differential equations is presented. Singh *et al.* [141] presented Haar wavelet approximation to solve a singular integral equations.

Review shows that the solution of integral equations obtained by Haar wavelet method is fast, more accurate than that obtained by other well known methods.

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