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A New Set of 32 In-equivalent Hadamard Matrices of Order 404 of Goethals-Seidel Type

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ABSTRACT

This research introduces a new set of 32 in-equivalent Hadamard matrices of order 404 of Goethals Seidel type. To apply the Goethals Seidel method, four Turyn type sequences of lengths 34 are found by a computer search. These sequences are used to construct base sequences of lengths 67 and are used to generate a set of four T-sequences of length 101. There were 16 possible ways of the linear combinations of these T- sequences and 1820 possible ways of choosing four sequences. Among them only 32 possible choices gave the new set of in-equivalent Hadamard matrices of order 404.

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Introduction

The orthogonal matrices are used in many applied problems, such as, the construction of discrete equipments realizing orthogonal constructions, one needs consideration of orthogonal matrices with the elements ± 1 . A special type of square orthogonal matrices with the elements ± 1 are called the Hadamard matrices.[17]

Investigation of Hadamard matrices were initially connected with linear algebra problems. Later, it turned out to the number of questions in information theory. Recently, a considerable increase of investigation devoted to Hadamard matrices has occurred. Some problems are still unanswered, so till now, it is not known if there exist a Hadamard matrix of order n for all n divisible by 4[17].

Historically, first work devoted to Hadamard matrices was constructed due to Sylvester in 1867 [12] who proposed a recurrent method for construction of Hadamard matrices of order 2^k.In 1933,Paley started that the order of any Hadamard matrix is divisible by 4.[17]A principal difficulty of this problem is the lack of unified methods for construction of Hadamard matrix of order 4n for all n.The known methods of construction are applicable only to relatively sequences on n.For many n it is usually necessary to develop a direct method of construction using machine access.

The known methods of constructing Hadamard matrices can be found as Williamson[3][11],Baumert-Hall-Goethals-Seidel[8],Paley-Wallis-Whiteman methods [10] & Golay-Turyn, Plotkin [12] approaches. All these methods are based on a set of orthogonal sequences.

Here in this work, the ultimate purpose is to construct the Hadamard matrix of order 404, which is known to be exist as it is a multiple of 4. It is understood that this matrix cannot be constructed using the preliminary method, which is known as Sylvester construction since there does not exist any Hadamard matrix of order 101. Thus, the major goal is to construct the Hadamard matrix of order 404 using any other method mentioned above.

For this construction, the Goethals Seidel method is chosen which also uses a set of orthogonal sequences with all entries being ± 1 . Here it provides a search method to construct such a set of orthogonal sequences known as Turyn type sequences. Such sequences of length 34 can be used to construct four circulant type matrices of order 101 and those matrices applied to the Goethal's Seidel method will follow the required result. The search method of constructing Turyn Type sequences of length 34 is given in [2*].

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Definition : (Hadamard matrix)

A square matrix whose entries are either +1 or -1 and whose rows are mutually orthogonal is called the Hadamard matrix. Hadamard matrix H of order n satisfies

 $HH^{T} = nI_{n}$

where I_n is the $n \times n$ identity matrix and H^T is the transpose of H.[7]

Further, it was proved that a Hadamard matrix has maximal determinant among matrices. Hence, if H is a Hadamard matrix, then det $H = +n^{\frac{n}{2} \cdot [14]}$

The Hadamard matrices were initially constructed by James Joseph Sylvester in 1867.[12] Let H be a Hadamard matrix of order *n*. Then the partitioned matrix $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$ is a Hadamard matrix of order 2n.

This observation can be applied repeatedly and leads to the following sequence of Hadamard matrices .[12]

$$\begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix} = H_2 \otimes H_{2^{k-1}}$$

where \otimes denotes the Kronecker product.

The following lemma states some of the properties of Hadamard matrices.

Lemma 1[17]:

Let H be an Hadamard matrix of order n. Then:

1. Hadamard matrices may be changed into other Hadamard matrices by permuting rows and columns and by multiplying rows and columns by -1. Matrices which can be obtained from one another by these methods are referred to as H-equivalent.

2. Every Hadamard matrix is H-equivalent to an Hadamard matrix which has every element of its first row and column equal to +1; matrices of this latter form are called normalized.

3. If H_{4n} is a normalized Hadamard matrix of order 4n, then every row (column), except the first, has 2n minus ones and 2n plus ones in each row (column).

4. The order of an Hadamard matrix is 1, 2, or 4n, where n is a positive integer.

The definition given below explains about the non- periodic autocorrelation value for a particular sequence, which guarantees the orthogonality property of that sequences.

A. Definition: (Non-periodic autocorrelation function)

Given A = $\{a_0, a_1, \dots, a_{n-1}\}$ be a sequences of length n, the non-periodic autocorrelation function N_A is defined by,

$$N_{A}(k) = \begin{cases} \sum_{i=0}^{n-k} a_{i}a_{i+k} \text{ for } k = 0, 1, ..., n-1 [8] \\ N_{A} = 0 \text{ for } k \ge n \end{cases}$$

Moreover, this idea can be used to verify the orthogonal property of the following sets of sequences.

B. Definition : (Turyn-type sequences)

A set of four $\{-1,1\}$ sequences A, B, C, D with lengths n, n, n, n - 1 is defined to be of Turyn type if

 $(N_A + N_B + 2N_C + 2N_D)(k) = 0$, for $k \ge 1[12]$ (1)

The following theorems and definitions illustrate the facts about the orthogonal sequences that are used in the construction of Hadamard matrix of order 404.

Theorem 01:

If X, Y, Z, W are $\{-1,1\}$ sequences of lengths n, n, n, n - 1 respectively and satisfy the condition (1), i.e if the sequences X, Y, Z, Z, W, W are 6-Base sequences (6 - BS(n, n, n, n; n - 1, n - 1)) then the following $\{-1,1\}$ sequences A = (Z; W), B = (Z, -W), C = X, D = Y are of length 2n - 1, 2n - 1, n, n respectively are Base sequences (BS(3n - 1)).[10]

This theorem introduces set of orthogonal sequences satisfying (1) known as Base sequences whose entries are ± 1 . Moreover, this set consists of four sequences, for which the first two sequences are of the length 2n - 1 and the other two are of length n, where n is the length of the Six base sequences (Turyn type sequences).[6]

These sequences can be used to construct T- sequences, that can be used to construct orthogonal matrices.

C. Definition 2: (T- sequences)

The four sequences X, Y, Z, W of length n with entries $\{0, -1, 1\}$ are said to be T-sequences ,[17]

$$\begin{split} 1. & |x_i| + |y_i| + |z_i| + |w_i| = 1 \text{ ; } i = 1,2, \dots, n \\ 2. & (N_X + N_Y + N_Z + N_W)(k) = \begin{cases} 0; & k = 1, \dots, n-1 \\ n; & k = 0 \end{cases} \end{split}$$

The following theorem states the method of constructing T-Sequences using Base sequences.

Theorem 02:

If A, B, C, D are Base sequences of lengths n + p, n + p, n, n then the set of T sequences of length 2n + p is given by

$$X = \left(\frac{1}{2}(A + B), 0_n\right),$$
$$Y = \left(\frac{1}{2}(A - B), 0_n\right),$$

 $Z = \left(0_{n+p}, \frac{1}{2}(C+D)\right),$ $W = \left(0_{n+p}, \frac{1}{2}(C-D)\right)'$

where 0_{m} denotes the sequence of zeros of length m.[13]

These sequences can be used to construct matrices in several ways. Among them, the most important type of constructing matrices is the circulant type. The following definition illustrates the method of obtaining the circulant type matrices using sequences. This method is more useful in constructing T-matrices which is defined in the next step.

[4]

D. Definition : (Circulant Matrix)

An $n \times n$ circulant matrix C of the array $c = (c_0, c_1, c_2, ..., c_{n-2}, c_{n-1})$ takes the form

$$C = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & \dots & c_3 & c_2 \\ \vdots & \ddots & & \vdots \\ c_{n-2} & c_{n-3} & \dots & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{bmatrix}$$

For an example, the circulant matrix of the array l = (1,2,3,4) is given by

$$C = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

The above method can be applied to the set of T-sequences illustrated in theorem 2 to obtain a set of orthogonal matrices which is known as T-matrices. These matrices have special characteristics which we illustrate in the next definition.

E. Definition : (T- matrices)

A set of four T-matrices T_{i} , i = 1,2,3,4 of order t are four circulant matrices that have the entries $0, \pm 1$ and that satisfy;

- i. $T_i * T_j = 0$ for $i \neq j$ where * denotes the Hadamard product.
- ii. $\sum_{i=1}^{4} T_i$ is a (1, -1) matrix.
- iii. $\sum_{i=1}^{4} T_i T_i^T = t I_t$.
- iv. $t = t_1^2 + t_2^2 + t_3^2 + t_4^2$ where t_i 's are the row (column)sum of each T_i .[17]

The following result, in a slightly different form was also discovered by R.J.Turyn. It is the simplest most useful method of constructing orthogonal designs.

Theorem 03:

Suppose there exists circulant T-matrices T_i ; i = 1,2,3,4 of order n.Let a, b, c, d be commuting variables. Then,

 $A = aT_1 + bT_2 + cT_3 + dT_4$

$$B = -bT_1 + aT_2 + dT_3 - cT_4$$
$$C = -cT_1 - dT_2 + aT_3 + bT_4$$
$$D = -dT_1 + cT_2 - bT_3 + aT_4$$

can be used in Goethals Seidel array to obtain a Hadamard matrix of order 4n.[17]

The next section explicates the Goethal Seidel method to construct Hadamard matrices.

Goethals-Seidel Method for Constructing Hadamard Matrices.

Theorem 04:

The Goethals Seidel array is of the form,

$$H = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & D'R & -C'R \\ -CR & -D'R & A & B'R \\ -DR & C'R & -B'R & A \end{bmatrix},$$

A, B, C, Dare circulant matrices of order n with

 $AA' + BB' + CC' + DD' = 4nI_n$

and R is the back diagonal identity matrix of order n,

i.e.
$$R = (r_{ij}) = \begin{cases} 1; ifi + j = n + 1 \\ 0 ; otherwise \end{cases}$$
; i, j = 1,2,3, ..., n

Then H is a Hadamard matrix of order 4n.[17]

The search for the initial rows of the circulant matrices A, B, C and D is related to sequences with zero autocorrelations. To initiate the relations, we may use the sequences defined above.

Methodology

Search method of Turyn Type sequences

A quadruple set of Turyn type sequences of length 34 consists of 3 sequences of length 34 and a sequence of length 33.So,first, sequences of length 34 whose elements are +1s and -1s were generated. These sequences are similar to the binary sequences whose elements are 0 and 1.This property was used in writing the computer program using C/C++ programming languages in generating the sequences of length 34.Since, there are 34 bits in one sequence, there is a total of 234 (=17179869184) number of sequences and in constructing sequences of length 33, there are 233 (=8589934592) number of sequences.

Then, the values 0,0,10,1 were selected for a, b, c&d respectively such that $a^2 + b^2 + 2c^2 + 2d^2 = 202$, where a, b, c, d are the sums of entries of the sequences A, B, C, D above. Then, all the possible solutions for each sequence A, B, C&D with above selected values a, b, c&d are filtered and save each result in four separate files.

From these sequences, all sequences C with sum of entries equal to c and for which $f_C(\theta) = N_C(0) + 2\sum_{k=1}^{n-1} N_C(k) \cos(k\theta) \le 101$ for all $\theta \in \left\{\frac{j\pi}{500}: j = 1, 2, \dots 500\right\}$ are found and the same process is done for the set of sequences D with the sum of the entries. Now, every possible combination from the recently filtered two sequences C&D are found so that, $f_C(\theta) + f_D(\theta) \le 101$.

Next, two sequences must be selected from the set of sequences of sum of entries a and b and the correlation value of each pair of sequence was determined. Then, each of these couples of sequences were combined with each of the results obtained in previous step and the non periodic auto correlation condition was checked.

For further details of the construction of these sequences we refer the reader to [2].

Search method of the Circulant matrices

Using the details given in theorem 1 and the results obtained in constructing Turyn type sequences of length 34, another set of sequences known as Base sequences of length 67 whose entries are ± 1 was constructed. Then, using the theorem 02, the set of T-sequences of length 101 with entries $\pm 1 \& 0$ was constructed.

Also, each sequence gives a circulant matrix so that each matrix is of order 101. Let those circulant matrices to be T_1, T_2, T_3 and T_4 . Note that any Hadamard matrix consists of only the entries ± 1 . In order to remove 0s from the above four circulant matrices, any linear combination of these four matrices can be used as they satisfy the property that $\sum_{i=1}^{4} T_i$ is a (1, -1) matrix.

Then, applying the definition given under E and F of the section I to the above constructed T-sequences, a quadruple set of Tmatrices were obtained. Then, letting all the commuting variables to be one in theorem 3, all the possible linear combinations of the Tmatrices were considered to construct the circulant matrices which can be applied in Goethal-Seidel array.

The following table shows all the possible ways of the linear combinations of the T-matrices that are used to construct the circulant matrices. In the following table, 1 denotes the addition and 0 denotes the subtraction. For an example, the matrices A_1 and A_5 are of the form $T_1 + T_2 + T_3 + T_4$ and $T_1 - T_2 + T_3 + T_4$ respectively.

Matrix	<i>T</i> ₁	T_2	<i>T</i> ₃	T_4
A ₁	1	1	1	1
A ₂	1	1	1	0
A ₃	1	1	0	1
A_4	1	1	0	0
A ₅	1	0	1	1
A ₆	1	0	1	0
A ₇	1	0	0	1
A ₈	1	0	0	0
A ₉	0	1	1	1
A ₁₀	0	1	1	0
A ₁₁	0	1	0	1
A ₁₂	0	1	0	0
A ₁₃	0	0	1	1
A ₁₄	0	0	1	0
A ₁₅	0	0	0	1
A ₁₆	0	0	0	0

Table 01-The possible linear combinations of T_1 , T_2 , $T_3 \& T_4$.

All these possible permutations were tested and among them only 32 matrices were suitable to apply in Goethals Seidel array. Thus, by this method, 32 in-equivalent Hadamard matrices of order 404 were obtained. The following figures show the design patterns of some Hadamard matrices obtained in this method..

Results

From the first part of the process, the following set of Turyn type of sequences were obtained. In each of the following sets of sequences + refers to +1 and _ refers to -1. For further details about this result refer [12].

Next, by using the details given in theorem 3 and the above Turyn type sequences, a set of Base sequences of length 67 was obtained.

To obtain the set of T-sequences of length 101, the above set of Base sequences was used.

Then, four T-matrices were constructed using these four sequences t_1, t_2, t_3 and t_4 . When considering all the possible linear combinations of these four T-matrices, there were $\binom{16}{4} = 1820$ permutations. All these possible permutations were tested and among them only 32 matrices were suitable to apply in Goethals Seidel array. Thus, by this method, 32 in-equivalent Hadamard matrices of order 404 were obtained. The following figures show the design patterns of some Hadamard matrices obtained in this method.



Conclusion

The primary goal of this project is to find a set of Turyn type sequences of length 34.Since this process needs more time, several partitions of the algorithm was used. So when the first result was obtained, the process was terminated due to the lack of space in the computer. By a complete machine search, an infinite number of Turyn type sequences of length 34 would have been obtained.

Further, the use of grid computers, cluster computers, mainframe computers or super computers can reduce the usage time in this process. Also, usage of a group of core I7 computers would have been given more than one result during this 6 months time.

This program can be rearranged to obtain skew Turyn type sequences so that one may result with the skew Hadamard matrices which can be used in several applications such as constructing graphs[1], Bush type Hadamard matrices[15] etc.

Let $A = \{a_{1,}a_{2,...,}a_{n,}\}$ and $B = \{b_{1,}b_{2,...,}b_{m,}\}$ be two sequences. The concatenation of these two sequences is the single sequence given by $X = \{a_{1,}a_{2,...,}a_{n,}b_{1,}b_{2,...,}b_{m,}\}$ or $= \{a_{1,}b_{1,}a_{2,}b_{2,}...,a_{n,}b_{m,}\}$. In this construction process, a set of T sequences of length 101 had been constructed using the set of Turyn type sequences obtained in this method. To construct them, the concatenation method given in the sequence X was used. One can use the other method of concatenation and obtain another set of T-sequences and proceed further.

Also, by using the Sylvester construction, there will be an infinite class of Hadamard matrices that begins with the order 404. Further, one may obtain many equivalent Hadamard matrices for each matrix given above by multiplying row/column by -1 or interchanging row/column. Any Hadamard matrix is said to be in-equivalent if it is not equivalent. Thus, the following diagrams illustrate that they are not equivalent to any Hadamard matrix obtained in this method.

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