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New Operational Matrices of Shifted Fourth Chebyshev wavelets

Suha N. Shihab[1] and Mohammed Abdulhadi Sarhan[2]

[1]Applied Science Department, University of Technology

[2]College of Science, Al-mustansiriah University.

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ABSTRACT

This paper presents some important operational matrices of fourth kind Chebyshev wavelets. New formula of operational matrix of derivative for fourth kind Chebyshev polynomials is first obtained and then operational matrices of derivative and integration of the fourth Chebyshevwavelets basis are derived. The proposed results are of direct interest in many applications.

Introduction

First and second kind Chebyshev wavelets were used in good number of areas such as approximation, solution of differential and integral equations. A numerical method for one-dimensional Bratu's problem based on Chebyshev wavelets of the first kind was presented in [1], Fariborzi Araghi [2] proposed a method to approximate the solution of a linear Fredholmintegro-differential equation via Chebyshev wavelets of the first kind . Ali [3] applied Chebyshev wavelets method for delay differential equations. See [4-9] for other works.

The idea that there are four kinds of Chebyshev wavelets leads to an extended range of application. There properties had been used in improving the performance of many numerical methods. For instance [10-12] applied the Chebyshev wavelets of the third kind.

In this paper however, our aim is to derive the operational matrices of both derivative and integration of fourth Chebyshev wavelets. Which will be used to solve calculus of variational problems in the next work.

Definition of Fourth Chebyshev Polynomials

In this section, the basic definition of fourth Chebyshev polynomials and some important properties, this is needed in the next sections.

The Fourth Kind Chebyshev Polynomial $W_n(x)$

The Chebyshev polynomial $W_n(x)$ has trigonometric definitions involving the half angle $/2$, $x = \cos\theta$. Gautsichireferred to this "Air-flow polynomials"as fourth kind Chebyshev polynomials [13]. It is the defined by:

$$W_n(x) = \sin\left(n+\frac{1}{2}\right)\theta/\sin\left(\frac{\theta}{2}\right) \qquad (1)$$

where $x = \cos\theta$. And they many be generated by using the recurrence relation $W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x)$ (2)

$$W_0(x) = 1, W_1(x) = 2x - 1$$

The Fourth Kind Shifted Chebyshev Polynomial $w_n^*(x)$

The fourth kind Chebyshev polynomials $w_n^*(x)$ appropriate to finite range $0 \le x \le 1$ of x by making the interval correspond to the interval $-1 \le x \le 1$ of anew variable t under the linear transformation

$$t = 2x - 1 \qquad (3)$$

The fourth kind Chebyshev polynomial to $0 \le x \le 1$ are thus given by $W_n(t)$, where t is given in equation (2). Using equations (2) and (3) yields:

Tele:

E-mail addresses: alrawy1978@yahoo.com

$$w_0^*(x) = 1$$

$$w_1^*(x) = 4x + 1 \quad (4)$$

with the recursive formula given as:

$$w_n^*(x) = 2(2x-1)w_{n-1}^*(x) - w_{n-2}^*(x) \quad (5)$$

with the initial values given in eq.(4).coefficients of x^kin $w_n^*(x)$ are listed in table (1)

Table (1) coefficients of x^k in $w_n^*(x)$.

k/n	0	1	2	3	4	5	6
0	1	-1	1	-1	1	-1	1
1		4	-12	24	-40	60	-84
2			16	-80	240	-560	1120
3				64	-448	1792	-5376
4					256	-2304	11520
5						1024	-11264
6							4096

New Operational Matrix of Derivative of $w_n^*(x)$

The following new theorems are needed hereafter.

Theorem(1): The derivative of fourth kind shifted Chebyshev polynomials is a linear combination of lower order fourth kind shifted Chebyshev polynomials and the relation is given throughout the following formulas

$$\dot{w}_n^* = 2\begin{cases} -\sum_{\substack{i=0 \\ i\ even}}^{n-2} (n-i)w_i^* + \sum_{\substack{i=1 \\ i\ odd}}^{n-1} (n+i+1)w_i^* & n\ even; \\ \sum_{\substack{i=0 \\ i\ even}}^{n-1} (n+i+1)w_i^* - \sum_{\substack{i=1 \\ i\ odd}}^{n-2} (n-i)w_i^* & n\ odd; \end{cases} \quad (6)$$

Theorem(2):the first derivative of fourth kind shifted Chebyshev polynomials in terms of Chebyshev polynomials is given by

$$\frac{dw_n^*(x)}{dx} = D_{w^*}w^*(x) \quad (7)$$

where $w^*(x) = [w_1^* w_2^*\, w_3^* \,\ldots w_n^*]^T$ and D_{w^*} is the $n \times n$ operational matrix of derivative when n=6 the operational matrix D_{w^*} defines as follow

$$D_{w^*} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ -4 & 8 & 0 & 0 & 0 & 0 \\ 8 & -4 & 12 & 0 & 0 & 0 \\ -8 & 12 & -4 & 16 & 0 & 0 \\ 12 & -8 & 16 & -4 & 20 & 0 \\ -12 & 16 & -8 & 20 & -4 & 24 \end{pmatrix} \quad (8)$$

In general, the elements of $n \times n$ operational matrix is defined as follow

$$(d_{ij})_{w^*} = \begin{cases} 0 & if & i < j \\ 4i & if & i = j \\ -4(k+1) & if & i-j = 2k+1 \\ 4(k+j) & if & i-j = 2k \end{cases}$$

Not that $\frac{dw_0^*(x)}{dx} = 0$. (9)

It is seen that D_{w^*} is sparse matrix.

Corollary : using eq.(7), the operational matrix for nth derivative can be derived as

$$\frac{d^n w^*(x)}{dx} = D_{w^*}^n w^*(x)$$

where $D_{w^*}^n$ is the nth power of matrix D_{w^*}.

Fourth Kind ChebyshevWavelets

Wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelets when the dilation parameter a and the we have the following family of continuous wavelets

$$\Psi_{a,b}(t) = |a|^{-1/2}\,\Psi\left(\frac{t-b}{a}\right) \quad a, b \in R, a \neq 0$$

Chebyshev wavelets of the fourth kind $\Psi^{4}_{n,m}(t) = \Psi(k,n,m,t)$have four arguments n argument, k can assume any positive integer, m is the order for Chebyshev polynomials $w_n(t)$ and t is the normalized time, they are defined on are interval $[0,1]$ by

$$\Psi^{4}_{n,m}(t) = \begin{cases} 2^{(k+1)/2} \quad P_m\left(2^k t - 2n + 1\right) \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0 \qquad\qquad otherwise \end{cases} \tag{10}$$

where $P_m(t) \frac{1}{\sqrt{\pi}} W_m(t)$ and m=0,1,….,M n=1,2,…,2^{k-1}

If we take k=1 and M=2. and using eq.(10),one can obtain

$$\left.\begin{aligned} \Psi^{4}_{0,0}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}} \\ \Psi^{4}_{0,1}(t) &= 2\sqrt{\frac{2}{\pi}}(8t-1) \\ \Psi^{4}_{0,2}(t) &= 2\sqrt{\frac{2}{\pi}}((8t-1)^2 - (8t-1) - 1) \end{aligned}\right\} \quad 0 \le t < \frac{1}{2}$$

$$\left.\begin{aligned} \Psi^{4}_{1,0}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}} \\ \Psi^{4}_{1,1}(t) &= 2\sqrt{\frac{2}{\pi}}(8t-5) \\ \Psi^{4}_{1,2}(t) &= 2\sqrt{\frac{2}{\pi}}((8t-5)^2 - (8t-5) - 1) \end{aligned}\right\} \frac{1}{2} \le t < 1$$

we should note that in dealing with Chebyshev wavelets the weight function $w(x)$ have to be dilated and translated as

$$w_n(x) = w\left(2^k t - 2n + 1\right)$$

Function Approximation

Suppose that H=L^2[0,1] and $\{\Psi^{4}_{10}(t), \Psi^{4}_{11}(t), \dots, \Psi^{4}_{M-1}(t), \Psi^{4}_{20}(t), \dots, \Psi^{4}_{2M-1}(t), \Psi^{4}_{2^k 0}(t), \dots, \Psi^{4}_{2^k M-1}(t)\}$ ⊂H be the set of fourth kind Chebyshev wavelets and

Y=span$\{\Psi^{4}_{10}(t), \Psi^{4}_{11}(t), \dots, \Psi^{4}_{M-1}(t), \Psi^{4}_{20}(t), \dots, \Psi^{4}_{2M-1}(t), \Psi^{4}_{2^k 0}(t), \dots, \Psi^{4}_{2^k M-1}(t)\}$

and f be an arbitrary element in H. since Y is a finite dimensional vector space,

f has the unique best approximation out of Y such that f_0∈Y,that is

$$\|f - f_0\| \le \|f - y\|, \qquad \forall y \in Y$$

Since $f_0 \in Y$, there exists the unique coefficients $f_{10}, f_{11}, \dots, f_{2^k M-1}$ such that $f(t) \cong f_{2^k M-1}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} f_{nm} \Psi^{4}_{nm}(t) = F^T \Psi^4(t)$ (11)

where F and $\Psi(t)$ are $2^k M \times 1$ matrices given by

$$F = \left[f_{10}, f_{11}, f_{1M-1}, f_{20}, \dots, f_{2M-1}, \dots f_{2^k 0}, \dots, f_{2^k M-1}\right]^T \tag{12}$$

and

$$\Psi^4(t) = \left[\Psi^{4}_{10}(t), \Psi^{4}_{11}(t), \dots, \Psi^{4}_{M-1}(t), \Psi^{4}_{20}(t), \dots, \Psi^{4}_{2M-1}(t), \Psi^{4}_{2^k 0}(t), \dots, \Psi^{4}_{2^k M-1}(t)\right]^T$$

(13)

Operational matrix of Derivative of Ψ^{4}_{nm}

A New operational matrix of derivative of fourth kind Chebyshev wavelets is introduced in this section.

In the interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$, we have

$$\Psi^{4}_{n,m}(t) = \frac{1}{\sqrt{\pi}} 2^{(k+1)/2} \quad w^*_m\left(2^k t - 2n + 1\right) \tag{14}$$

Using eq.(10), one can obtain the derivative of eq.(14) to be

$$\dot{\Psi}^4_{nm} = \frac{1}{\sqrt{\pi}} 2^{(k+1)/2}.2^k \begin{cases} -\sum_{\substack{i=0 \\ i\ even}}^{n-2} (n-i)w_i^* + \sum_{\substack{i=1 \\ i\ odd}}^{n-1} (n+i+1)w_i^* & n\ even; \\ \sum_{\substack{i=0 \\ i\ even}}^{n-1} (n+i+1)w_i^* - \sum_{\substack{i=1 \\ i\ odd}}^{n-2} (n-i)w_i^* & n\ odd; \end{cases} \quad (15)$$

The function $\Psi_i^4(t)$ is zero outside the interval $\left[\frac{i-1}{2^{k-1}}, \frac{i}{2^{k-1}}\right)$, so

$$\dot{\Psi}_i^4(t) = D_{\psi^4}\psi_i(t) \text{i=1,2,…,} 2^{k-1} \quad (16)$$

where the $2^k(M+1)$operational matrix of derivative is defined as follows

$$D_{\psi^4} = diag(M, M, \dots, M)$$

$$M = 2^{k+2}\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & 0 & 0 & \cdots & 0 \\ 2 & -1 & 3 & 0 & \cdots & 0 \\ -2 & 2 & -1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & 0 \\ -n & n+2 & -(n-2) & n+4 & \cdots & 2n \end{pmatrix}$$

For even n

and $$M = 2^{k+2}\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & 0 & 0 & \cdots & 0 \\ 2 & -1 & 3 & 0 & \cdots & 0 \\ -2 & 2 & -1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & 0 \\ n+1 & -(n-1) & n+3 & -(n-3) & \cdots & 2n \end{pmatrix}$$

for odd n.

Another definition of the operational matrix of derivative o fourth Chebyshev wavelets is given throughout the following lemma

Lemma(1):

Let $\Psi(t)$ be the fourth Chebyshev wavelets vector defined in (13).

The derivative of this vector can be obtained as

$$\frac{d\Psi^4(t)}{dx} = D_{\Psi^4}\Psi^4(t) \quad (17)$$

where D is the $2^k\ (M+1)$ operational matrix of derivative defined as follows

$$D_{\Psi^4} = \begin{pmatrix} F & 0 & 0 & \cdots & 0 \\ 0 & F & 0 & \cdots & 0 \\ 0 & 0 & F & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & F \end{pmatrix},$$

In which F is $(M+1)\times(M+1)$ matrix and its elements are defined as follow $\left(F_{ij}\right)_{\Psi^4} = 2^{k+2}\begin{cases} 0 & if & i<j \\ i & if & i=j \\ -(h+1) & if & i-j=2h+1 \\ (h+j) & if & i-j=2h \end{cases}$

Note that $\frac{d\Psi_0^4(x)}{dx} = 0$.

Operational Matrix of Integration of Ψ^4_{nm}

The integration of the vector $\Psi^4\ (x)$, defined in (6) , can be achieved as

$$\int_0^x \Psi^4\ (t)dt = P\Psi^4$$

where P is $2^{k-1}\ M \times 2^{k-1}\ M$ matrix, named operational matrix of integration of $\Psi^4_{nm}(x)$. This matrix is determined as follows

$$P=\begin{pmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ 0 & 0 & L & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & F \\ 0 & 0 & 0 & 0 & L \end{pmatrix}$$

where the F, O and L are $M \times M$ matrices and given by

$$F=\frac{1}{2^{k-1}}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{1}{M} & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{if M is odd}$$

$$F=\frac{1}{2^{k-1}}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{1}{M-1} & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{if M is even}$$

$$P=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $$L=\frac{1}{2^{k}}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \cdots & 0 \\ -\frac{1}{6} & -\frac{1}{4} & \frac{1}{12} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{(-1)^{M+1}}{2M} & \frac{(-1)^{M+1}}{2(M-1)} & \frac{1}{4M} & \cdots & \frac{1}{4M} \end{pmatrix}$$

Conclusion

In this paper, a new general formula for shifted fourth kind Chebyshev polynomials operational matrix of derivatives was first presented. Then, this formula is employed for deriving a general procedure for forming shifted fourth kind Chebyshevwavelets operational matrix of derivatives. Also, a general formulation for shifted fourth kind Chebyshev polynomials operational matrix of integration has been derived in this work. The application of the obtained operational matrices will be discussed in the next work.

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