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A study on Q-fuzzy normal subgroups and cosets

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ABSTRACT

In this paper, we study some of the properties of Q-fuzzy normal subgroups, cosets and prove some results on these.

Introduction

After the introduction of fuzzy sets by L.A.Zadeh[18], several researchers explored on the generalization of the notion of fuzzy set. Azriel Rosenfeld[3] defined a Fuzzy groups. Anthony. J.M. and Sherwood.H[2] defined a fuzzy groups redefined. Choudhury.F.P. and Chakraborty.A.B. and Khare.S.S. [5] defined a fuzzy subgroups and fuzzy homomorphism. A.Solairaju and R.Nagarajan [14] have introduced and defined a new algebraic structure called Q-fuzzy subgroups. We introduce the concept of Q-fuzzy normal subgroups, cosets and established some results.

Preliminaries

Definition: Let X be a non-empty set. A **fuzzy subset A** of X is a function $A : X \to [\,0, 1\,]$.

Definition: Let X be a non-empty set and Q be a non-empty set. A Q-**fuzzy subset A** of X is a function $A : XxQ \to [\,0, 1\,]$.

Example: Let X = { a, b, c } be a set and Q = { p }. Then A = { ⟨ (a, p), 0.4 ⟩, ⟨ (b, p), 0.2 ⟩, ⟨ (c, p), 0.5 ⟩ } is a Q-fuzzy subset of X.

Definition: A subgroup (H, **.**) of a group (G, **.**) is called **a normal subgroup** of G if aH = Ha, for all a in G.

Definition: Let (G, ·) be a group and Q be a set. A Q-fuzzy subset A of G is said to be a Q-**fuzzy subgroup(QFSG)** of G if the following conditions are satisfied:

(i) $A(xy, q) \geq \min \{A(x, q), A(y, q)\}$,

(ii) $A(x^{-1}, q) \geq A(x, q)$, for all x and y in G and q in Q.

Definition: Let (G, ·) be a group and Q be a non-empty set. A Q-fuzzy subgroup A of G is said to be a Q-**fuzzy normal subgroup** (QFNSG) of G if A(xy, q) = A(yx, q), for all x and y in G and q in Q.

Definition: Let (G, ·) be a group and Q be a non-empty set. A Q-fuzzy subgroup A of G is said to be a **Q-fuzzy characteristic subgroup(QFCSG)** of G if A(x, q) = A(f(x, q)), for all x in G and f in Q-AutG and q in Q.

Definition: A Q-fuzzy subset A of a set X is said to be **normalized** if there exist x in X such that A(x, q) =1.

Definition: Let A be a Q-fuzzy subgroup of a group (G, ·). For any a in G, aA defined by $(aA)(x, q) = A(a^{-1}x, q)$, for every x in G and q in Q, is called a **Q-fuzzy coset** of G.

Definition: Let A be a Q-fuzzy subgroup of a group (G, ·) and H = { x∈G / A(x, q) = A(e, q) }, then O(A) **,order of A** is defined as O(A) = O(H).

Definition: Let A be a Q-fuzzy subgroup of a group (G, ·). Then for any a and b in G, a **Q-fuzzy middle coset aAb** of G is defined by $(aAb)(x, q) = A(a^{-1}x\, b^{-1}, q)$, for every x in G and q in Q.

Definition: Let A be a Q-fuzzy subgroup of a group (G, ·) and a in G. Then the **pseudo Q-fuzzy coset** $(aA)^p$ is defined by ($(aA)^p$)(x, q) = p(a)A(x, q), for every x in G and for some p in P and q in Q.

Definition: A Q-fuzzy subgroup A of a group G is called a **generalized characteristic Q-fuzzy subgroup** (GCQFSG) if for all x and y in G, O(x) = O(y) implies A(x, q) = A(y, q), q in Q.

Some properties of Q-fuzzy normal subgroupS:

Proposition: Let (G, ·) be a group and Q be a non-empty set. If A and B are two Q-fuzzy normal subgroups of G, then their intersection $A \cap B$ is a Q-fuzzy normal subgroup of G.

Proof: Let x and y in G and q in Q and A ={ ⟨ (x, q), A(x, q) ⟩ / x in G and q in Q } and B = { ⟨(x, q), B(x, q) ⟩ / x in G and q in Q } be a Q-fuzzy normal subgroups of G. Let $C = A \cap B$ and C ={⟨(x, q), C(x, q) ⟩ / x in G and q in Q}. Where C(x, q) = min{A(x, q), B(x, q)}. Then, Clearly C is a Q-fuzzy subgroup of G, since A and B are two Q-fuzzy subgroups of G. And, C(xy, q) = min {A(xy, q), B(xy, q) }, = min{ A(yx, q), B(yx, q) } = C(yx, q). Therefore, C(xy, q) = C(yx, q) , for all x and y in G and q in Q.

Hence $A \cap B$ is a Q-fuzzy normal subgroup of a group G.

Proposition: Let (G, ·) be a group and Q be a non-empty set. The intersection of a family of Q-fuzzy normal subgroups of G is a Q-fuzzy normal subgroup of G.

Proof: Let $\{ A_i \}_{i\in I}$ be a family of Q-fuzzy normal subgroups of G and $A = \bigcap_{i\in I} A_i$. Then for x and y in G and q in Q, clearly the intersection of a family of Q-fuzzy subgroups of a

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group G is a Q-fuzzy subgroup of a group G. Now, A(xy, q) = $\inf_{i\in I} A_i(xy,q)$ = $\inf_{i\in I} A_i(yx,q)$ = A(yx, q).

Therefore, A(xy, q) = A(yx, q) , for all x and y in G and q in Q. Hence the intersection of a family of Q-fuzzy normal subgroups of a group G is a Q-fuzzy normal subgroup of G.

Proposition: If A is a Q-fuzzy characteristic subgroup of a group G, then A is a Q-fuzzy normal subgroup of a group G.

Proof: Let A be a Q-fuzzy characteristic subgroup of a group G, x and y in G and q in Q. Consider the map f : GxQ → GxQ defined by f(x, q) =(yxy^{-1}, q).

Clearly, f in Q-AutG. Now, A(xy, q) = A(f(xy, q)) = A($y(xy)y^{-1}$, q) = A(yx, q).Therefore, A(xy, q) = A(yx, q) , for all x and y in G and q in Q.

Hence A is a Q-fuzzy normal subgroup of a group G.

Proposition: A Q-fuzzy subgroup A of a group G is a Q-fuzzy normal subgroup of G if and only if A is constant on the conjugate classes of G.

Proof : Suppose that A is a Q-fuzzy normal subgroup of a group G. Let x and y in G and q in Q. Now, A($y^{-1}xy$, q) = A(xyy^{-1}, q) = A(x, q). Therefore, A($y^{-1}xy$,q) = A(x,q) , for all x and y in G and q in Q.Hence (x) ={$y^{-1}xy$/ y$\in$G}.

Hence A is constant on the conjugate classes of G. Conversely, suppose that A is constant on the conjugate classes of G. Then, A(xy, q) = A($xyxx^{-1}$, q) = A($x(yx)x^{-1}$, q) = A(yx, q). Therefore, A(xy, q) = A(yx, q) , for all x and y in G and q in Q. Hence A is a Q-fuzzy normal subgroup of a group G.

Proposition: Let A be a Q-fuzzy normal subgroup of a group G. Then for any y in G and q in Q, we have A(yxy^{-1}, q) = A($y^{-1}xy$, q), for every x in G.

Proof: Let A be a Q-fuzzy normal subgroup of a group G. For any y in G and q in Q, we have, A(yxy^{-1}, q) = A(x, q) = A(xyy^{-1}, q) = A($y^{-1}xy$, q).

Therefore, A(yxy^{-1}, q) = A($y^{-1}xy$, q) , for all x and y in G and q in Q.

Proposition: A Q-fuzzy subgroup A of a group G is normalized if and only if A(e, q) = 1, where e is the identity element of the group G and q in Q.

Proof: If A is normalized, then there exists x in G such that A(x, q) = 1 , but by properties of a Q-fuzzy subgroup A of G, A(x, q) $\leq$ A(e, q), for every x in G and q in Q. Since A(x,q) = 1and A(x,q) $\leq$ A(e,q),1$\leq$ A(e,q) . But 1 $\geq$ A(e, q).

Hence A(e, q) = 1. Conversely, if A(e, q) = 1 , then by the definition of normalized Q-fuzzy subset, A is normalized.

Proposition: Let A and B be Q-fuzzy subgroups of the groups G and H, respectively. If A and B are Q-fuzzy normal subgroups, then AxB is a Q-fuzzy normal subgroup of GxH.

Proof: Let A and B be Q-fuzzy normal subgroups of the groups G and H respectively. Clearly AxB is a Q-fuzzy subgroup of GxH, since A and B are Q-fuzzy subgroups G and H. Let x_1 and x_2 be in G, y_1 and y_2 be in H and q in Q. Then (x_1, y_1) and (x_2, y_2) are in GxH. Now, AxB [(x_1, y_1)(x_2, y_2), q] = AxB((x_1x_2, y_1y_2), q) = min { A(x_1x_2, q), B(y_1y_2, q) }= min { A(x_2x_1, q), B(y_2y_1, q) }= AxB((x_2x_1, y_2y_1), q) = AxB [(x_2, y_2)(x_1, y_1), q]. Therefore, AxB [(x_1, y_1)(x_2, y_2), q] = AxB [(x_2, y_2)(x_1, y_1), q]. Hence AxB is a Q-fuzzy normal subgroup of GxH.

Proposition: Let a Q-fuzzy normal subgroup A of a group G be conjugate to a Q-fuzzy normal subgroup M of G and a Q-fuzzy normal subgroup B of a group H be conjugate to a Q-fuzzy normal subgroup N of H. Then a Q-fuzzy normal subgroup AxB of a group GxH is conjugate to a Q-fuzzy normal subgroup MxN of GxH.

Proof: It is trivial.

Properties of q-fuzzy cosets

Proposition: Let A be a Q-fuzzy subgroup of a finite group G, then O(A) / O(G).

Proof: Let A be a Q-fuzzy subgroup of a finite group G with e as its identity element. Clearly H ={ x$\in$G / A(x, q) = A(e, q)} is a subgroup of G for H is a α-level subset of G where α = A(e, q). By Lagranges theorem O(H) / O(G).

Hence by the definition of the order of the Q-fuzzy subgroup of G, we have O(A) / O(G).

Proposition: Let A and B be two Q-fuzzy subsets of an abelian group G. Then A and B are conjugate Q-fuzzy subsets of the abelian group G if and only if A = B.

Proof: Let A and B be conjugate Q-fuzzy subsets of abelian group G, then for some y in G, we have A(x, q) = B($y^{-1}xy$, q), for every x in G and q in Q = B($yy^{-1}x$, q) ,since G is an abelian group, = B(ex, q) = B(x, q).Therefore, A(x, q) = B(x, q) , for every x in G and q in Q. Hence A = B. Conversely, if A = B, then for the identity element e of G, we have, A(x, q) = B($e^{-1}xe$, q), for every x in G and q in Q. Hence A and B are conjugate Q-fuzzy subsets of G.

Proposition: If A and B are conjugate Q-fuzzy subgroups of the normal group G, then O(A) = O(B).

Proof: Let A and B be conjugate Q-fuzzy subgroups of G. Now, O(A) = order of { x$\in$G / A(x, q) = A(e, q) } = order of { x$\in$G / B($y^{-1}xy$, q) = B($y^{-1}ey$, q) } = order of { x$\in$G / B(x, q) = B(e, q)}= O(B). Hence O(A) = O(B).

Proposition: Let A be a Q-fuzzy subgroup of a group G, then the pseudo Q-fuzzy coset $(aA)^p$ is a Q-fuzzy subgroup of a group G, for every a in G.

Proof: Let A be a Q-fuzzy subgroup of a group G. For every x and y in G and q in Q, we have,(($(aA)^p$)(xy^{-1}, q) = p(a)A(xy^{-1},q) $\geq$ p(a) min{A(x, q) , A(y, q)} = min { p(a)A(x, q), p(a)A(y, q) } =min { ($(aA)^p$)(x, q), ($(aA)^p$)(y, q) }. Therefore, ($(aA)^p$)(xy^{-1}, q) $\geq$ min { ($(aA)^p$)(x, q), ($(aA)^p$)(y, q) }, for x and y in G and q in Q. Hence $(aA)^p$ is a Q-fuzzy subgroup of a group G.

Proposition: If A is a Q-fuzzy subgroup of a group G, then for any a in G the Q-fuzzy middle coset aAa^{-1} of G is also a Q-fuzzy subgroup of G.

Proof: Let A be a Q-fuzzy subgroup of G and a in G. To prove aAa^{-1} is a Q-fuzzy subgroup of G. Let x and y in G and q in Q. Then (aAa^{-1})(xy^{-1}, q) = A($a^{-1}xy^{-1}a$, q), = A($a^{-1}xaa^{-1}y^{-1}a$,q) =A(($a^{-1}xa$)($a^{-1}ya$) $^{-1}$, q) $\geq$ min { A($a^{-1}xa$, q), A(($a^{-1}ya$)$^{-1}$, q) } $\geq$ min { A($a^{-1}xa$, q), A($a^{-1}ya$, q), since A is a QFSG of G. = min{(aAa^{-1})(x, q), (aAa^{-1})(y, q)}. Therefore, (aAa^{-1})(xy^{-1},q) $\geq$ min{(aAa^{-1})(x, q), (aAa^{-1})(y, q) }. Hence aAa^{-1} is a Q-fuzzy subgroup of a group G.

Proposition: Let A be a Q-fuzzy subgroup of a group G and aAa^{-1} be a Q-fuzzy middle coset of G, then O(aAa^{-1}) = O(A), for any a in G.

Proof: Let A be a Q-fuzzy subgroup of G and a in G. By proposition 3.5, the Q-fuzzy middle coset aAa^{-1} is a Q-fuzzy subgroup of G.

Further by the definition of a Q-fuzzy middle coset of G, we have, (aAa^{-1}) (x, q) = A($a^{-1}xa$, q), for every x in G and q in Q. Hence for any a in G , A and aAa^{-1} are conjugate Q-fuzzy subgroups of a group G as there exists a in G such that (aAa^{-1})(x, q) = A($a^{-1}xa$, q) , for every x in G and q in Q. By proposition 3.3, O(aAa^{-1}) = O(A), for any a in G.

Proposition: Let A be a Q-fuzzy subgroup of a group G and B be a Q-fuzzy subset of a group G. If A and B are conjugate Q-fuzzy subsets of the group G, then B is a Q-fuzzy subgroup of a group G.

Proof: Let A be a Q-fuzzy subgroup of a group G and B be a Q-fuzzy subset of G. And, let A and B be conjugate Q-fuzzy subsets of G. To prove B is a Q-fuzzy subgroup of G. Let x and y in G and q in Q. Then xy^{-1} in G. Now, $B(xy^{-1}, q) = A(g^{-1}xy^{-1}g, q)$, for some g in G $=A(g^{-1}xgg^{-1}y^{-1}g, q) = A((g^{-1}xg)(g^{-1}yg)^{-1}, q) \geq \min\{A(g^{-1}xg, q), A((g^{-1}yg)^{-1}, q)\} \geq \min\{A(g^{-1}xg, q), A(g^{-1}yg, q)$, since A is a QFSG of G $= \min\{B(x, q), B(y, q)\}$. Therefore, $B(xy^{-1}, q) \geq \min\{B(x, q), B(y, q)\}$, for x and y in G and q in Q. Hence B is a Q-fuzzy subgroup of the group G.

Proposition: Let A be a Q-fuzzy subgroup of a group G. Then $(x, q)A = (y, q)A$, for x,y in G if and only if $A(x^{-1}y, q) = A(y^{-1}x, q) = A(e, q)$.

Proof: Let A be a Q-fuzzy subgroup of a group G. Let $(x, q)A = (y, q)A$, for x and y in G and q in Q. Then, $(x, q)A(x, q) = (y, q)A(x, q)$ and $(x, q)A(y, q) = (y, q)A(y, q)$, $\Rightarrow A(x^{-1}x, q) = A(y^{-1}x, q)$ and $A(x^{-1}y, q) = A(y^{-1}y, q)$. Hence $A(e, q) = A(y^{-1}x, q)$ and $A(x^{-1}y, q) = A(e, q)$. Therefore, $A(x^{-1}y, q) = A(y^{-1}x, q) = A(e, q)$, for x and y in G and q in Q. Conversely, let $A(x^{-1}y, q) = A(y^{-1}x, q) = A(e, q)$, for x and y in G and q in Q. For every g in G and we have, $(x, q)A(g, q) = A(x^{-1}g, q) = A(x^{-1}yy^{-1}g, q) \geq \min\{A(x^{-1}y, q), A(y^{-1}g, q)\} = \min\{A(e, q), A(y^{-1}g, q)\} = A(y^{-1}g, q) = (y, q)A(g, q)$. Therefore, $(x, q)A(g, q) \geq (y, q)A(g, q)$ ------- (1).And , $(y, q)A(g, q) = A(y^{-1}g, q) = A(y^{-1}xx^{-1}g, q) \geq \min\{A(y^{-1}x, q), A(x^{-1}g, q)\} = \min\{A(e, q), A(x^{-1}g, q)\} = A(x^{-1}g, q) = (x, q)A(g, q)$. Therefore, $(y, q)A(g, q) \geq (x, q)A(g, q)$ -----(2). From (1) and (2) we get, $(x, q)A(g, q) = (y, q)A(g, q)$ -----(3). We get, $(x, q)A = (y, q)A$, for all x and y in G and q in Q.

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