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Star partial ordering of secondary k-normal matrices

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ABSTRACT

Let $C_{n\times n}$ denote the set of all complex $n\times n$ matrices .By the star partial ordering $\leq^*$ .So $A\leq^* B$ means that $A^*A = A^*B$ and $AA^* = BA^*$. We find several characterizations for $A\leq^* B$ in the case of s-k normal matrices. As an applications, we study how $A<^* B$ relates to $A^2\leq^* B^2$. The results can be extended to study how $A\leq^* B$ relates to $A^n\leq^* B^n$, where $n\geq 2$ is an integer.

Introduction

Preliminaries and definitions:

Let $C_{n\times n}$ be the space of $n\times n$ complex matrices ($n\geq 2$). We order it by the Star partial ordering $\leq^*$. So $A\leq^* B$ means that $AA^* = A^*B$ and $AA^* = BA^*$. Let k be a fixed product of disjoint transposition in S_n ={1,2,…n}.(hence, involutary) and let K be the associated permutation matrix of k and let V be the units in the secondary diagonal. A matrix $A\in C_{n\times n}$ is said to be secondary k-normal(s-k normal) if $A(KVA^*VK) = (KVA^*VK)A$. A matrix $A\in C_{n\times n}$ is said to be secondary k-unitary(s-k unitary) if $A(KVA^*VK) = (KVA^*VK)A = I$. A matrix $A\in C_{n\times n}$ is said to be s-k hermitian if $A = KVA^*VK$.

We will study how $A<^* B$ relates to $A^2\leq^* B^2$. In the case of secondary k-normal matrices. We will see theorem(3.1) that the "if part of theorem(1.1)remains valid". However it is valid for all matrices.

Theorem:

Let A and B be secondary k-hermitian and non-negative definite then $A^2\leq^* B^2$ iff $A\leq^* B$.

Proof: Proof is obvious.

Example: Let

$$A=\begin{pmatrix} i & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & i\end{pmatrix},\quad B=\begin{pmatrix} -i & 0 & 0 & 0\\ 0 & i & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & i\end{pmatrix}$$

then $A^2\leq^* B^2$,but not $A\leq^* B$.

Let

$$A=\begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix},\quad B=\begin{pmatrix} 1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -1\end{pmatrix}$$

then $A\leq^* B$, but not $A^2\leq^* B^2$.

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Characterizations of $A \leq^{*} B$.

Hartwing and styan [4,theorem 2] presented eleven characterizations of $A \leq^{*} B$ for general matrices. One of them uses singular value decompositions. In the case of s-k normal matrices, spectral decompositions are more accessible.

Theorem:

Let A and B be s-k normal matrices with $1 \leq$ rank(A) $<$ rank(B). Then the following conditions are equivalent.

(a) $A \leq^{*} B$

(b) There is a s-k unitary matrix U such that $(KVU^{*}VK)AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, $(KVU^{*}VK)BU = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix}$ where D is a non-singular s-k diagonal matrix and $E \neq 0$ is a s-k unitary matrix.

(c) There is a s-k unitary matrix U such that

$(KVU^{*}AU)AU = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, $(KVU^{*}VK)BU = \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix}$

Where F is a non singular square matrix and $G \neq 0$.

(d) If a s-k unitary matrix U satisfies $(KVU^{*}VK)AU = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, $(KVU^{*}VK)BU = \begin{pmatrix} F' & 0 \\ 0 & G \end{pmatrix}$,where F is a non-singular square matrix. F' is a square matrix of the same dimension and $G \neq 0$, then $F = F'$.

(e) If a s-k unitary matrix U satisfies

$(KVU^{*}VK)AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, $(KVU^{*}VK)BU = \begin{pmatrix} D' & 0 \\ 0 & 0 \end{pmatrix}$,

Where D is a non-singular s-k diagonal matrix, D' is a s-k diagonal matrix of the same dimension and $E \neq 0$ is a s-k diagonal matrix, then $D = D'$.

(f) If a s-k unitary matrix U satisfies $(KVU^{*}VK)AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$,

where D is a non-singular s-k diagonal matrix then

$(KVU^{*}VK)BU = \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix}$, where $G \neq 0$.

(g) All s-k eigen vectors corresponding to non-zero s-k eigen values of A are s-k eigen vector of B corresponding to the same s-k eigen values.

Proof:

We prove this theorem in four parts.

Part(1): (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a)

(a) $\Rightarrow$ (b): Assume (a). Then by the normality A^{*} and B commutes and therefore simultaneously s-k diagonalizable. Since A^{*} and A have the same eigen vectors also Aand B are simultaneously s-k diagonalizable. Then A and B commutes.

Suppose, let D' be a s-k diagonable matrix of B there exist a s-k unitary matrix U such that

$$A = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(KVU^* VK)$$

$$B = U\begin{pmatrix} D' & 0 \\ 0 & E \end{pmatrix}(KVU^* VK)$$

where D is a non-singular s-k diagonal, $E \neq 0$. Now,

$$A^* = (KVU\,VK)\begin{pmatrix} D^* & 0 \\ 0 & 0 \end{pmatrix}U^*$$

Therefore,

$$A^*A = (KVU\,VK)\begin{pmatrix} D^* & 0 \\ 0 & 0 \end{pmatrix}U^*U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(KVU^* VK)$$

$$= (KVUVK)\begin{pmatrix} D^*D & 0 \\ 0 & 0 \end{pmatrix}(KV\,U^* VK) \qquad \ldots\ldots\ldots..\ldots\ldots..(1)$$

and

$$= (KVUVK)\begin{pmatrix} D^*D' & 0 \\ 0 & 0 \end{pmatrix}(KV\,U^* VK) \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

From (1) and (2), we get

$A^*A = A^*B \Rightarrow D^*D = D^*D' \Rightarrow D = D'$ Therefore, D is non-singular s-k diagonal matrix and D' is a s-k diagonal matrix of same dimension and $E \neq 0$.

(b) $\Rightarrow$ (c) : Trivial.

(c) $\Rightarrow$ (a) : Direct calculation.

Part 2: (a) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a)

This is a trivial modification of part-I.

Part 3:

(b) $\Rightarrow$ (f) : Assume(b),

Let U be a s-k unitary matrix satisfies

$$(KVU^* VK)\,AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

By (b), there exists a s-k unitary matrix W such that

$$(KVW^* VK)\,AW = \begin{pmatrix} D' & 0 \\ 0 & 0 \end{pmatrix}, \quad (KVW^* VK)\,BW = \begin{pmatrix} D' & 0 \\ 0 & E \end{pmatrix}$$

where D' is a non-singular s-k diagonal matrix and $E \neq 0$ is a s-k diagonal matrix. Interchanging the columns of W if necessary, we assume $D' = D$.

Let $U = (U_1, U_2)$ be such a partition that,

$$(KVU^* VK)\,AU = \left(KV\begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix}VK\right)A(U_1, U_2) =$$

$$\begin{pmatrix} (KVU_1^* VK)\,AU_1 & (KVU_1^* VK)\,AU_2 \\ (KVU_2^* VK)\,AU_1 & (KVU_2^* VK)\,AU_2 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \qquad \ldots\ldots..(3)$$

Then, for the corresponding partition W= (W_1, W_2) we have,

$$(KVW^{*}VK)AW = \left(KV\begin{pmatrix}W_1^{*}\\ W_2^{*}\end{pmatrix}VK\right)A(W_1, W_2) =$$

$$\begin{pmatrix}(KVW_1^{*}VK)AW_1 & (KVW_1^{*}VK)AW_2\\ (KVW_2^{*}VK)AW_1 & (KVW_2^{*}VK)AW_2\end{pmatrix} = \begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix} \qquad \ldots\ldots(4)$$

and

$$(KVW^{*}VK)BW = \left(KV\begin{pmatrix}W_1^{*}\\ W_2^{*}\end{pmatrix}VK\right)B(W_1, W_2) =$$

$$\begin{pmatrix}(KVW_1^{*}VK)BW_1 & (KVW_1^{*}VK)BW_2\\ (KVW_2^{*}VK)BW_1 & (KVW_2^{*}VK)BW_2\end{pmatrix} = \begin{pmatrix}D & 0\\ 0 & E\end{pmatrix} \qquad \ldots\ldots(5)$$

Noting that, (4) $\Rightarrow$

$$A = W\begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix}(KVW^{*}VK)$$

$$\Rightarrow A = (W_1\ \ W_2)\begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix}\left(KV\begin{pmatrix}W_1^{*}\\ W_2^{*}\end{pmatrix}VK\right)$$

$$= (W_1 D\ \ \ 0)\begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix}\left(KV\begin{pmatrix}W_1^{*}\\ W_2^{*}\end{pmatrix}VK\right)$$

$$= (W_1 D\ \ \ 0)\begin{pmatrix}KVW_1^{*}VK\\ KVW_2^{*}VK\end{pmatrix}$$

$$A = W_1 D(KV\,W_1^{*}VK)$$

(5) $\Rightarrow$

$$B = W\begin{pmatrix}D & 0\\ 0 & E\end{pmatrix}(KVW^{*}VK)$$

$$(KVU^{*}VK)BU = (KVU^{*}VK)W\begin{pmatrix}D & 0\\ 0 & E\end{pmatrix}(KVW^{*}VK)U$$

$$= \left(KV\begin{pmatrix}U_1^{*}\\ U_2^{*}\end{pmatrix}VK\right)(W_1\ \ W_2)\begin{pmatrix}D & 0\\ 0 & E\end{pmatrix}\left(KV\begin{pmatrix}W_1^{*}\\ W_2^{*}\end{pmatrix}VK\right)(U_1\ \ U_2)$$

$$= \begin{pmatrix}(KVU_1^{*}VK)W_1 & (KVU_1^{*}VK)W_2\\ (KVU_2^{*}VK)W_1 & (KVU_2^{*}VK)W_2\end{pmatrix}\begin{pmatrix}D(KVW_1^{*}VK)U_1 & D(KVW_1^{*}VK)U_2\\ E(KVW_2^{*}VK)U_1 & E(KVW_2^{*}VK)U_2\end{pmatrix}$$

By the normality condition,

$$= \begin{pmatrix}(KVU_1^{*}VK)W_1 & 0\\ 0 & (KVU_2^{*}VK)W_2\end{pmatrix}\begin{pmatrix}D(KVW_1^{*}VK)U_1 & 0\\ 0 & E(KVW_2^{*}VK)U_2\end{pmatrix}$$

$$= \begin{pmatrix}(KVU_1^{*}VK)W_1 D(KVW_1^{*}VK)U_1 & 0\\ 0 & (KVU_2^{*}VK)W_2 E(KVW_2^{*}VK)U_2\end{pmatrix}$$

$$= \begin{pmatrix} (KVU_1^* VK) A U_1 & 0 \\ 0 & (KVU_2^* VK) W_2 E (KVW_2^* VK) U_2 \end{pmatrix}$$

Since $(KVU_1^* VK) A U_1 = D$,

let us take $(KVU_2^* VK) W_2 E (KVW_2^* VK) U_2 = G$, $G \neq 0$.

Therefore $(KVU^* VK) BU = \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix}$ where D is a non-singular and $G \neq 0$.

Hence the proof (f).

(f) $\Rightarrow$ (b):

Assume (f), Let U be a s-k unitary matrix such that $(KVU^* VK) AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ where D is a non-singular s-k diagonal matrix.

Then by (f), $(KVU^* VK) BU = \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix}$ where $G \neq 0$. Since G is s-k normal, there exists a s-k unitary matrix X such that $E = (KVW^* VK) GW$ is a s-k diagonal matrix.

Let W= $UKV \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \Rightarrow W^* = \begin{pmatrix} I & 0 \\ 0 & X^* \end{pmatrix} VKU^*$

Then , $(KVW^* VK) AW = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ and $(KVW^* VK) BW = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix}$

Hence the proof (b).

Part 4: (a) $\Leftrightarrow$ (g)

Proof is obvious.

Theorem:

Let A and B be s-k normal matrices. If $A \leq^* B$ then AB=BA.

Proof:

Let A and B be s-k normal matrices,

then $A(KVA^* VK) = (KVA^* VK) A$ and

$B(KVB^* VK) = (KVB^* VK) B$.

Let us take $(KVU^* VK) AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ and

$$(KVU^* VK) BU = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix}$$

$\Rightarrow$ $A = (KV\,UVK) \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} U^*$ and

$$B = (KVUVK)\begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} U^*$$

$$\Rightarrow \quad A^* = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(KVU^*VK)$$

Therefore , $A^*A = A^*B = U\begin{pmatrix} D^2 & 0 \\ 0 & 0 \end{pmatrix}U^* \Rightarrow A \le^* B$

Similarly ,we can prove AB=BA.

Remark:

The converse does not hold(even assuming rank(A) $<$ rank(B), see Example(2.4). The normality assumption cannot be dropped out, see Example(2.5).

Example:

Let $A=\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then AB=BA and rank(A) $<$ rank(B), but $A <^* B$ does not hold. However $\frac{1}{2} A \le^* B$, which makes its look for a counter example such that $nA \le^* B$ does not hold for any $n \ne 0$.

It is easy to see that we must have $n \le 3$. The matrices

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ obviously have the required properties.

Relationship between $A \le^* B$ and $A^2 \le^* B^2$

We will see that $A \le^* B \Rightarrow A^2 \le^* B^2$ for s-k normal matrices, but the converse needs an extra condition, which we first present using s-k eigen values.

Theorem:

Let A and B be s-k normal matrices with $1 \le$ rank(A) $<$ rank(B). Then

(a) $A \le^* B$ is equivalent to the following.

(b) $A^2 \le^* B^2$ and if A and B have non-zero s-k eigen values α and respectively β such that α^2 and β^2 are s-k eigen values of A^2 and B^2 respectively with a common s-k eigen vector X, then $\alpha = \beta$ and Y is a common eigen vector of A and B.

Proof:

Assume(a); $A \le^* B \Rightarrow A^*A = A^*B$. Let U be a s-k unitary matrix such that $(KVU^*VK)AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ and $(KVU^*VK)BU = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix}$ and also by the theorem (2.1) of (b)

$$(KVU^*VK)A^2U = \begin{pmatrix} D^2 & 0 \\ 0 & 0 \end{pmatrix} \text{ and .. } (KVU^*VK)B^2U = \begin{pmatrix} D^2 & 0 \\ 0 & E^2 \end{pmatrix}.$$

Let a and b have non-zero s-k eigen values of A and B respectively. Therefore a^2 and b^2 have non-zero s-k eigen values of A^2 and B^2 respectively. Suppose X be the common s-k eigen vector of A^2 and B^2, then $a = b$ and X is a common eigen vector of A and B.

Conversely, Assume (b), Then $(KVU^*VK)A^2U = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ and $(KVU^*VK)B^2U = \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix}$ where U, D, G are matrices obtained by applying(b) of Theorem(2.1) to A^2 and B^2.

Let $u_{sk(1)}, u_{sk(2)} \ldots\ldots\ldots u_{sk(n)}$ be the column vectors of U and denote r=rank(A).

For i=1,2,3…….r we have $A^2 u_{sk(i)} = B^2 u_{sk(i)} = \delta_{sk(i)} u_{sk(i)}$ where $\delta_{sk(i)} = \text{diag}\, D$.

So by the second part of (b), these exist complex numbers $d_{sk(1)}, d_{sk(2)} \ldots\ldots d_{sk(r)}$ such that, for all i=1,2,3,…..r we have $d^2_{sk(i)} = \delta_{sk(i)}$ and $Au_{sk(i)} = Bu_{sk(i)} = \delta_{sk(i)} u_{sk(i)}$.

Let D be the s-k diagonal matrix with $d_{sk(i)} = \text{diag}\, D$. For i=r+1,r+2,…n.

We have $B^2 u_{sk(i)} = \gamma_{sk(i-r)} u_{sk(i)}$ where $\gamma_{sk(i)} = \text{diag}\, G$.

Take complex numbers $e_{sk(1)}, e_{sk(2)}, \ldots\ldots\ldots e_{sk(n-r)}$ satisfying $e^2_{sk(i)} = \gamma_{sk(i)}$ for i=1,2,…..n-r.

Let E be the s-k diagonal with $e_{sk(i)} =$ diag(E).

Then $(KVU^*VK)BU = \begin{pmatrix} DH & 0 \\ 0 & 0 \end{pmatrix}$, $(KVU^*VK)BU = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix}$

This equation satisfies condition(a). Therefore $A^2 \le^* B^2 \Rightarrow A \le^* B$.

Corollary:

Let A and B be s-k normal matrices whose all s-k eigen values have non-negative real parts. Then $A^2 \le^* B^2$ iff $A \le^* B$.

Theorem:

Let A and B be s-k normal matrices with $1 \le$ rank(A)<rank(B). Then

(a) $A \le^* B$ is equivalent to the following

(b) $A^2 \le^* B^2$ and if $(KVU^*VK)AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, $(KVU^*VK)BU = \begin{pmatrix} DH & 0 \\ 0 & 0 \end{pmatrix}$ where U is a s-k unitary matrix, D is a non-singular s-k diagonal matrix, H is a s-k unitary diagonal matrix and $E \neq 0$ is a s-k diagonal matrix then H =I.

Proof:

For[a] $\Rightarrow$ the first part of [b], see the proof of theorem(3.1).

For[a] $\Rightarrow$ the second part of [b], see(e) of theorem(2.1).

Conversely, Assume (b) As in the proof of theorem(3.1),

we have, $(KVU^*VK)A^2U = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ $(KVU^*VK)B^2U = \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix}$, Hence,

$$(KVU^*VK)AU=\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

$$(KVU^*VK)BU=\begin{pmatrix} D' & 0 \\ 0 & E \end{pmatrix}$$

Where D and D' are s-k diagonal matrices satisfying $D^2=(D')^2=V$ and E is a s-k diagonal matrices satisfying $E^2=G$.

Devoting $d_{sk(i)}$ =diag D, $d'_{sk(i)}$ =diag D'.

r=rank(A), are therefore have $d^2_{sk(i)}=(d'_{sk(i)})^2$, for all i=1,2,….r.

Hence there are complex numbers. $h_{sk(1)}h_{sk(2)}..........h_{sk(r)}$ such that $|h_{sk(1)}|=|h_{sk(2)}|.......=|h_{sk(r)}|$ and $d'_{sk(i)}=d_{sk(i)}h_{sk(i)}$ for all i=1,2,3,…..r. Let H be the s-k diagonal matrix with $h_{sk(i)}$=diag H. Then $D'=DH$ and so $D'=D$ by the second part of (b). Thus (b) of theorem(2.1) is satisfied and so (a) follows.

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