Application Functional equations on quasigroups

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ABSTRACT

In mathematics or its applications, a functional equation is an equation in terms of independent variables, and also unknown functions, which are to be solved for. Many properties of functions can be determined by studying the types of functional equations they satisfy. Usually the term *functional equation* is reserved for equations that are not in some simple sense reducible to algebraic equations, often because two or more known functions of the variables are substituted as arguments into an unknown function to be solved for. For example: The commutative and associative laws are functional equations. When the associative law is expressed in its familiar form, one lets some symbol between two variables represent a binary operation, thus:

$(a * b) * c = a * (b * c)$

But if we write $f(a, b)$ instead of $a * b$, then the associative law looks more like what one conventionally thinks of as a functional equation:

$f(f(a, b), c) = f(a, f(b, c))$We study generalized quadratic functional equations on quasigroups. These are equations $s = t$, where each variable appears exactly twice in $s = t$ and each Operational symbol is assumed to be a quasigroup operation on a (fixed) set. A fundamental problem in this class of equations is to investigate their structure and classify them accordingly.

Introduction

Quasi groups:

A set with a binary operation (usually called multiplication) in which each of the equations $ax = b$and ya=b has a unique solution, for any elements a,b of the set. A quasi-group with a unit is called a loop.

A quasi-group is a natural generalization of the concept of a group. Quasi-groups arise in various areas of mathematics, for example in the theory of projective planes, non-associative division rings, in a number of questions in combinatorial analysis, etc. The term "quasi-group" was introduced by R. Moufang; it was after her work on non-Desarguesian planes (1935), in which she elucidated the connection of such planes with quasi-groups that the development of the theory of quasi-groups properly began. A quasigroup is a natural generalization of the concept of group. Quasigroups Differ from groups in that they need not be associative

Definition :If $(S; \cdot)$ and $(T; \times)$ are quasigroups and $f, g, h : S \rightarrow T$ arebijections such that $f(xy) = g(x) \times h(y)$, then we say that $(S; \cdot)$ and $(T; \times)$ are *isotopic* and that (f, g, h) is an *isotopic*Isotopy is a generalization of isomorphism. The isotopic image of aquasigroup is again a quasigroup. Every quasigroup is isotopic to some loop. A loop isotopic to a group is isomorphic to it. If two quasigroups are isotopic, so are their Corresponding parastrophes

Definition :Two quasigroups are *isostrophic* if one of them is isotopic to a parastrophe of the other. All these relations are equivalences between quasigroups. Isomorphism is a finer relation than isotopy, which in turn is finer than isostrophy

Systems of quasi-groups and functional equations

Suppose that a system of quasi-groups is defined on a set Q In this case the operations are more conveniently denoted by letters: e.g., instead of ab=c one writes A(a,b) = c. The quasi-group operations on Q are assumed to be related in some way, most

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often by identities, called in this case "functional equations" . One can usually solve the problem of finding a system of quasi-groups on Q satisfying given functional equations. For example, the equation of general associativity has been solved; namely, it has proved that if four quasi-groups satisfy (1), then they are isotopic to one and the same group Q(.) , and the general solution is given by the equalities:

$$A_1 [A_2 (x, y), z] = A_3 [x, A_4 (y, z)] \qquad (1)$$

$$A_1 (x, y) = \alpha x. \beta y, \qquad A_2 (x, y) = \alpha^{-1} (\phi x. \psi y),$$

$$A_3 (x, y) = \phi x. \theta y, \qquad A_4 (x, y) = \theta^{-1} (\psi x. \beta y),$$

where $\alpha, \beta, \theta, \phi \cdot \varphi$ are arbitrary permutations of Q. The equation of general mediality is similarly solved:

$$A_1 [A_2 (x, y), A_3 (u, v)] = A_4 [A_5 (x, u), A_6 (y, v)] \ .$$

Here all six quasi-groups turn out to be isotopic to one and the same Abelian group.

n-quasi-groups.

A set with an n-ary operation is called an n-quasi-group if each of the equations

$$a_1 \ldots a_{i-1} x a_{i+1} \ldots a_n = b$$

(where $b, a_1, \ldots a_i \epsilon Q$, $i = 1, \ldots, n$) has a unique solution. The basic concepts (isotopy, parastrophy, etc.) of the theory of quasi-groups carry over to n-quasi-groups. Each n-quasi-group is isotopic to a certain n-loop (see Loop).

Certain classes of ordinary binary quasi-groups (such as the classes of medial and TS-quasi-groups, etc.) have an analogue in the n-ary case. An n-ary operation A is reducible if there exist two operations B, C, of arity at least 2, such that

$$A(x_1, \ldots, x_n) =$$
$$= B(x_1, \ldots, x_{i-1}, C(x_i, \ldots, x_j), x_{j+1}, \ldots, x_n)$$

(written $A = B \overset{i}{+} C$ for short). Otherwise A is said to be irreducible. An analogue of the theorem on the canonical factorization of positive integers into prime numbers holds for n-quasi-groups.

In thespecial case ofthe generalapplication thefunctional equation ofthe quasi-groups will discuss.

i)Let $s = t$ be a quadratic equation with n variables. The equation is fully determined by the binary tree of the equation $s = t$ and the order in which variables occur in the equation. The tree and the order are independent of each i) It is well known that the number of different binary trees with n leaves is

c_{n-1}, where c_n ($n \geq 0$) is the sequence of *Catalan numbers*. The sequence satisfies .The formula: $c_n = (2n)! / (n + 1)!n!$ And is denoted by A000108 in Sloane's "The On-Line Encyclopedia of Integer Sequences" [14]. The first ten members of the Sequence c_n ($0 \leq n \leq 9$) are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862.

ii) The order of variables in $s = t$ is determined by the word of length $2n$ in which every one of the letters $x_1, \ldots, x_n$ appears exactly twice. Let us denote by w_n the number of such words. It is easy to see that the sequence w_n satisfies the recurrence relation:

$w_1 = 1$, $w_n = n(2n - 1) w_{n-1}$

w_n is the Sloane sequence A000680, and the general formula $w_n = (2n)!/2n$.

iii) As noted before, the number of all quadratic equations with n variables is $E_n = T_n W_n = C_{2n-1} W_n = (4n - 2)!/2n(2n - 1)!$. The recurrence relation is $E_{n+1} = 2(16n2 - 1) E_n$ Combinatorial questions.

The multiplication table of a finite quasi-group, that is, its Cayley table, is known in combinatorics as a Latin square. One of the problems of the combinatorial theory of quasi-groups, finding systems of mutually orthogonal quasi-groups on a given set, is important for the construction of finite projective planes. Two quasi-groups A and B defined on a set Q are orthogonal if the system of

equations $A(x,y) = \alpha, B(x,y) = b$ has a unique solution for any α and b in Q. Orthogonality of finite quasi-groups is equivalent to that of their Latin squares. It has been proved that a system of mutually orthogonal quasi-groups defined on a set of n elements cannot contain more than $n-1$ quasi-groups.

Another combinatorial concept related to that of a quasi-group is that of a full permutation. A permutation ϕ of a quasi-group $Q(.)$ is said to be full if the mapping $\phi' \to \phi x \phi$ is also a permutation of Q. Not every quasi-group has a full permutation. A quasi-group admitting a full permutation is called admissible. For an admissible group there exists a quasi-group orthogonal to it and conversely: If a group has a quasi-group orthogonal to it, then the group is admissible. If a finite quasi-group of order n is admissible, then one can obtain from it a quasi-group of order $n+1$ by a special process (extension).

There are at least two equivalent formal definitions of quasigroup. One definition casts quasigroups as a set with one binary operation, and the other is a version from universal algebra which describes a quasigroup by using three primitive operations. We begin with the first definition, which is easier to follow.

A quasigroup $(Q, *)$ is a set Q with a binary operation * (that is, a magma), obeying the Latin square property. This states that, for each a and b in Q, there exist unique elements x and y in Q such that:

$a * x = b$

$Y * a = b$

(In other words: Each element of the set occurs exactly once in each row and exactly once in each column of the quasigroup's multiplication table, or Cayley table. This property ensures that the Cayley table of a finite quasigroup is a Latin square.) The unique solutions to these equations are written $x = a \backslash b$ and $y = b / a$. The operations '\' and '/' are called, respectively, left and right division.

The empty set equipped with the empty binary operation satisfies this definition of a quasigroup. Some authors accept the empty quasigroup but others explicitly exclude it

A quasigroup $(Q, *, \backslash, /)$ is a type $(2, 2, 2)$ algebra satisfying the functional equations identitie:

$y = x * (x \backslash y)$

$y = x \backslash (x * y)$

$y = (y / x) * x$

$y = (y * x) / x$

Hence if $(Q, *)$ is a quasigroup according to the first definition, then $(Q, *, \backslash, /)$ is the same quasigroup in the sense of universal algebra.

A right-quasigroup $(Q, *, /)$ is a type $(2, 2)$ algebra satisfying the identities:

$y = (y / x) * x$

$y = (y * x) / x$

Similarly, a left-quasigroup $(Q, *, \backslash)$ is a type $(2, 2)$ algebra satisfying the identities:

$y = x * (x \backslash y)$

$y = x \backslash (x * y)$

Result:

1)-The set of all solutions of the functional equation of generalized pseudo-mediality over the set of quasigroup operations on an arbitrary set A is described by the realtions$\Longrightarrow$The set of all solutions of the functional equation of generalized associativity over the set of quasigroup operations on an arbitrary set A is described by the relations

2)-The total number of generalized quadratic quasigroup functional equations with n variables is

$$En \leq \left(\binom{4n-2}{2n-1}\right)\frac{1}{2^n} < \binom{4n-2}{2n-1}$$

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