Available online at www.elixirpublishers.com (Elixir International Journal)

Advances in Pure Mathematics

Elixir Adv. Pure Math. 70C (2014) 24188-24193

ISSN: 2229-712X

Approximation of Summation Formula of the Series Involving a Set of Polynomials and its Applications

Hemant Kumar

Department of Mathematics, D. A-V. (P. G.) College Kanpur-208001,(U. P.) India.

ARTICLE INFO

Article history:

Received: 28 March 2014;

Received in revised form: 25 April 2014;

Accepted: 10 May 2014;

Keywords

A set of polynomials,

Bosanquet and Kestelman convergence theorem,

Approximation formula,

Generating functions.

ABSTRACT

In this paper, we define a set of polynomials and two dimensional gamma density function to obtain its expectation formula. Again, we derive the generalized Bosanquet and Kestleman convergence theorem and then use it to evaluate an approximation formula of the series involving this set of polynomials. Finally, we make an application of this approximation formula to derive the generating functions of various hypergeometric functions.

Introduction

From many years ago, summability of the series has been studied by many authors (for example see, Prasad [10], Bhatt [1], Srivastava [13], Pathan [8]). Here, we make an interest of this subject to derive generalized Bosanquet and Kestleman convergence theorem [2] due to Kumar and Yadav [5], and then use it to obtain the approximation formula of the series involving a set of polynomials. Again we use this approximation formula to find out the known and unknown generating functions.

In this research work we consider following formulae:

The two dimensional gamma density function is defined by (see, Jain [3])

$$f(u,v)=\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'},0\le u<\infty,0\le v<\infty,\Re(\alpha)>0,$$

$\Re e(\beta')>0,\Re e(\alpha+\beta'+\mu')>0,\lambda>0$, here, the beta function is $B(\alpha,\beta')=\frac{\Gamma(\alpha)\Gamma(\beta')}{\Gamma(\alpha+\beta')},\Re(\alpha)>0,\Re(\beta')>0,$

and $\Gamma(\alpha),\Re(\alpha)>0$, is the gamma function.

Otherwise, $f(u,v)=0$. (1)

For other properties of beta and gamma functions see Rainville (11).

Then, the distribution DF with respect to two dimensional gamma density function defined in Eqn. (1) is given by (see Kanwal [4], p. 391)

$$DF=f(u,v)dudv \tag{2}$$

The generalization of Bosanquet and Kestleman convergence theorem [2] is presented by (also see, Kumar and Yadav [5], and Kumar, Pathan and Srivastava [6]):

Suppose that $(x_1,\dots,x_k)$ is a k-dimensional random variable with density $f(x_1,\dots,x_k)$ and the sequence of functions $g_n(x_1,\dots,x_k)$ is measurable in the region $-\infty<x_1<\infty$, ...,, $-\infty<x_k<\infty,(\forall\, n=0,1,2,3,\dots)$ and if

$$\sum_{n=0}^{\infty}\left|\int_{-\infty}^{\infty}.(k).\int_{-\infty}^{\infty}g_n(x_1,\dots,x_k)f(x_1,\dots,x_k)dx_1\dots dx_k\right|\le\eta \tag{3}$$

Tele:+9105122683120

E-mail addresses: palhemant2007@rediffmail.com

Then, $\sum_{n=0}^{\infty}|g_n(x_1,\dots,x_k)| \le \eta$ (4)

Here, every $(x_1,\dots,x_k) \in \mathbb{R}^k$ ($\mathbb{R}$ is set of real numbers) and η is absolutely constant.

The expectation of function $G(u,v)$ due to gamma distribution $f(u,v)$, defined in Eqn. (1), is given by (see, Kanwal [4])

$$\langle G(u,v)\rangle = \int_{\mathbb{R}^2} G(u,v)f(u,v)dudv, 0 \le u < \infty, 0 \le v < \infty. \tag{5}$$

In our investigations, we define a set of polynomials

$$H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{\Gamma(\alpha+\beta'+\mu'+\beta n+1+(\mu+m)k)}\delta_k x^k,$$

$n \ge 0, m > 0, a\ bounded\ sequence\ \delta_k (k \ge 0)$ and $\alpha,\beta,\beta',\mu,\mu'$ be arbitrary fixed numbers. (6)

Main Theorems

Theorem 1

If $0 \le u < \infty, 0 \le v < \infty, \Re(\alpha) > 0, \Re(\beta') > 0, \Re e(\alpha+\beta'+\mu') > 0, \lambda > 0, \mu \ge 0, n \ge 0, m > 0$, and then due to the two dimensional gamma density function given in Eqn. (1), the expectation $\langle \frac{\{\lambda(u+v)\}^{\beta n+n+1}}{n!} H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x(u+v)^{\mu})\rangle$ exists and is equals to $G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}\left(\frac{x}{\lambda^{\mu}}\right)$,

where $G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x)$ is other set of polynomials defined by

$$G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x) = \frac{1}{\Gamma(\alpha+\beta'+\mu')}\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}\Gamma(\alpha+\beta'+\mu'+\beta n+n+1+\mu k)}{\Gamma(\alpha+\beta'+\mu'+\beta n+1+(\mu+m)k)}\delta_k x^k. \tag{7}$$

Proof: Make an appeal to the Eqns. (1), (5) and (6), we get the expectation

$$\langle \frac{\{\lambda(u+v)\}^{\beta n+n+1}}{n!} H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x(u+v)^{\mu})\rangle = \frac{(\lambda)^{\alpha+\beta'+\mu'+\beta n+n+1}}{n!B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')}\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{\Gamma(\alpha+\beta'+\mu'+\beta n+1+(\mu+m)k)}\delta_k x^k$$
$$\times \int_0^{\infty}\int_0^{\infty} e^{-\lambda(u+v)} u^{\alpha-1} v^{\beta'-1}(u+v)^{\mu'+\beta n+n+1+\mu k} dudv \tag{8}$$

Now in right hand side of Eqn. (8) use the integral formula due to Jain [3] (see also, Srivastava and Manocha [16, p. 252]), we obtain the result (7).

Theorem 2

If $0 \le u < \infty, 0 \le v < \infty, \Re e(\alpha) > 0, \Re e(\beta') > 0, \Re e(\alpha+\beta'+\mu') > 0, \lambda > 0, \mu \ge 0, n \ge 0, m > 0$, and then for $|T| < 1$, and for the two dimensional gamma density function given in Eqn. (1), a series involving a set of polynomials defined in (6) is summable and has the approximation formula (this theorem is the generalization of the Bosanquet and Kestleman theorem [2])

$$\sum_{n=0}^{\infty}\left|\frac{T^n\{\lambda(u+v)\}^{\beta n+n+1}}{n!} H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x(u+v)^{\mu})\right| \le \left|\frac{1}{\Gamma(\alpha+\beta'+\mu')}\right| \left|\frac{(1+\zeta)^{1+\alpha+\beta'+\mu'}}{1-\beta\zeta}\right| \left|\Psi\left[x(-\zeta)^m\left\{\frac{1}{\lambda}(1+\zeta)\right\}^{\mu}\right]\right|,$$

where, $\zeta = \zeta(T) = T(1+\zeta)^{\beta+1}, \zeta(0) = 0,\ \Psi(x) = \sum_{k=0}^{\infty}\delta_k x^k, \delta_0 \ne 0.$ (9)

Proof: Make an appeal to the Theorem 1, and we have

$$\sum_{n=0}^{\infty}\langle \frac{\{\lambda(u+v)\}^{\beta n+n+1}}{n!} H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x(u+v)^{\mu})\rangle T^n$$
$$= \sum_{n=0}^{\infty}\frac{(T)^n}{\Gamma(\alpha+\beta'+\mu')}\sum_{k=0}^{\left[\frac{n}{m}\right]}(-1)^{mk}\binom{\alpha+\beta'+\mu'+(\beta+1)n+\mu k}{n-mk}\delta_k\left(\frac{x}{\lambda^{\mu}}\right)^k$$
$$= \frac{1}{\Gamma(\alpha+\beta'+\mu')}\sum_{k=0}^{\infty}\delta_k\left(\frac{x(-T)^m}{\lambda^{\mu}}\right)^k\sum_{n=0}^{\infty}\binom{\alpha+\beta'+\mu'+\mu k+(\beta+1)mk+(\beta+1)n}{n}(T)^n \tag{10}$$

Now in right hand side of series (10) set $\zeta = \zeta(T) = T(1+\zeta)^{\beta+1}, \zeta(0) = 0$ and apply the result due to Polya and Szego [9, p. 349, Problem 216] with $\Psi(x) = \sum_{k=0}^{\infty}\delta_k x^k, \delta_0 \ne 0.$ we get

$$\sum_{n=0}^{\infty}\langle \frac{T^n\{\lambda(u+v)\}^{\beta n+n+1}}{n!} H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x(u+v)^{\mu})\rangle = \left(\frac{1}{\Gamma(\alpha+\beta'+\mu')}\right)\left(\frac{(1+\zeta)^{1+\alpha+\beta'+\mu'}}{1-\beta\zeta}\right)\Psi\left[x(-\zeta)^m\left\{\frac{1}{\lambda}(1+\zeta)\right\}^{\mu}\right] \tag{11}$$

Then, apply the definitions (1), (5) and (6) in Eqn. (11), that may be written by

$$\lim_{N\to\infty}\left|\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}\int_0^{\infty}\int_0^{\infty} e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'}\right.$$

$$\times \sum_{n=0}^{N} \frac{T^n\{\lambda(u+v)\}^{\beta n+n+1}}{n!} H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x(u+v)^{\mu})\, dudv \Big| =$$

$$\Big|\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}\int_0^{\infty}\int_0^{\infty} e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'}$$

$$\times\left(\frac{1}{\Gamma(\alpha+\beta'+\mu')}\right)\left(\frac{(1+\zeta)^{1+\alpha+\beta'+\mu'}}{1-\beta\zeta}\right)\Psi\left[x(-\zeta)^m\left\{\frac{1}{\lambda}(1+\zeta)\right\}^{\mu}\right]dudv\Big| \qquad (12)$$

Now, in Eqn. (12) set

$$F(x,\zeta)=\left(\frac{1}{\Gamma(\alpha+\beta'+\mu')}\right)\left(\frac{(1+\zeta)^{1+\alpha+\beta'+\mu'}}{1-\beta\zeta}\right)\Psi\left[x(-\zeta)^m\left\{\frac{1}{\lambda}(1+\zeta)\right\}^{\mu}\right],$$

and

$$K_N(x;u,v)=\sum_{n=0}^{N}\frac{T^n\{\lambda(u+v)\}^{\beta n+n+1}}{n!}H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x(u+v)^{\mu})$$

and then for $|T|<1$and the two dimensional gamma density function given in Eqn. (1), there exists an equality

$$\lim_{N\to\infty}\Big|\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}\int_0^{\infty}\int_0^{\infty} e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'}\{K_N(x;u,v)-F(x,\zeta)+F(x,\zeta)\}dudv\Big|$$

$$=\Big|\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}\int_0^{\infty}\int_0^{\infty} e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'}F(x,\zeta)dudv\Big| \qquad (13)$$

The Eqn. (13) implies an inequality given by

$$0=\lim_{N\to\infty}\Big|\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}\int_0^{\infty}\int_0^{\infty} e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'}\{K_N(x;u,v)-F(x,\zeta)+F(x,\zeta)\}dudv\Big|$$

$$-\Big|\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}\int_0^{\infty}\int_0^{\infty} e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'}F(x,\zeta)dudv\Big|$$

$$\le\lim_{N\to\infty}\Big|\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}\int_0^{\infty}\int_0^{\infty} e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'}\{K_N(x;u,v)-F(x,\zeta)\}dudv\Big| \qquad (14a)$$

Again, the inequality (14a) on application of the theorem for integration of series of positive terms (see, Natanson, [7]) and with help of Eqns. (3) and (4) gives a result equivalent to generalized Bosanquet and Kestleman convergence theorem due to Kumar and Yadav [5] in the form

$$\lim_{N\to\infty}\Big|\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}\int_0^{\infty}\int_0^{\infty} e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'}\{K_N(x;u,v)-F(x,\zeta)\}dudv\Big|$$

$$\le\lim_{N\to\infty}\Big|\left[B(\alpha,\beta')\Gamma(\alpha+\beta'+\mu')(\lambda)^{-\alpha-\beta'-\mu'}\right]^{-1}\Big|\int_0^{\infty}\int_0^{\infty}\left|e^{-\lambda(u+v)}u^{\alpha-1}v^{\beta'-1}(u+v)^{\mu'}\right|\left|\{K_N(x;u,v)-F(x,\zeta)\}\right||dudv|$$

$$\le 0.$$

If $\lim_{N\to\infty}|\{K_N(x;u,v)-F(x,\zeta)\}|\le 0.$ (14b)

Then, from the result (14b) with above definitions of $F(x,\zeta)$ and $K_N(x;u,v)$, we find the result of theorem 2.

Applications

In this section, we apply following Lebesgue convergence theorem to obtain generating functions:

Theorem 3. From the inequality given in Eqn. (9), there exists a generating function

$$\sum_{n=0}^{\infty}\frac{T^n}{n!}G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}\left(\frac{x}{\lambda^{\mu}}\right)=\left(\frac{1}{\Gamma(\alpha+\beta'+\mu')}\right)\left(\frac{(1+\zeta)^{1+\alpha+\beta'+\mu'}}{1-\beta\zeta}\right)\Psi\left[x(-\zeta)^m\left\{\frac{1}{\lambda}(1+\zeta)\right\}^{\mu}\right],$$

where, $\zeta=\zeta(T)=T(1+\zeta)^{\beta+1}, \zeta(0)=0,$ ***and*** $\Psi(x)=\sum_{k=0}^{\infty}\delta_k x^k, \delta_0\ne 0,$ **and** $G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x)$ is defined in Eqn. (7) (15)

Proof: The inequality (9) implies that

$$\sum_{n=0}^{\infty}\left|\frac{T^n\{\lambda(u+v)\}^{\beta n+n+1}}{n!}H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x(u+v)^{\mu})\right|\le\left|\frac{1}{\Gamma(\alpha+\beta'+\mu')}\right|\left|\frac{(1+\zeta)^{1+\alpha+\beta'+\mu'}}{1-\beta\zeta}\right|\left|\Psi\left[x(-\zeta)^m\left\{\frac{1}{\lambda}(1+\zeta)\right\}^{\mu}\right]\right|$$

$$with, \zeta = \zeta(T) = T(1+\zeta)^{\beta+1}, \zeta(0) = 0, and\ \Psi(x) = \sum_{k=0}^{\infty} \delta_k x^k, \delta_0 \neq 0,$$

and then apply Lebesgue convergence theorem with distribution DF given in Eqn. (2), we get the generating function

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \frac{T^n \{\lambda(u+v)\}^{\beta n+n+1}}{n!} H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x(u+v)^{\mu}) DF = \frac{1}{\Gamma(\alpha+\beta'+\mu')} \frac{(1+\zeta)^{1+\alpha+\beta'+\mu'}}{1-\beta\zeta} \Psi\left[x(-\zeta)^m \left\{\frac{1}{\lambda}(1+\zeta)\right\}^{\mu}\right] \int_{\mathbb{R}^2} DF \quad (16)$$

Then use Eqns. (2) and (7) in that of (16), we get the result (15).

Special Cases

In theorem 3 set $\alpha+\beta'+\mu' = a > 0, \lambda = 1, \mu = 0, \delta_k = \frac{\delta_k}{k!}$, it becomes a result equivalent to Srivastava and Manocha [16, theorem 2, p.359]. (17)

Again set $\alpha+\beta'+\mu' = a > 0, \lambda = 1, \mu = 0, \delta_k = \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j k!}$, in theorem 3, then by Eqn. (7) we have

$$\frac{1}{n!} G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x) = \frac{1}{\Gamma(a)} \binom{a+(\beta+1)n}{n} {}_{p+m}F_{q+m}\left[\begin{matrix} \Delta(m;-n), (a_p); \\ \Delta(m; a+\beta n+1), (b_q); \end{matrix} x\right],$$

$\Delta(m;a)$ **abbreviates the array** $\frac{a}{m}, \frac{a+1}{m}, \dots, \frac{a+m-1}{m}$, $m \geq 1$, and then we get a generating function equivalent to Srivastava [14] in the form

$$\sum_{n=0}^{\infty} \frac{T^n}{n!} G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x) = \frac{1}{\Gamma(a)} \frac{(1+\zeta)^{1+a}}{1-\beta\zeta} {}_pF_q\left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta)^m\right] \quad (18)$$

Again then, put $\beta = 1$ in theorem 3, we get the quadratic equation $\zeta^2 + \left(2 - \frac{1}{T}\right)\zeta + 1 = 0$ which gives us $\zeta = \frac{(1-2T) \pm \sqrt{(1-4T)}}{2T}$, now set $\alpha+\beta'+\mu' = a > 0, \lambda = 1, \beta = 1, \mu = r-1, \delta_k = \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j k!}$, r **being a positive integer, and now take** $\zeta = \frac{(1-2T) - \sqrt{(1-4T)}}{2T}$, we get

$$\frac{1}{n!} G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x) = \frac{1}{\Gamma(a)} \binom{a+2n}{n} {}_{p+m+r-1}F_{q+m+r-1}\left[\begin{matrix} \Delta(m;-n), (a_p), \Delta(r-1; a+2n+1); \\ \Delta(r+m-1; a+n+1), (b_q); \end{matrix} x \frac{m^m (r-1)^{r-1}}{(r+m-1)^{r+m-1}}\right] \quad (19)$$

and therefore we find a generating function equivalent to the Srivastava and Buschman [15, p. 363]

$$\sum_{n=0}^{\infty} \frac{T^n}{n!} G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x) = \frac{1}{\Gamma(a)} \left(\frac{1}{\sqrt{1-4T}}\right) \left(\frac{2}{1+\sqrt{1-4T}}\right)^a {}_pF_q\left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-T)^m \left\{\frac{2}{1+\sqrt{1-4T}}\right\}^{2m+r-1}\right] \quad (20)$$

Now, in Eqns. (19) and (20), put $r = m$ *and then take* $m \to 1$, again replace a *by* $a-1$, we get the generating function studied by Shively [12, p. 55, Eqn. (48)] (see also Rainville [11, p. 298, Eqn. (5)] in the form

$$\sum_{n=0}^{\infty} \frac{T^n}{n!} \frac{(a)_{2n}}{(a)_n} {}_{p+1}F_{q+1}\left[\begin{matrix} -n, (a_p); \\ a+n, (b_q); \end{matrix} x\right] = \left(\frac{1}{\sqrt{1-4T}}\right) \left(\frac{2}{1+\sqrt{1-4T}}\right)^{a-1} {}_pF_q\left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} \frac{-4xT}{\left(1+\sqrt{1-4T}\right)^2}\right] \quad (21)$$

Further, in theorem 3, set $\alpha+\beta'+\mu' = a > 0, \lambda = 1, \beta = 1, \mu = r-1, \delta_k = \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j k!}$, r **being a positive integer, and now take** $\zeta = \frac{(1-2T) + \sqrt{(1-4T)}}{2T}$, we get

$$\frac{1}{n!} G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x) = \frac{1}{\Gamma(a)} \binom{a+2n}{n} {}_{p+m+r-1}F_{q+m+r-1}\left[\begin{matrix} \Delta(m;-n), (a_p), \Delta(r-1; a+2n+1); \\ \Delta(r+m-1; a+n+1), (b_q); \end{matrix} x \frac{m^m (r-1)^{r-1}}{(r+m-1)^{r+m-1}}\right] \quad (22)$$

Therefore, we find another form of generating function equivalent to the Srivastavaand Buschman [15, p. 363]

$$\sum_{n=0}^{\infty}\frac{T^n}{n!}G_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x)=\frac{1}{\Gamma(a)}\left(\frac{-1}{\sqrt{1-4T}}\right)\left(\frac{2}{1+\sqrt{1-4T}}\right)^{a}\ {}_pF_q\left[\begin{matrix}(a_p);\\(b_q);\end{matrix}x(-T)^m\left\{\frac{2}{1+\sqrt{1-4T}}\right\}^{2m+r-1}\right] \tag{23}$$

References

[1] R. C. Bhatt, A summation formula for Appell's function F_2,Israel J. Math. 3 (1965), 87-88.

[2] L. S. Bosanquet, and H. Kestelman, The absolute convergence of series of integrals, Proceedings of the London Mathematical Society,Vol. 2, no. 1, (January 1, 1939) pp. 88-97.

[3]R. N. Jain, Some double integral transformations of certain hypergeometric functions, Math. Japan, 10 (1965), 17-26.

[4]R. P. Kanwal, Generalized Functions: Theory and Technique, Academic Press Inc., New York, 1983, p. 391.

[5] H. Kumar and P. Yadav, On summability of the series involving Exton's quadruple hypergeometric function K_{12}, *Gen. Math. Notes*, Vol. 2 (1) (2011), 45-54.

[6] H. Kumar, M. A. Pathan and Harish Srivastava, Joint-moments due to Dirichlet density of three dimensional space and their applications in summability of hypergeometric functions, (communicated).

[7] I. P. Natanson, Theory of Functions of a Real Variable, 2nd. rev. ed., Ungar, New York, 1961.

[8] M. A. Pathan, A note on the generalization of the summation formulas for Appell's and Kampe' de Fe'riet's hypergeometric functions, Journal of Mathematical Analysis and Applications , 72 (1979), 783-781.

[9] G. Polya and G. Szego, Problems and Theorems in Analysis, Vol. I, (Translated from the German by D. Aeppli) Springer-Verlag, New York, 1972, p. 349.

[10] B. N. Prasad, On the summability of Fourier series and the bounded variation of power series, Proceedings of the Edinburgh Mathematical Society (Series 2)/Volume 2/Issue 3/ January (1931), pp 129-134.

[11] E. D. Rainville, Special Functions, Macmillan, New York, 1960; Reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

[12] R. L. Shively, On Pseudo Laguerre Polynomials, Ph. D. Thesis, Univ. of Michigan, 1953.

[13] H. M. Srivastava, On a summation formula for the Appell function F_2, Proc. Cambridge Philos. Soc. 63 (1967), 1087-1089.

[14] H. M. Srivastava, A class of generating functions for generalized hypergeometric polynomials, J. Math. Anal. Appl. 35, (1971) 230-235.

[15] H. M. Srivastava and R. G. Buschman, Some polynomials defined by generating relations, Trans. Amer. Math. Soc. 205 (1975), 360-370.

[16] H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions, John Wiley and Sons, New York, 1984.

Annexure-1

On Dated 16 May 2014**,** I have sent this published paper titled**: "Approximation of Summation Formula of the Series Involving a Set of Polynomials and its Applications"** ***Hemant Kumar/ Elixir Adv. Pure Math. 70C (2014) 24188-24192*** to *Professor M. A. Pathan,* Centre for Mathematical Sciences, Pala Campus Arunapuram, P. O. Pala-686574, Kerala India, , for further reviewing. He raises a question in definition of a set of polynomials (Eqn. 6) given by

$H_{n,m}^{\alpha,\beta,\beta',\mu,\mu'}(x)=\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-n)_{mk}}{\Gamma(\alpha+\beta'+\mu'+\beta n+1+(\mu+m)k)}\delta_k x^k,$ that why you consider three parameters in addition in gamma function $\Gamma(\alpha+\beta'+\mu'+\beta n+1+(\mu+m)k)$ that is $\alpha+\beta'+\mu'$, it may be taken as one parameter.

I clarify him by giving an example of permutation and further analyse him that by this process we obtain many ways to find out our same results. Also, I clear that in Eqns. (17)-(23) of this paper, it is taken by $\alpha+\beta'+\mu'=a$ as a one parameter and again express beauty of this work that for getting 12 ways our same results, we have to take $a=6, \textit{with the restrictions } -1\le\alpha\le 2, 1\le\beta', \textit{and } \mu'\le 4.$

I am very thankful to Prof. Pathan to get a chance for clarifying of this work and then add its further applications in permutation and probability theory.

Example:

(See, Generating Functions in the site, http://www.mathdb.org/ notes_download/elementary/algebra/ae_A11.pdf (p. 10))

If the equation a + b + c = 6, is such that it satisfies $-1 \leq a \leq 2, 1 \leq b, and\ c \leq 4$, then this equation has 12 integer solutions. (24)

Solution: Since $-1 \leq a \leq 2$, the variable a contributes a term $x^{-1} + x^0 + x^1 + x^2$, to the generating function. Similarly, each b and c contributes a term $x^1 + x^2 + x^3 + x^4$. Hence the generating function is

$$G(x) = (x^{-1} + x^0 + x^1 + x^2)(x^1 + x^2 + x^3 + x^4)^2 = x(1 + x + x^2 + x^3)^3$$

$$= x\left(\frac{1-x^4}{1-x}\right)^3 = x(1 - 3x^4 + 3x^8 - x^{12})(1-x)^{-3} = \left(x - 3x^5 + 3x^9 - x^{13}\right)(1-x)^{-3}.$$

In this generating function, the coefficient of x^6 is 12. Hence it has 12 ways to get integral solution. (25)

Analysis:

By above concept, in Eqn. (17) if we put $a = 6, such\ that\ -1 \leq \alpha \leq 2, 1 \leq \beta', and\ \mu' \leq 4, then\ by\ Eqns.,\ \alpha + \beta' + \mu' = 6 > 0, \lambda = 1, \mu = 0, \delta_k = \frac{\delta_k}{k!}$, we may find 12 ways equivalent result to Srivastava and Manocha [16, theorem 2, p.359].

Further if we put in Eqn. (18) $a = 6, -1 \leq \alpha \leq 2, 1 \leq \beta', and\ \mu' \leq 4,$ **then by Eqns.**, $\alpha + \beta' + \mu' = 6 > 0, \lambda = 1, \mu = 0, \delta_k = \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j k!}$, we get 12 ways following equality

$$\frac{1}{n!} G_{n,m}^{-1\leq\alpha\leq 2,\beta,1\leq\beta',0,\mu'\leq 4}(x) = \frac{1}{\Gamma(6)} \binom{6 + (\beta+1)n}{n} {}_{p+m}F_{q+m}\left[\begin{matrix} \Delta(m; -n), (a_p); \\ \Delta(m; 7+\beta n), (b_q); \end{matrix} x\right]. \quad (26)$$

Similarly, by putting $a = 6,$ **with the restrictions** $-1 \leq \alpha \leq 2, 1 \leq \beta', and\ \mu' \leq 4$, in Eqns. (19)-(23), we obtain our results as same as 12 ways.

Other beautiful results of permutation may be found by putting different values of a and with other restrictions of $\alpha, \beta', and\ \mu'$ in our results and they may be used in probability theory also.