Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 71 (2014) 24990-24998

ISSN: 2229-712X

# Computation of discrete generalized Hankel-Clifford transform

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ARTICLE INFO

**Article history:**

Received: 15 April 2014;
Received in revised form: 6 June 2014;
Accepted: 14 June 2014;

**Keywords**

Generalized Hankel-Clifford transform,
Discrete transforms,
Fourier-Bessel series,
Lommel's transforms and numerical computation.

**ABSTRACT**

An algorithm for computing a discrete generalized Hankel-Clifford transform is presented in this paper. In the solution of certain types of differential equations this algorithm can provide a major improvement in speed and accuracy over previously described methods. An application to a particular form of the transport equation relevant to a class of problems in electron scattering is given.



## Introduction

Malgonde [1] investigated the following variant of the generalized Hankel-Clifford transform defined by

$$F(y)=\int_0^\infty (y/x)^{-(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right) f(x)dx\ ,\ \ (\alpha-\beta)\ge -1/2 \qquad (1)$$

where $J_{\alpha-\beta}(x)$, being the Bessel function of the first kind of order $(\alpha-\beta)$, in spaces of generalized functions. Note that (1) reduces to well-known Hankel-Clifford transform for suitable values of the parameters viz. for $\alpha=0$ and $\beta=-\mu$, a transform studied in [4].

We can write generalization Hankel-Clifford of the Bessel-Clifford transform of $f(x)$ as

$$F_{\alpha,\beta}(y)=\int_0^\infty f(x)J_{\alpha-\beta}\left(2\sqrt{xy}\right)x^{(\alpha+\beta)/2}dx\ ,\ \ (\alpha-\beta)\ge -1/2 \qquad (2)$$

The reverse transform from [2] is given by

$$f(x)=\int_0^\infty F(y)J_{\alpha-\beta}\left(2\sqrt{xy}\right)y^{-(\alpha+\beta)/2}dy\ ,\ (\alpha-\beta)\ge -1/2 \qquad (3)$$

where $\int_0^\infty f(x)x^{(\alpha+\beta)/2}dx$ must exist must exist and be absolutely convergent, and where $f(x)$ satisfies Dirichlet's conditions (of limited total fluctuation) in the interval $[0,\infty]$.

For the purpose of determining a discrete transform as in [3], assume $f(x)=0$ that for all $x>T$ and define $r=yT/j_N\left(j_N=N^{\text{th}}\text{ zero of }J_{\alpha-\beta}\left(2\sqrt{xy}\right)\right)$ so that the forward transform can be written as

$$F_{\alpha,\beta}(rj_N/T)=T^{-\frac{(\alpha+\beta)}{2}+1}\int_0^1 f(xT)J_{\alpha-\beta}\left(2\sqrt{xrj_N}\right)x^{(\alpha+\beta)/2}dx\ ,\ \ (\alpha-\beta)\ge -1/2 \qquad (4)$$

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And the reverse transform is written as

$$f(xT)=\int_0^\infty F\left(rj_N/T\right)J_{\alpha-\beta}\left(2\sqrt{xrj_N}\right)\left(\frac{rj_N}{T}\right)^{-(\alpha+\beta)/2}\left(\frac{j_N}{T}\right)dr\ ,\ \left(\alpha-\beta\right)\geq -1/2$$

$$f(xT)=\left(\frac{j_N}{T}\right)^{-\frac{(\alpha+\beta)}{2}+1}\int_0^\infty F\left(rj_N/T\right)J_{\alpha-\beta}\left(2\sqrt{xrj_N}\right)r^{-(\alpha+\beta)/2}dr\ ,\ \left(\alpha-\beta\right)\geq -1/2 \tag{5}$$

We can now expand $f(xT)$ over $[0,1]$, using Lommel's generalized version of the Fourier-Bessel series [8], namely,

$$f(xT)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{C_m \mathcal{J}_{\alpha,\beta}(j_m x)}{j_m\ \mathcal{J}^2_{\alpha.\beta-1}(j_m)}, & 0\leq x\leq 1,\\ 0, & 1\leq x\leq\infty\end{cases} \tag{6}$$

where $j_m$ are the zeros of $J_{\alpha-\beta}(x)$ arranged in ascending order, and where the coefficients $C_m$ are given by

$$C_m=\int_0^1 x^{(\alpha+\beta)/2}f(xT)J_{\alpha-\beta}\left(2\sqrt{xj_m}\right)dx. \tag{7}$$

Considering the additional assumption that $C_m=0$ for all $m\geq N$ choose $N$ and $T$ arbitrarily large, with no loss of generality by having imposed this additional assumption on the properties of $f(xT)$. Taking the transform of eq. (6) utilizing eq. (4), the well established result is obtained:

$$F_{\alpha,\beta}(j_m/T)=T^{-\frac{(\alpha+\beta)}{2}+1}C_m\ \text{ and }\ r=j_m/j_N. \tag{8}$$

and

$$F_{\alpha,\beta}(rj_N/T)=\sum_{m=1}^{N-1}\frac{F_{\alpha,\beta}(j_m/T)\mathcal{J}_{\alpha,\beta}(rj_N)j_m}{\mathcal{J}_{\alpha,\beta-1}(j_m)\left(j_m^2-\left(rj_N\right)^2\right)}\ ,\ \ 0\leq r\leq\infty. \tag{9}$$

Applying eq. (8) to eq. (6) it is seen that

$$f(xT)=\begin{cases}\displaystyle\sum_{m=1}^{N-1}\frac{F_{\alpha,\beta}(j_m/T)\mathcal{J}_{\alpha,\beta}(j_m x)}{T^{-\frac{(\alpha+\beta)}{2}+1}j_m\ \mathcal{J}^2_{\alpha.\beta-1}(j_m)}, & 0\leq x\leq 1,\\ 0, & 1\leq x\leq\infty.\end{cases} \tag{10}$$

**A new discrete generalized Hankel-Clifford transform algorithm**

This last equation gives an exact relationship between $f(xT)$ and the values of its transform at particular values of $r$. We now need a similar relationship, relating $F_{\alpha,\beta}(rj_N/T)$ to values of $f(xT)$ at particular values of $x$. Setting $x=j_p/j_N$ in eq. (10), multiplying to both the sides by $J_{\alpha-\beta}\left(2\sqrt{\left(j_p j_m\right)/j_N}\right)$ and assuming, one obtains

$$F_{\alpha,\beta}(j_m/T)=\begin{cases}\displaystyle\frac{T^{-\frac{(\alpha+\beta)}{2}+1}}{j_N^2}\sum_{p=1}^{N-1}\frac{\mathcal{J}_{\alpha,\beta}(j_m j_p/j_N)}{j_m\ \mathcal{J}^2_{\alpha.\beta-1}(j_p)}f(j_pT/j_N), & 0\leq x\leq 1,\\ 0, & 1\leq x\leq\infty\end{cases} \tag{12}$$

Therefore, exact pair of discrete transform equations are written as

$$f(i)=\frac{1}{T^{2-(\alpha+\beta)}}\sum_{m=1}^{N-1}Y_{\alpha,\beta}(m,i)F_{\alpha,\beta}(m);F_{\alpha,\beta}(m)=\frac{T^{2-(\alpha+\beta)}}{j_N^2}\sum_{i=1}^{N-1}Y_{\alpha,\beta}(m,i)\,f(i),$$

where

$$F_{\alpha,\beta}(m)=F_{\alpha,\beta}\left(j_m/T\right);\, f\left(i\right)=f\left(j_pT/j_N\right);Y_{\alpha,\beta}\left(i,m\right)=\frac{\mathcal{J}_{\alpha,\beta}(j_m j_i/j_N)}{j_i\,\mathcal{J}^2_{\alpha,\beta-1}(j_m)}.$$

This now can be taken a step further. Inserting eq. (12) into eq. (9), the relationship between $f\left(p\right)$ and $F_{\alpha,\beta}\left(r\,j_N/T\right)$ at continuous values of $r$ is obtained.

$$F_{\alpha,\beta}(rj_N/T)=\begin{cases}\dfrac{T^{2-(\alpha+\beta)}}{j_N^2}\displaystyle\sum_{p=1}^{N-1}\frac{f\left(j_pT/j_N\right)}{\mathcal{J}^2_{\alpha,\beta-1}\left(j_p\right)}\sum_{m=1}^{N-1}\frac{\mathcal{J}_{\alpha,\beta}\left(j_m j_p/j_N\right)\mathcal{J}_{\alpha,\beta}\left(r\,j_N\right)j_m}{\left(j_m-rj_N\right)\mathcal{J}_{\alpha,\beta-1}\left(j_m\right)}, & 0\le x\le 1,\\ 0, & 1\le x\le\infty\end{cases}\tag{13}$$

This can be further simplified by recognizing that the inner sum on the right-hand side of the above equation is the first $\left(N-1\right)$ terms in the Fourier-Bessel series for $\mathcal{J}_{\alpha,\beta}\left(r\,j_p\right)$ in [5]. Therefore for $r<1$,

$$F_{\alpha,\beta}(rj_N/T)=\begin{cases}\dfrac{T^{2-(\alpha+\beta)}}{j_N^2}\displaystyle\sum_{p=1}^{N-1}\frac{f\left(j_pT/j_N\right)}{\mathcal{J}^2_{\alpha,\beta-1}\left(j_p\right)}\mathcal{J}_{\alpha,\beta}\left(r\,j_p\right)+\varepsilon; & 0\le x\le 1\\ 0, & 1\le x\le\infty\end{cases}\tag{14}$$

where

$$\varepsilon=\frac{T^{T^{2-(\alpha+\beta)}}}{j_N^2}\sum_{p=1}^{N-1}\frac{f\left(j_pT/j_N\right)}{\mathcal{J}^2_{\alpha,\beta-1}\left(j_p\right)}\sum_{m=N}^{\infty}\frac{\mathcal{J}_{\alpha,\beta}\left(j_m j_p/j_N\right)\mathcal{J}_{\alpha,\beta}\left(r\,j_N\right)j_m}{\left(j_m-rj_N\right)\mathcal{J}_{\alpha,\beta-1}\left(j_m\right)}.$$

For most functions $f\left(xT\right)$, the error in approximation (14) rapidly becomes small for increasing values of $N$. In fact, for values of $N$ greater than 10 this error can be less than 1%. In [7], Oppenheim et al presented an algorithm for the numerical evaluation of the Hankel transform.

Calculations in [9] give,

$$\sum_{m=1}^{N-1}\frac{\mathcal{J}_{\alpha,\beta}\left(\lambda_m j_r/j_N\right)\mathcal{J}_{\alpha,\beta}\left(\lambda_m j_p/j_N\right)}{\mathcal{J}^2_{\alpha,\beta-1}\left(\lambda_m\right)j_N^2}=\delta_{p,r}\,\frac{\mathcal{J}^2_{\alpha,\beta-1}\left(j_p\right)}{2}-\frac{2\left(\alpha+\beta+1\right)j_r^{\alpha+\beta}j_p^{\alpha+\beta}}{\left(j_N^{2(\alpha+\beta+1)}\right)}+\varepsilon_1\left(N\right),$$

$$\lambda_m=m^{th}\text{ zero of }\mathcal{J}_{\alpha,\beta-1}\left(x\right),$$

$$\sum_{m=1}^{N}\frac{\mathcal{J}_{\alpha,\beta-1}\left(j_m\lambda_k/\lambda_N\right)\mathcal{J}_{\alpha,\beta-1}\left(j_m\lambda_p/\lambda_N\right)}{\mathcal{J}^2_{\alpha,\beta-1}\left(j_m\right)\lambda_N^2}=\delta_{p,k}\,\frac{\mathcal{J}^2_{\alpha,\beta}\left(\lambda_p\right)}{2}+\varepsilon_2\left(N\right),$$

$$\sum_{m=1}^{N-1}\frac{\mathcal{J}_{\alpha,\beta-1}\left(\lambda_m j_p/\lambda_N\right)\mathcal{J}_{\alpha,\beta}\left(\lambda_m j_r/\lambda_N\right)}{\mathcal{J}^2_{\alpha,\beta-1}\left(j_m\right)\lambda_N^2}=\delta_{p,r}\,\frac{\mathcal{J}^2_{\alpha,\beta}\left(j_r\right)}{2}-\frac{\mathcal{J}_{\alpha,\beta-1}\left(j_r\right)\mathcal{J}_{\alpha,\beta-1}\left(j_p\right)}{2N}+\varepsilon_3\left(N\right),$$

$$\sum_{m=1}^{N-1}\frac{j_N}{j_m}\frac{\mathcal{J}_{\alpha,\beta-1}\left(\lambda_k j_m/j_N\right)\mathcal{J}_{\alpha,\beta}\left(j_m j_p/j_N\right)}{\mathcal{J}^2_{\alpha,\beta-1}\left(j_m\right)j_N^2}=\int_0^1\mathcal{J}_{\alpha,\beta}\left(xj_p\right)\mathcal{J}_{\alpha,\beta-1}\left(x\,\lambda_k\right)dx+\varepsilon_4\left(N\right),$$

where numerical analysis indicates that for $\alpha-\beta=0.$

$\varepsilon_1\left(N\right)<j_m j_p/j_N^3;$ where $k,\ p<N, N>3,$

$\varepsilon_2\left(N\right)<j_k j_p/j_N^3;$ where $k,\ p<N, N>3,$

$\varepsilon_3\left(N\right)<j_N^{1.5};$ where $k,\ p<N, N>3,$

$\varepsilon_4\left(N\right)<\left(j_k j_p\right)^{1.5}/j_N^3;$ where $k,\ p<N, N>3,$

These additional relations combined with the original orthogonality relation in eq. (11) may prove to be useful in numerically computing the solutions to various types of differential equations which require generalized Hankel-Clifford transforms.

**Computations**

The discontinuous function as presented in [6],

$$f_1(\xi)=\begin{cases}1, \xi<a\\ 0, \xi>a\end{cases}$$

which has analytic transform

$$F_{\alpha,\beta}(r)=\frac{4}{3}\sqrt{a}\,J_0\left(2\sqrt{ap}\right)(ap)^{\frac{1}{4}}-\frac{2}{3}\sqrt{2a}\,J_0\left(2\sqrt{ap}\right)LommelS1\left(\frac{3}{2},1,2\sqrt{ap}\right)$$
$$+\sqrt{2a}\,J_1\left(2\sqrt{ap}\right)LommelS1\left(\frac{1}{2},0,2\sqrt{ap}\right). \tag{15}$$

and the reciprocal of the above, namely

$$f_2(\xi)=\frac{4}{3}\sqrt{a}\,J_0\left(2\sqrt{a\xi}\right)(a\xi)^{\frac{1}{4}}-\frac{2}{3}\sqrt{2a}\,J_0\left(2\sqrt{a\xi}\right)LommelS1\left(\frac{3}{2},1,2\sqrt{a\xi}\right)$$
$$+\sqrt{2a}\,J_1\left(2\sqrt{a\xi}\right)LommelS1\left(\frac{1}{2},0,2\sqrt{a\xi}\right). \tag{16}$$

which has the transform

$$F_{\alpha,\beta}(r)=\begin{cases}1, r\le a,\\ 0, r>a.\end{cases}$$

For the purpose of separating $f_1(\xi)$ and $f_2(\xi)$ into discrete points we choose

$$f_1(n)=\begin{cases}1, n\le h,\\ 0, n>h,\end{cases}$$

where

$n=1,\dots,N, T=\pi N/2$ for our transform,

$n=-M/2,-M/2+1,\dots,M/2-1,M/2; T'=\pi M$ for Candel's transform as in [6]

and

$$f_2(n)=\frac{4}{3}J_0\left(2\sqrt{j_nT/j_N}\right)(j_nT/j_N)^{\frac{1}{4}}-\frac{2}{3}\sqrt{2}\,J_0\left(2\sqrt{j_nT/j_N}\right)LommelS1\left(\frac{3}{2},1,2\sqrt{j_nT/j_N}\right)$$
$$+\sqrt{2}\,J_1\left(2\sqrt{j_nT/j_N}\right)LommelS1\left(\frac{1}{2},0,2\sqrt{j_nT/j_N}\right). \tag{17}$$

$n=1,\dots,N, T=\pi N/4$ for our transform,

And

$$f_2(n)=\begin{cases}\frac{4}{3}J_0\left(2\sqrt{2nT'/M}\right)(2nT'/M)^{\frac{1}{4}}\\ -\frac{2}{3}\sqrt{2}\,J_0\left(2\sqrt{2nT'/M}\right)LommelS1\left(\frac{3}{2},1,2\sqrt{2nT'/M}\right)\\ +\sqrt{2}\,J_1\left(2\sqrt{2nT'/M}\right)LommelS1\left(\frac{1}{2},0,2\sqrt{2nT'/M}\right); n=1,\dots,M/2,\\ 1/2. \qquad ; n=0,\\ \frac{4}{3}J_0\left(2\sqrt{(2n-1)T'/M}\right)((2n-1)T'/M)^{\frac{1}{4}}\\ -\frac{2}{3}\sqrt{2}\,J_0\left(2\sqrt{(2n-1)T'/M}\right)LommelS1\left(\frac{3}{2},1,2\sqrt{(2n-1)T'/M}\right)\\ +\sqrt{2}\,J_1\left(2\sqrt{(2n-1)T'/M}\right)LommelS1\left(\frac{1}{2},0,2\sqrt{(2n-1)T'/M}\right); n=-1,\dots,-M/2.\end{cases} \tag{18}$$

for Candel's transform.

$f_3(\xi)=(\xi^2-1)^{-1/2}$ which has applications in the problems with elastic scattering of electrons. The analytic function is found by MATHEMATICA 5.0 gives these results

$$\text{If } \sqrt{p}\in \mathrm{Re}\,als \,\&\&\, \sqrt{p}=0 \,\&\&\, \mathrm{Re}[\beta]<\frac{3}{2} \| \sqrt{p}\in \mathrm{Re}\,als \,\&\&\, p\geq 0 \,\&\&\, \mathrm{Re}[\beta]<\frac{3}{2},$$

$$\left[ i2^{-4-\alpha+\beta} p^{\frac{\alpha+\beta}{2}} \left(p^2\right)^{\frac{1}{4}(1+\alpha-\beta)} \pi^{3/2} \left( \begin{array}{l} -4\Gamma\left[\frac{3}{4}-\frac{\beta}{2}\right] {}_1F_4\left[ \begin{array}{l} \left\{\frac{1}{4}(3-2\beta)\right\}, \\ \left\{\frac{1}{2},\frac{1}{4}(5-2\beta),\frac{1}{2}(2+\alpha-\beta)\frac{1}{2}(3+\alpha-\beta)\right\},\frac{p^2}{16} \end{array} \right] \\ +p\Gamma\left[\frac{5}{4}-\frac{\beta}{2}\right] {}_1F_4\left[ \begin{array}{l} \left\{\frac{1}{4}(5-2\beta)\right\}, \\ \left\{\frac{3}{2},\frac{1}{4}(7-2\beta),\frac{1}{2}(3+\alpha-\beta),\frac{1}{2}(4+\alpha-\beta)\right\},\frac{p^2}{16} \end{array} \right] \end{array} \right) \right]$$

Now considering $(\alpha-\beta)\geq -1/2$ ; the transform of the function is shown below:

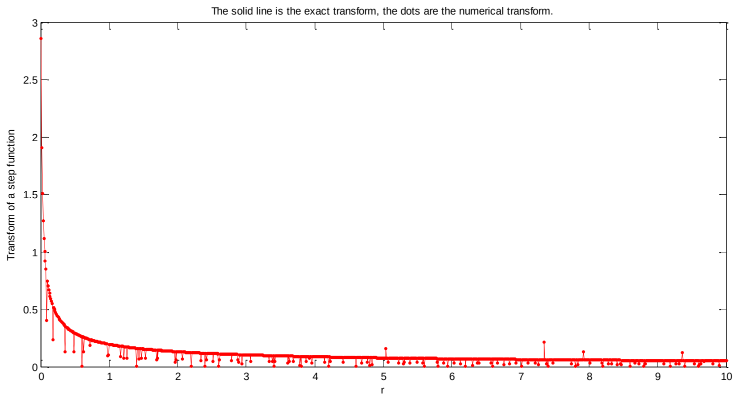


**Figure 1: Results of the exact transforms and numerical transforms**

For $f_1(\xi)=\begin{cases}1, \xi<1\\ 0, \xi>1\end{cases}$; step function

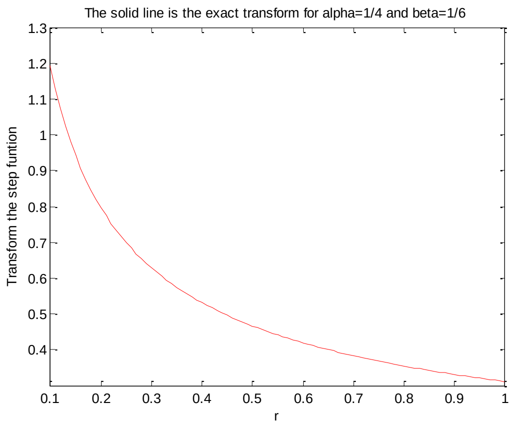


**Figure 2: Results show the step function considering the exact transform at** $\alpha=\frac{1}{4};\beta=\frac{1}{6}$.

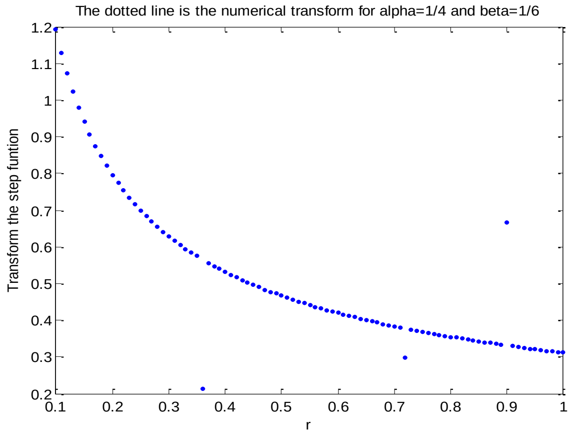


**Figure 3: Results show the step function considering the numerical transform at** $\alpha = \frac{1}{4}; \beta = \frac{1}{6}.$

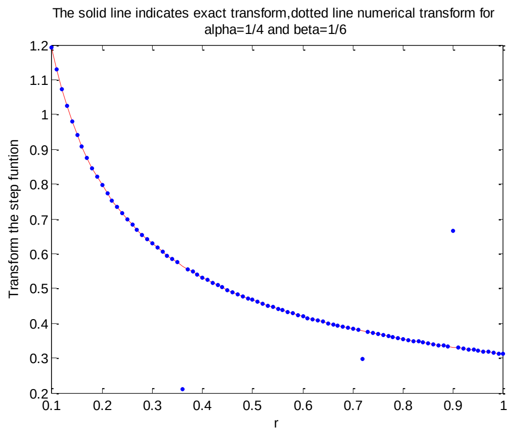


**Figure 4: Results of the exact transform and numerical at** $\alpha = \frac{1}{4}; \beta = \frac{1}{6}.$

For $f_1 = r;$

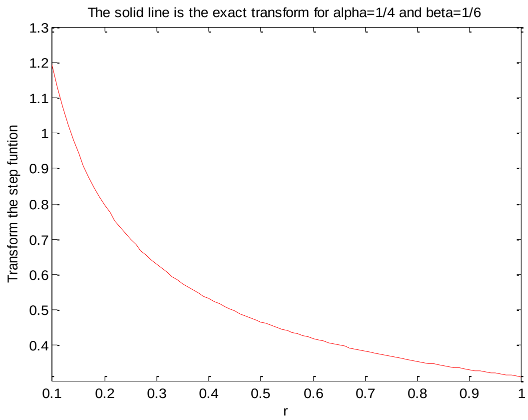


**Figure 5: Graph of exact transform of** $f_1 = r;$

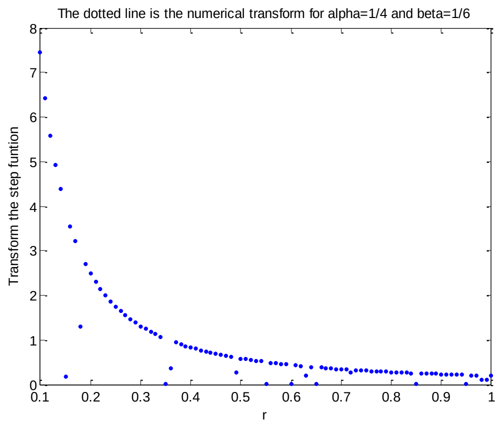


**Figure 6: Results show the function $f_1 = r$ the numerical transform at $\alpha = \frac{1}{4}; \beta = \frac{1}{6}$.**

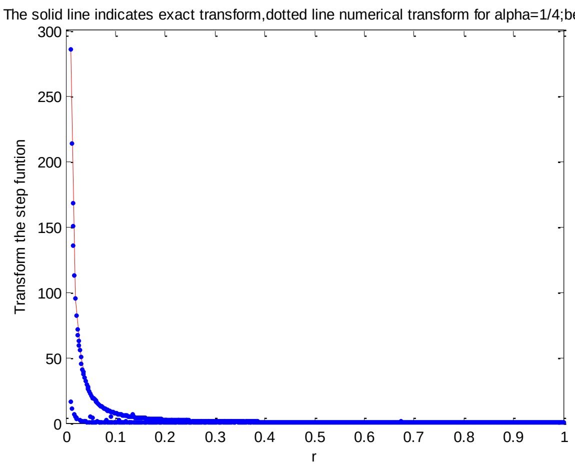


**Figure 8: Results show the function exact transform and numerical transform at $\alpha = \frac{1}{4}; \beta = \frac{1}{6}$.**

For $f_1 = r^{\frac{3}{2}};$

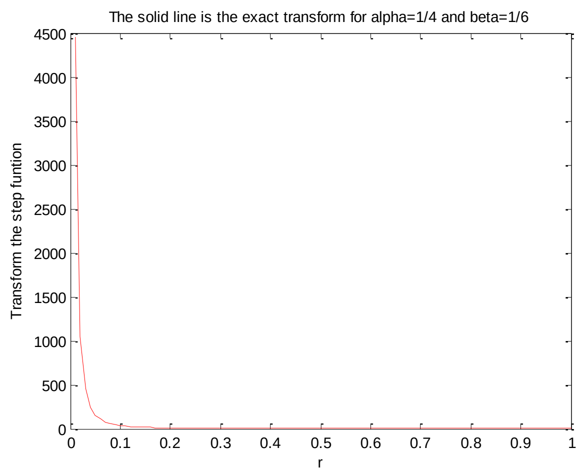


**Figure 9: Results show the function $f_1 = r^{\frac{3}{2}}$ the exact transform at $\alpha = \frac{1}{4}; \beta = \frac{1}{6}$.**

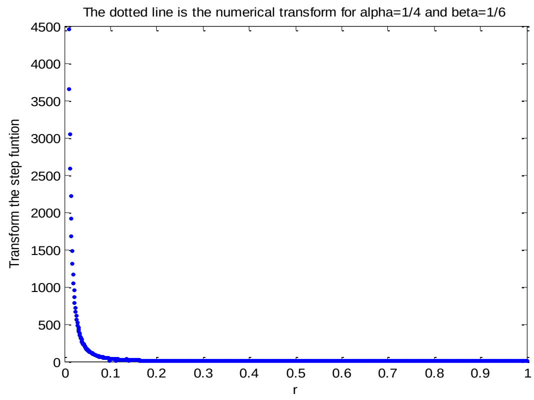


**Figure 10: Results show the function $f_1 = r^{\frac{3}{2}}$ the numerical transform at $\alpha = \frac{1}{4}; \beta = \frac{1}{6}$.**

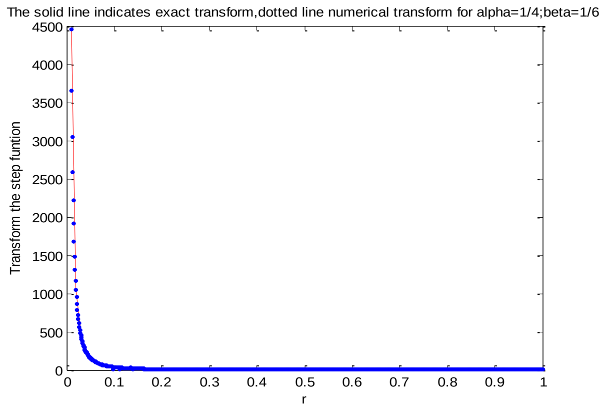


**Figure 11: Results show the function $f_1 = r^{\frac{3}{2}}$ the exact and numerical transform at $\alpha = \frac{1}{4}; \beta = \frac{1}{6}$.**

## Results and discussion

In some of the applications it may be necessary to find an analytic solution where equation (15) will help to solve generalized Hankel-Clifford Transforms. It may additionally be possible to use one of the relationships in equation (16)-(18) in determining a solution. In solving a differential equation requiring generalized Hankel-Clifford transforms, one may be able to dramatically improve the speed of calculation utilizing some of the fundamental principles outlined in this paper.

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