Available online at www.elixirpublishers.com (Elixir International Journal)

Discrete Mathematics

Elixir Dis. Math. 71 (2014) 24783-24787

ISSN: 2229-712X

S – Domination in Semigraphs

D. K. Thakkar[1] and A. A. Prajapati[2]

[1]Department of Mathematics, Saurashtra University Campus, Rajkot – 360 005, India.

[2]Mathematics Department, Government Engineering College, Dahod – 389 151, India.

ARTICLE INFO

Article history:

Received: 16 April 2014;

Received in revised form: 25 May 2014;

Accepted: 6 June 2014;

Keywords

Semigraph,

s - *dominating set*,

s - *dominating* Number,

$i-set$.

ABSTRACT

In this paper we introduce the variant of domination called s - *domination* for semigraphs. The concept of s - *domination* is stronger than the concept of domination in semigraphs. We prove that the s - *domination* number of a semigraphs does not increase when a vertex is removed in a semigraphs. This is unlike in the case of domination. We also consider maximal independent sets in semigraphs and observe that they are exactly independent s - *dominating sets* in semigraphs. We prove some related results.

1. Introduction:

Semigraphs have been studied by several authors [1], [3]. The concept of domination in semigraphs has also appeared in research papers [2], [3], [4], and [5]. However it must be noted that the concept of domination in semigraphs does not reflect the fact that the edges of the semigraph can have more than two vertices. In order to appreciate this factor we define a new concept called s - *domination* for semigraphs. We will observe every s - *dominating set* is a dominating set. Also we will see that any maximal independent subset of a semigraph is a (minimal) s - *dominating set* of the semigraph.

2. Preliminaries:

Definition: 2.1 s **- *dominating set***

Let be G a semigraph, $v \in V(G)$ and $S \subseteq V(G)$ then S is said to be a s - *dominating set* in G if for every vertex v in $V(G) - S$, there is an edge e such that $v \in e$ and $e - \{v\}$ is subset of S.

Note that s - *dominating set* is always dominating set, but converse is not true. For example:

Consider the semigraph G, whose vertex set $V(G) = \{1,2,3\}$ and edge set $E(G) = \{(1,2,3)\}$. Then $S = \{1\}$ is a dominating set but not s - *dominating set*. Since $e = (1,2,3)$, $V[e - \{2\}] = \{1,3\} \not\subset S$.

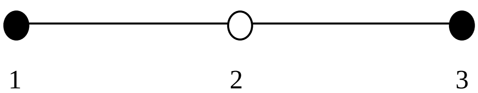

Figure: 1

Definition: 2.2 Minimum s **- *dominating set***

A s - *dominating set* with minimum cardinality is called minimum s - *dominating set*. A minimum s – *dominating set* is also called a $\gamma_s - set$.

Definition: 2.3 s **- *domination* Number**

The cardinality of a minimum s - *dominating set* is called s - *domination* number of the semigraph G. It is denoted as $\gamma_s(G)$.

Definition: 2.4 s **- *adjacent***

Let G be a semigraph and $S \subseteq V(G)$ and $u, v \in S$. We say that u and v are s - *adjacent* in S, if there is an edge e of G such that

(1) $u, v \in e$

(2) $e \subseteq S$.

Definition: 2.5 Minimal s **- *dominating set***

Let G be a semigraph and S be a s - *dominating set* of G, then S is said to be minimal s - *dominating set* if for every vertex v in S, $S - \{v\}$ is not a s - *dominating set*.

Definition: 2.6 Maximum Independent Set

An independent set S of a semigraph G is said to be a maximum independent set if for every vertex $v \notin S, S \cup \{v\}$ is not an independent set.

Definition: 2.7 Strong Private Neighbourhood

Let G be a semigraph, $v \in V(G)$, $S \subset V(G)$ and $v \in S$. Let $w \in V(G)$ then w is in the strong private neighborhood of v with respect to S, if one of the following conditions holds.

Tele:

E-mail addresses: dkthakkar1@yahoo.co.in

1. $w \in S$, $w = v$ and w is not strongly adjacent to any other vertex of S.

2. $w \notin S$ and

(i) There is an edge e which contains w and v such that $\left(e - \{v, w\}\right) \subseteq S$.

(ii) If f is an edge containing w and $f \neq e$, then f contains a vertex which is not in $S - \{v\}$.

The strong private neighbourhood of v with respect to S is denoted as $p_{ns}[v, S]$.

In this paper we will consider two types of subsemigraph on $V(G) - \{v\}$.

- In this subsemigraph those edges are included which are obtained by removing v from every edge of semigraph G. It is referred to as subsemigraph of type I.
- In second subsemigraph those edge of G are included which do not contain vertex v. It is referred to as subsemigraph of type II.

3. Main Result:

Theorem: 3.1 A s- *dominating set* S is minimal s- *dominating set* if and only if for each vertex v in S, one of the following statements holds.

(1) v is not S- *adjacent* to any other vertex of S.

(2) There is a vertex v such that $u \notin S$ and

i. There is an edge e containing u and v such that $\left(e - \{u, v\}\right) \subseteq S$ and

ii. If f is any other edge of G containing u then f contains a vertex which is not in $S - \{v\}$.

Proof: Suppose S is minimal and $v \in S$, then $S - \{v\}$ is not a s- *dominating set*.

Case: I For every edge e containing v, $\left(e - \{v\}\right) \not\subset S$.

In this case v is not strongly adjacent to any other vertex of S.

Case: II Suppose there is a vertex u such that $u \notin S$ and for every edge f containing u, $\left(f - \{u\}\right) \not\subset \left(S - \{v\}\right)$. However, S is a s- *dominating set* of G. Hence there is an edge e such that $\left(e - \{u\}\right) \subset S$. Also by the above statement e must contain v and thus, $\left(e - \{u, v\}\right) \subseteq S$. If f is an edge of G such that $u \in f$ and $f \neq e$, and if $\left(f - \{u\}\right) \subseteq S$ then note that $v \notin S$ (because otherwise $e \cap f$ will contains two vertices u and v which is not possible). Therefore, $\left(f - \{u\}\right) \subset \left(S - \{v\}\right)$. This is again a contradiction. Thus, $\left(f - \{u\}\right)$ must contain a vertex outside $\left(S - \{v\}\right)$.

Conversely, suppose S is s- *dominating set*, $v \in S$ and any one of conditions (1) and (2) holds.

Case: I Suppose condition (1) holds.

Then for every edge e containing v, $\left(e - \{v\}\right)$ contains a vertex outside S, and thus, $\left(S - \{v\}\right)$ is not a s- *dominating set*.

Case: II Suppose condition (2) holds.

Let u be the vertex outside S, which satisfies the conditions. Then for every edge g containing u, $\left(g - \{u\}\right)$ contains a vertex outside $\left(S - \{v\}\right)$. Hence $\left(S - \{v\}\right)$ is not a s- *dominating set* since v is an arbitrary vertex of S.

Then theorem is proved. □

Here, if G is a semigraph and $v \in V(G)$ then $G - \{v\}$ will be the subsemigraph whose vertex set $V(G) - \{v\}$ and edges are obtained by removing v from every edge of the semigraph G. (That is subsemigraph of type I).

Theorem: 3.2 Let G be a semigraph and $v \in V(G)$ then $\gamma_s\left(G - \{v\}\right) \leq \gamma_s(G)$.

Proof: Let S be a minimum s- *dominating set* of G.

First suppose that $v \notin S$. Let x be any vertex of $G - \{v\}$, which is not in S. There is an edge e of G such that $\left(e - \{x\}\right) \subseteq S$. Obviously, $\left(e' - \{x\}\right) \subseteq S$, $\left(e' = e - \{v\}\right)$.

Thus, S is a s- *dominating set* of $G - \{v\}$.

Therefore, $\gamma_s\left(G - \{v\}\right) \leq \gamma_s(G)$.

On the other hand suppose S is a minimum s- *dominating set* of G such that $v \in S$. Consider the set $S - \{v\}$.

Let x be any vertex of $G - \{v\}$ which is not in $S - \{v\}$. Now there is an edge e such that $\left(e - \{x\}\right) \subseteq S$, then $\left(e' - \{x\}\right) \subseteq \left(S - \{v\}\right)$, $\left(e' = e - \{v\}\right)$. Therefore, $S - \{v\}$ is a s- *dominating set* of $G - \{v\}$.

Hence, $\gamma_s\left(G - \{v\}\right) \leq \left|S - \{v\}\right| < |S| = \gamma_s(G)$.

Thus, $\gamma_s\left(G - \{v\}\right) \leq \gamma_s(G)$. □

Theorem 3.3 Let G be a semigraph and $v \in V(G)$, then $\gamma_s\left(G - \{v\}\right) < \gamma_s(G)$ if and only if there is a minimum s- *dominating set* S of G such that $v \in S$.

Proof: Suppose there is a minimum s- *dominating set* S of G such that $v \in S$. Then from the proof of theorem 3.2 it follows that $\gamma_s\left(G - \{v\}\right) < \gamma_s(G)$.

Conversely, suppose $\gamma_s\left(G - \{v\}\right) < \gamma_s(G)$. Let S_1 be a minimum s- *dominating set* of $G - \{v\}$. Then S_1 cannot be a s- *dominating set* in G. So there is a vertex $x \in V(G)$ such that for every edge e containing x, $e - \{x\}$ contains a vertex which is not in S_1. Also, there is an edge e' of $G - \{v\}$ such that $x \in e'$ and $\left(e'-\{x\}\right) \subseteq S_1$. If $e-\{v\}=e'$ then $e-\{x\}$ is a subset of $S_1 \cup \{v\}$. Thus, $S=\left(S_1 \cup \{v\}\right)$ is a s - *dominating set* of G. Since $\gamma_s\left(G-\{v\}\right)<\gamma_s\left(G\right)$, S is a minimum s - *dominating set* of G and $v \in S$. □

Now we consider G is a semigraph and $v \in V(G)$ then $G-\{v\}$ will be the subsemigraph whose vertex set $V(G)-\{v\}$ and edges are those edges of G are included which do not contain vertex v. (That is subsemigraph of type II).

Theorem: 3.4 Let G be a semigraph and $v \in V\left(G\right)$, then $\gamma_S\left(G-\{v\}\right)>\gamma_S\left(G\right)$ if and only if each of the following three conditions are satisfied.

(1) v is not an isolated vertex.

(2) $v \in S$, for every minimum $s-$*dominating set* of G.

(3) There is no subset S such that $N[v]\cap\left(\left(V(G)-\{S\}\right)\neq\phi\right.$, $|S| \le \gamma_s\left(G\right)$ and S is a s - *dominating set* of $G-\{v\}$.

Proof: Suppose $\gamma_S\left(G-\{v\}\right)>\gamma_S\left(G\right)$.

(1) If v is an isolated vertex then $v \in S$, for every minimum $s-$*dominating set* S, then $S-\{v\}$ is an $s-$*dominating set* of the subsemigraph $G-\{V\}$. This implies that $\gamma_S\left(G-\{v\}\right)<\gamma_S\left(G\right)$. Which is contradiction.

(2) Suppose there is a minimum $s-$*dominating set* S of G such that $v \notin S$. Let x be a vertex of $G-\{v\}$ such that $x \notin S$. Since S is an $s-$*dominating set* in G, there is an edge e of G such that $\left(e-\{x\}\right)\subseteq S$. Obviously, $v \notin e$. Thus, e is an edge of $G-\{v\}$ such that $x \in e$ and $\left(e-\{x\}\right)\subseteq S$. Hence S is an $s-$*dominating set* of $G-\{v\}$. Therefore, $\gamma_S\left(G-\{v\}\right)\le|S|=\gamma_S\left(G\right)$, which is a contradiction.

(3) If such a set exist then $\gamma_S\left(G-\{v\}\right)\le\gamma_S\left(G\right)$, which is a contradiction.

Conversely, suppose conditions (1), (2) and (3) are satisfied.

First suppose that $\gamma_S\left(G-\{v\}\right)\le\gamma_S\left(G\right)$. Let S be a minimum $s-$*dominating set* in $G-\{v\}$, then $|S|=\gamma_S\left(G\right)$. Suppose there is an edge e of G such that $v \in e$ and $\left(e-\{v\}\right)\subseteq S$. Then S is a minimum $s-$*dominating set* in G not containing v, which contradicts condition (2). On the other hand suppose for every edge e of G, which contains the vertex v, $\left(e-\{v\}\right)$ contains a vertex outside S, then it implies that $N[v]\cap\left(V(G)-S\right)\neq\phi$, $|S|=\gamma_s(G)$ and S is a $s-$*dominating set* of $G-\{v\}$. This contradicts condition (3). So it is impossible that $\gamma_S\left(G-\{v\}\right)=\gamma_S\left(G\right)$.

Suppose $\gamma_S\left(G-\{v\}\right)<\gamma_S\left(G\right)$.

Let S be a minimum $s-$*dominating set* of $G-\{v\}$. Then $|S|<\gamma_s\left(G\right)$. If there is an edge e of G such that $v \in e$ and $\left(e-\{v\}\right)\subseteq S$, then S is a $s-$*dominating set* of G with $|S|<\gamma_s\left(G\right)$, which is a contradiction. Therefore, for every edge e containing v, $\left(e-\{v\}\right)\not\subset S$. Thus, $e-\{v\}$ contain a vertex outside S. Hence, $N[v]\cap\left(V\left(G\right)-S\right)\neq\phi$, $|S|\le\gamma_s\left(G\right)$ and S is an $s-$*dominating set* of $G-\{v\}$, which is again contradicts the condition (3). Thus, $\gamma_s\left(G-\{v\}\right)<\gamma_s\left(G\right)$ is an impossible. Thus, only possibility left is $\gamma_s\left(G-\{v\}\right)>\gamma_s\left(G\right)$. Hence, the theorem is proved. □

Theorem: 3.5 Suppose G be a semigraph and $v \in V\left(G\right)$. If $\gamma_s\left(G-\{v\}\right)<\gamma_s\left(G\right)$ then $\gamma_s\left(G-\{v\}\right)=\gamma_s\left(G\right)-1$.

Proof: Let S_1 be a minimum $s-$*dominating set* in $G-\{v\}$, then S_1 cannot be $s-$*dominating set* in G. Consider the set $S=S_1\cup\{v\}$. Then S is a $s-$*dominating set* of G. Obviously, S is a minimum $s-$*dominating set* of G. Thus, $\gamma_s\left(G\right)=|S|=|S_1|+1=\gamma_s\left(G-\{v\}\right)+1$. Therefore, $\gamma_s\left(G-\{v\}\right)=\gamma_s\left(G\right)-1$. Hence the theorem is proved. □

Theorem: 3.6 Let G be a semigraph and $v \in V\left(G\right)$, then $\gamma_s\left(G-\{v\}\right)<\gamma_s\left(G\right)$ if and only if there is a minimum s - *dominating set* of G such that $v \in S$ and $p_{ns}\left[v,S\right]=\{v\}$.

Proof: Suppose $\gamma_s\left(G-\{v\}\right)<\gamma_s\left(G\right)$. Let S_1 be a minimum s - *dominating set* of $G-\{v\}$, then as proved theorem 3.3 $S=S_1\cup\{v\}$ is a minimum s - *dominating set* of G. Since S_1 not an s - *dominating set* in G for every edge e containing v, $\left(e-\{v\}\right)$ containing a vertex of $\left(V\left(G\right)-S\right)$. Thus v is not strongly adjacent with any other of S. Thus, $v \in p_{ns}\left[v,S\right]$.

Suppose, $w \neq v$ and $w \in p_{ns}\left[v,S\right]$, then $w \notin S$. Now there is an edge e_1 which contains v and w such that $\left(e_1-\{v,w\}\right)\subset S$. Also, since S_1 is an s - *dominating set* of $G-\{v\}$, there is an edge e_2 such that $\left(e_2-\{w\}\right)\subset S_1$.

Therefore, $v \notin e_2$. Hence, $e_1 \neq e_2$. Since $w \in p_{ns}[v,S]$ and $e_1 \neq e_2$, $(e_2 - \{w\})$ must contain a vertex x outside $S-\{v\}$. This is a contradiction, because as mentioned above, $(e_2-\{v\}) \subseteq (S-\{v\})$. Thus, we have proved that if $w \neq v$ then $w \notin p_{ns}[v,S]$.

Thus, it follows that $p_{ns}[v,S]=\{v\}$.

Conversely, suppose there is a minimum s- *dominating set* of G such that $v \in S$ and $p_{ns}[v,S]=\{v\}$.

Consider the set $S_1 = S-\{v\}$. Let w be a vertex of $G-\{v\}$ which is not in $S-\{v\}$, then $w \notin S$. Therefore, $w \notin p_{ns}[v,S]$. Then one of the following possibilities holds.

(1) For every edge e of G containing v and w, $(e-\{v,w\})$ contains a vertex outside S. Therefore, for every edge e containing v and w, $(e-\{w\})$ contains a vertex outside S. Now S is a s- *dominating set* of G. Therefore, there is an edge e' of G containing w such that $(e'-\{w\}) \subseteq S$ by above statement e' cannot contain a vertex v. Therefore, e' is an edge of $G-\{v\}$ contain vertex w and $(e'-\{w\}) \subseteq S_1$.

Thus, we have proved that for every vertex w in $G-\{v\}$ which is outside S_1, there is an edge e' containing w such that $(e'-\{w\}) \subseteq S_1$. This proves that S_1 is a s- *dominating set* of $G-\{v\}$.

(2) There is an edge f containing w such that $(f-\{w\}) \subseteq (S-\{v\}) = S_1$. This also proves that S_1 is an s- *dominating set* of $G-\{v\}$.

Thus, $\gamma_s(G-\{v\}) \leq |S_1| < |S| = \gamma_s(G)$.

This completes the theorem. □

Corollary: 3.7 Let G be a semigraph $v \in V(G)$ and suppose $\gamma_s(G-\{v\}) < \gamma_s(G)$. If v is not an isolated vertex of G, then there is a minimum s- *dominating set* S of G such that $v \notin S$.

Proof: By above theorem 3.6 there is a minimum s- *dominating set* S of G such that $v \in S$ and $p_{ns}[v,S]=\{v\}$. Now there is no edge which containing v and it is a subset of S. Since v is not isolated. There is a vertex w not in S such that w is adjacent to v in G. Now let $T=(S-\{v\}) \cup \{w\}$, then it can be proved that T is a s- *dominating set* in G which does not contain vertex v. Hence the corollary is proved. □

Theorem: 3.8 Let G be a semigraph and $u,v \in V(G)$. Suppose $\gamma_s(G-\{v\}) < \gamma_s(G)$ and $\gamma_s(G-\{u\}) > \gamma_s(G)$ then there is a minimum s- *dominating set* S such that $u,v \in S$ and v is not strongly adjacent to u in S.

Proof: Let S be a minimum s- *dominating set* of G such that $v \in S$ and $p_{ns}[v,S]=\{v\}$.

Since $\gamma_s(G-\{u\}) > \gamma_s(G)$, $u \in S$. If u and v are strongly adjacent vertices in S then it will imply that $v \notin p_{ns}[v,S]$. Thus, u and v cannot be strongly adjacent vertices in S. □

Theorem: 3.9 Let G be a semigraph and u be a vertex of G such that $\gamma_s(G-\{u\}) > \gamma_s(G)$ then for every s- *dominating set* of S, $u \in S$ and $p_{ns}[u,S]$ contains at least two vertices.

Proof: Since $\gamma_s(G-\{v\}) > \gamma_s(G)$ and $u \in S$. Also, since S is a minimal s- *dominating set* of G and hence $p_{ns}[v,S] \neq \phi$. If $p_{ns}[u,S]=\{u\}$, then by theorem 3.5 $\gamma_s(G-\{u\}) < \gamma_s(G)$, which is a contradiction. So, $p_{ns}[u,S]$ contains at least one vertex different from u.

Suppose u' is a vertex of G such that $p_{ns}[u,S]=\{u,u'\}$. Now consider the set $S_1 = (S-\{u\}) \cup \{u'\}$. Obviously, $|S_1|=|S|$.

Now we prove that S_1 is an s- *dominating set* of G (not containing u).

First there is an edge e of G such that $u,u' \in e$ and $(e-\{u,u'\}) \subset S$ (because $u' \in p_{ns}[u,S]$).

Therefore, $(e-\{u\}) \subset S_1$. Let x be any vertex outside S_1 such that $x \neq u$. Now $x \notin p_{ns}[u,S]$. Now since S is a s- *dominating set* in G, there is an edge e of G such that $x \in e$ and $(e-\{x\}) \subset S$. Suppose, $u \in (e-\{x\})$ then $(e-\{x,u\}) \subseteq S$. Now $x \notin p_{ns}[u,S]$, therefore, there is an edge f such that $f \neq e$ and $(f-\{x\}) \subseteq (S-\{u\})$. That is $(f-\{x\}) \subset S_1$.

Suppose $u \notin (e-\{x\})$ then $(e-\{x\}) \subseteq S_1$. Thus, we have proved that there is an edge h of G such that $x \in h$, $(h-\{x\}) \subseteq S_1$. This proves that S_1 is an s- *dominating set* of G and it is minimum also, because $|S_1| = |S|$. Further, $u \notin S_1$. This is a contradiction because, $\gamma_s(G-\{v\}) > \gamma_s(G)$. Thus, $u \notin p_{ns}[u,S]$. Suppose, $p_{ns}[u,S] = \{u_1\}$ where $u_1 \neq u$, then $u_1 \notin S$. Now consider the set $S_1 = (S-\{u\}) \cup \{u_1\}$. Here also we can prove that S_1 is an s - *dominating set* not containing u. This is again a contradiction. Therefore the $p_{ns}[u,S]$ must contains at least two vertices u_1 and u_2 such that $u_1 \neq u_2$, $u_2 \neq u$.

This completes the proof of theorem.

References:

[1] E.Sampathkumarachar *"Semigraphs and their Applications"* "Report on the DST (Department of Science and Technology) submitted to DST, India, May 2000.

[2] S. S. Kamath and R. S. Bhat, "*Domination in semigraphs",* Reprint submitted to Elsevier Science, 12 September (2003).

[3] Shyam S. Kamath, Saroja R. Hebber and Ravishankar Bhat "*Domination in Semigraph"* (Part-I) Lecture Notes National Workshop on Semigraphs, June 04-07, 2010.

[4] S. S. Kamath and Saroja R. Hebbar, "*Domination Critical Semigraphs"*, Electronic Notes in Discrete Mathematics, Vol. 15, May 2003,113.

[5] S.S. Kamath and R. S. Bhat, "*On Strong/ Weak Independent Sets and Vertex Coverings of a graph"*, Discrete Mathematics (Elsevier), 307(9-10), (2007), 1136-1145.

[6] T. W. Hynes, S. T. Hedetniemi, P. J. Slater "*Fundamental of Domination in Graphs*", 1998, Marcel Dekkar. Inc.

[7] T. W. Hynes, S. T. Hedetniemi, P. J. Slater "*Advanced Topics on Domination in Graphs*", 1998,Marcel Dekkar. Inc.