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The convolution product associated with the Bessel type wavelet transform

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ABSTRACT
In this paper the convolution product associated with the Bessel type wavelet transformation is investigated. Certain norm inequalities for the convolution product are established.

Introduction

Hankel convolution has been studied by many authors in recent past. Following Cholewinski [1], Haimo [2], Hirschman Jr. [3], the Hankel type convolution for the following form of the Hankel type transformation of a function $f \in L^1_\sigma(I)$, where $I = (0,\infty)$ and

$$L^1_\sigma(I) = \left\{ f : \int_0^\infty |f(x)|\, d\sigma(x) < \infty \{, \qquad I = (0,\infty) \right\}.$$

Namely,

$$\left(h_{\alpha,\beta} f\right)(x) = \tilde{f}(x) = \int_0^\infty j_{\alpha-\beta}(xt)\, f(t)\, d\sigma(t), \tag{1.1}$$

where

$$j_{\alpha-\beta}(x) = 2^{-2\beta}\,\Gamma(2\alpha)\, x^{2\beta}\, J_{-2\beta}(x) \text{ and } J_\lambda(x)$$

is the Bessel function of first kind and of order λ. Here

$$d\sigma(t) = \frac{t^{2(\alpha-\beta)}}{2^{-2\beta}\,\Gamma(2\alpha)}\, dt.$$

We say that $f \in L^p_\sigma(I),\ 1 \le p < \infty,\ if$

$$\|f\|_{p,\sigma} = \left(\int_0^\infty |f(x)|^p\, d\sigma(x) \right)^{\frac{1}{p}} < \infty.$$

If $f \in L^1_\sigma(I)$ *and* $h_{\alpha,\beta} f \in L^1_\sigma(I)$ then the inverse Hankel type transform is given by

$$f(x) = \left(h^{-1}_{\alpha,\beta}\left[\tilde{f}\right]\right)(x) = \int_0^\infty j_{\alpha-\beta}(xt)\left(h_{\alpha,\beta} f\right)(t)\, d\sigma(t) \tag{1.2}$$

If $f \in L^1_\sigma(I)$, $g \in L^1_\sigma(I)$ then the Hankel type convolution is defined by

$$(f \# g)(x) = \int_0^\infty (\tau_x f)(y)\, g(y)\, d\sigma(y), \tag{1.3}$$

where the Hankel type translation τ_x is given by

$$(\tau_x f)(y) = \tilde{f}(x,y) = \int_0^\infty D(x,y,z)\, f(x)\, d\sigma(z), \tag{1.4}$$

where

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$$D\,(x,y,z) = \int_0^\infty j_{\alpha-\beta}\,(xt)\, j_{\alpha-\beta}\,(yt)\, j_{\alpha-\beta}\,(zt)\, d\sigma(t)$$

$$= 2^{2(\alpha-2\beta)}\,(\pi)^{-2(\alpha+4\beta)}\,[\Gamma(6\alpha+4\beta)]^2\,[\Gamma(\alpha-\beta)]^{-1}\,(xyz)^{4\beta}\,[\Delta(x,y,z)]^{4\beta}\ ,$$

for $(\alpha-\beta)>0$, where $\Delta\,(x,y,z)$ is the area of a triangle with sides x,y,z if such a triangle exists and zero otherwise.

Here we note that $D(x,y,z)$ is symmetric in x,y,z. Applying (1.2) to (1.4), we get the formula

$$\int_0^\infty j_{\alpha-\beta}\,(zt)\, D(x,y,z)\, d\sigma(z) = j_{\alpha-\beta}\,(xt)\, j_{\alpha-\beta}\,(yt)\,.$$

Setting $t=0$, we get

$$\int_0^\infty D(x,y,z)\, d\sigma\,(z) = 1.$$

Therefore in view of (1.4),

$$\left\|\tilde{f}\,(x,y)\right\|_{1,\sigma} \le \|f\|_{1,\sigma}. \qquad (1.5)$$

Now, using (1.4) we can write (1.3) in the following form:

$$(f\#g)\,(x) = \int_0^\infty\int_0^\infty D(x,y,z)\, f(z)\, g(y)\, d\sigma(z)\, d\sigma\,(y)\,.$$

Some important properties of the Hankel type convolution that are relevant are:

1. If $f,g \in L_\sigma^1\,(I)$ then from [2],

$$\|f\#g\|_{1,\sigma} \le \|f\|_{1,\sigma}\,\|g\|_{1,\sigma} \qquad (1.6)$$

2. With the same assumptions,

$$h_{\alpha,\beta}\,(f\#g)\,(x) = \left(h_{\alpha,\beta}f\right)(x)\,\left(h_{\alpha,\beta}g\right)(x) \qquad (1.7)$$

3. If $f \in L_\sigma^1\,(I)\ and\ g \in L_\sigma^p\,(I),\ p\ge 1.$ Then $(f\#g)$ exists, is continuous and from [7], we get the inequality

$$\|f\#g\|_{p,\sigma} \le \|f\|_{1,\sigma}\,\|g\|_{p,\sigma} \qquad (1.8)$$

4. Let $f \in L_\sigma^p\,(I)\,,\ g \in L_\sigma^p\,(I)\,, \frac{1}{p}+\frac{1}{q}=1.$ Then $f\#g$ exists, is continuous and from [7] we have

$$\|f\#g\|_{\infty,\sigma} \le \|f\|_{p,\sigma}\,\|g\|_{q,\sigma} \qquad (1.9)$$

5. Let $f \in L_\sigma^p\,(I)\ and\ g \in L_\sigma^p\,(I)\,,\ \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1\,.$ Then $(f\#g)$ exists, is continuous and from [7], we get the inequality :

$$\|f\#g\|_{r,\sigma} \le \|f\|_p\,\|g\|_q \qquad (1.10)$$

6. Let $f \in L_\sigma^p\,(I)\,,\ g \in L_\sigma^p\,(I)\ and\ h \in L_\sigma^r\,(I)\,.$ Then the weighted norm inequality

$$\left|\int_0^\infty f(x)\,(g\#h)\,(x)\,d\sigma(x)\right| \le \|f\|_{p,\sigma}\,\|g\|_{q,\sigma}\,\|h\|_{r,\sigma}$$

holds for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2\,.$

As indicated above the proof of the properties 1 to 5 are well known. Hence, we next give the proof of 6.

Using Holder's inequality, we get

$$\left|\int_0^\infty f(x)\,(g\#h)\,(x)\,d\sigma\,(x)\right| \le \|f\|_{p,\sigma}\,\|g\|_{q,\sigma}\,\|h\|_{s,\sigma}\,, \frac{1}{s}=\frac{1}{q}+\frac{1}{r}-1\,.$$

Therefore using (1.9), we have

$$\left|\int_0^\infty f(x)\,(g\#h)\,(x)\,d\sigma\,(x)\right| \le \|f\|_{p,\sigma}\,\|g\|_{q,\sigma}\,\|h\|_{s,\sigma}\ , \qquad \frac{1}{s}=\frac{1}{q}+\frac{1}{r}-1\,.$$

From [4], $h_{\alpha,\beta}$ is isometric on $L^2_\sigma\,(I)$, $\left(h^{-1}_{\alpha,\beta}\ h_{\alpha,\beta}\, f\right) = f$ then Parseval's formula of the Hankel type transformation for $f, g \in L^2_\sigma\,(I)$ is given by

$$\int_0^\infty f(x)\, g(x)\, d\sigma\,(x) = \int_0^\infty \left(h_{\alpha,\beta}\, f\right)(y) \left(h_{\alpha,\beta}\, g\right)(y)\, d\sigma\,(y)\,. \tag{1.11}$$

Furthermore, this relation also holds for $f, g \in L^1_\sigma\,(I)$, (see [8.]).

For $\psi \in L^1_\sigma\,(I)$, using translation τ given in (1.4) and dialation $D_a f\,(x,y) = f\,(ax,\ ay)$, the Bessel wavelet [6] is defined by

$$\tilde{\psi}\left(\frac{t}{a}, \frac{b}{a}\right) = D_{1/a}\,\tau_b\,\psi\,(t) = \int_0^\infty \psi\,(z)\, D\left(\frac{t}{a}, \frac{b}{a}, z\right) d\sigma(z). \tag{1.12}$$

The continuous Bessel wavelet transform [6] of a function $f \in L^1_\sigma\,(I)$ with respect to wavelet $\psi \in L^1_\sigma\,(I)$ is defined by

$$\left(B_\psi\, f\right)(b,a) = a^{4\beta-2} \int_0^\infty \tilde{\psi}\left(\frac{t}{a}, \frac{b}{a}\right) f(t)\, d\sigma\,(t), \quad a > 0. \tag{1.13}$$

By simple modification of (1.13), we can get

$$\left(B_\psi\, f\right)(b,a) = (f \# \psi)\left(\frac{b}{a}\right), \qquad a > 0.$$

From (1.3) and (1.4) the continuous Bessel type wavelet transform of a function $f \in L^1_\sigma\,(I)$ can be written in the form :

$$\left(B_\psi\, f\right)(b,a) = \int_0^\infty j_{\alpha-\beta}\,(bw) \left(h_{\alpha,\beta}\, f\right)(w) \left(h_{\alpha,\beta}\,\psi\right)(aw)\, d\sigma\,(w) \tag{1.14}$$

Now, we state the Parseval formula of the Bessel type Wavelet transform from [6, p.245].

$$\int_0^\infty \int_0^\infty \left(B_\psi\, f\right)(b,a) \left(B_\psi\, g\right)(b,a)\, \frac{d\sigma(b)\, d\sigma\,(a)}{a^{4\alpha}} = C_\psi\,\langle f, g\rangle\,, \tag{1.15}$$

for $f \in L^2_\sigma\,(I)$ and $g \in L^2_\sigma\,(I)$.

Now, we also state from [3, Theorem 2c, p. 312] and [3, Corollary 2c, p.313] which is useful for our approximation results:

Theorem 1.1: Suppose that

1. $k_n\,(x) \geq 0, \quad 0 < x < \infty,$
2. $\int_0^\infty k_n\,(x)\, d\sigma\,(x) = 1, \ n = 0,1,2,3, \dots\dots,$
3. $\lim_{n\to\infty} \int_\delta^\infty k_n(x)\, d\sigma(x) = 0 \ \ \textit{for each}\ \delta > 0,$
4. $\phi\,(x) \in L^\infty_\sigma(I),$
5. ϕ is continuous at $x_0, x_0 \in [x-\delta,\ x+\delta]$ and $\delta > 0$.

Then

$$\lim_{n\to\infty} (\phi \# k_n)\,(x_0) = \phi(x_0).$$

Corollary 1.1 : With the same assumptions on $k_n\,(x)$, if $f\,(x) \in L^1_\sigma\,(I)$ then

$$\lim_{n\to\infty} \|f \# k_n - f\|_1 = 0.$$

The Bessel wavelet type convolution product:

In this section, using properties (1.5), (1.11) and (1.12), we formally define the convolution product for the Bessel type Wavelet transformation by the relation

$$B_\psi\,(f \otimes g)\,(b,a) = \left(B_\psi\, f\right)(b,a) \left(B_\psi\, g\right)(b,a)\,, \tag{2.1}$$

and investigate its boundedness and approximation properties. This in turn implies that the product of the two Bessel type wavelet transforms could be wavelet transform under certain conditions.

Theorem 2.1: Let $f, g, \psi \in L^1_\sigma\,(I)\ \textit{and}\ h_{\alpha,\beta}(\psi)\,(w) \neq 0.$ Then the Bessel type Wavelet convolution can be written in the form

$$(f \otimes g)\,(z) = \int_0^\infty \left(\tau_{z,a} f\right)(y) g(y)\, d\sigma(y)\ ,$$

where

$$\left(\tau_{z,a}f\right)(y) = \int_0^\infty f(x)\, D_a\,(x,y,z)\, d\sigma\,(x),$$

$$D_a\,(x,y,z) = \int_0^\infty \int_0^\infty \left(h_{\alpha,\beta}\psi\right)(at)\left(h_{\alpha,\beta}\psi\right)\,(a\xi)\, j_{\alpha-\beta}\,(xt)\, j_{\alpha-\beta}\,(y\xi)$$

$$\times\ L_a\,(t,\xi,z)\, d\sigma\,(t)\, d\sigma(\xi)\,, \tag{2.2}$$

and

$$L_a(t,\xi,z) = \int_0^\infty j_{\alpha-\beta}\,(y\xi)\, j_{\alpha-\beta}\,(yt)\, Q_a\,(y,z)\, d\sigma(y)\,, \tag{2.3}$$

$$Q_a\,(y,z) = \int_0^\infty \frac{j_{\alpha-\beta}\,(wz)\, j_{\alpha-\beta}(wy)}{\left(h_{\alpha,\beta}\,\psi\right)(aw)}\, d\sigma(w). \tag{2.4}$$

Proof: From (1.14), we have

$$h_{\alpha,\beta}\left[\left(B_\psi\, f\right)(b,a)\right](w) = \left(h_{\alpha,\beta}\,\psi\right)(aw)\left(h_{\alpha,\beta}\, f\right)(w). \tag{2.5}$$

Using (2.1) and (2.5) we get

$$h_{\alpha,\beta}\,\left[\left(B_\psi\,(f\otimes g)\right)\,(b,a)\right](w)$$

$$=h_{\alpha,\beta}\left[\left(B_\psi\, f\right)(b,a)\left(B_\psi\, g\right)(b,a)\right](w)$$

$$= h_{\alpha,\beta}\left[h_{\alpha,\beta}^{-1}\left(\left(h_{\alpha,\beta}\,\psi\right)(a\,\cdot)\left(h_{\alpha,\beta}\, f\right)(\cdot)\right) h_{\alpha,\beta}^{-1}\left(\left(h_{\alpha,\beta}\psi\right)(a\,\cdot)\left(h_{\alpha,\beta}g\right)(\cdot)\right)\right](w)\,\cdot$$

By property (1.7) of the Hankel type convolution, we have

$$h_{\alpha,\beta}\,\left[\left(B_\psi(f\otimes g)\right)\,(b,a)\right](w) = \left[\left(h_{\alpha,\beta}\psi\right)(a\,\cdot)\left(h_{\alpha,\beta}f\right)(\cdot)\,\#\left(h_{\alpha,\beta}\psi\right)(a\,\cdot)\left(h_{\alpha,\beta}g\right)(\cdot)\right](w)\,.$$

Therefore by (2.5), we get

$$\left(h_{\alpha,\beta}\psi\right)(aw)\, h_{\alpha,\beta}\,[(f\otimes g)]\,(w)$$

$$= \left[\left(h_{\alpha,\beta}\psi\right)(a\,\cdot)\left(h_{\alpha,\beta}f\right)(\cdot)\#\left(h_{\alpha,\beta}\psi\right)(a\,\cdot)\left(h_{\alpha,\beta}g\right)(\cdot)\right](w) \tag{2.6}$$

This gives a relation between the Bessel type wavelet transform-convolution and the Hankel type transform-convolution.

Let us set

$$F_a = \left(h_{\alpha,\beta}\,\psi\right)(a\,\cdot)\left(h_{\alpha,\beta}\, f\right)(\cdot)\,,$$

$$G_a = \left(h_{\alpha,\beta}\,\psi\right)(a\,\cdot)\left(h_{\alpha,\beta}\, g\right)(\cdot)\,.$$

Then by (1.3) and (1.4) we get

$$\left(h_{\alpha,\beta}\,\psi\right)(aw)\, h_{\alpha,\beta}\,[(f\otimes g)]\,(w)$$

$$= \int_0^\infty (\tau_w\, G_a)\,(\eta)\, F_a\,(\eta)\, d\sigma\,(\eta)$$

$$= \int_0^\infty F_a\,(\eta)\left(\int_0^\infty D\,(w,\eta,\xi)\, G_a\,(\xi)\, d\sigma(\xi)\, d\sigma\,(\eta)\right)$$

$$= \int_0^\infty \int_0^\infty F_a\,(\eta)\, G_a\,(\xi)\, D\,(w,\eta,\xi)\, d\sigma(\xi) d\sigma\,(\eta)$$

$$= \int_0^\infty \int_0^\infty F_a\,(\eta)\, G_a\,(\xi)\left(\int_0^\infty j_{\alpha-\beta}(wy)\, j_{\alpha-\beta}\,(\eta y)\, j_{\alpha-\beta}\,(\xi y)\, d\sigma\,(y)\right) d\sigma\,(\xi)\, d\sigma\,(\eta)$$

$$= \int_0^\infty \left(\int_0^\infty F_a\,(\eta)\, j_{\alpha-\beta}\,(\eta y) d\sigma(\eta)\right)\left(\int_0^\infty G_a\,(\xi)\, j_{\alpha-\beta}(\xi y) d\sigma(\xi)\right) j_{\alpha-\beta}\,(wy)\, d\sigma(y)$$

$$= \int_0^\infty \left(h_{\alpha,\beta} F_a\right)(y) \left(h_{\alpha,\beta} G_a\right)(y)\, j_{\alpha-\beta}(wy)\, d\sigma(y).$$

Therefore by the inversion formula of the Hankel type transformation (1.2), we have

$$(f \otimes g)(z) = \int_0^\infty \frac{j_{\alpha-\beta}(wz)}{\left(h_{\alpha,\beta}\psi\right)(aw)} \left(\int_0^\infty \left(h_{\alpha,\beta} F_a\right)(y) \left(h_{\alpha,\beta} G_a\right)(y)\, j_{\alpha-\beta}(wy)\, d\sigma(y)\right) d\sigma(w)$$

$$= \int_0^\infty \left(h_{\alpha,\beta} F_a\right)(y) \left(h_{\alpha,\beta} G_a\right)(y) \left(\int_0^\infty \frac{j_{\alpha-\beta}(wz)\, j_{\alpha-\beta}(wy)}{\left(h_{\alpha,\beta}\psi\right)(aw)}\, d\sigma(w)\right) d\sigma(y)$$

$$= \int_0^\infty \left(h_{\alpha,\beta} F_a\right)(y) \left(h_{\alpha,\beta} G_a\right)(y)\, Q_a(y,z)\, d\sigma(y),$$

where $Q_a(y,z)$ is given by (2.4).

Then by the definition of the Hankel type transformation (1.1), $(f \otimes g)(z)$

$$= \int_0^\infty \int_0^\infty j_{\alpha-\beta}(yt) \left(h_{\alpha,\beta}\psi\right)(at) \left(h_{\alpha,\beta} f\right)(t)\, d\sigma(t)$$

$$\times \left(\int_0^\infty j_{\alpha-\beta}(y\xi) \left(h_{\alpha,\beta}\psi\right)(a\xi) \left(h_{\alpha,\beta} g\right)(\xi) d\sigma(\xi)\, Q_a(y,z)\, d\sigma(y)\right)$$

$$= \int_0^\infty \int_0^\infty \left(h_{\alpha,\beta}\psi\right)(at) \left(h_{\alpha,\beta}\psi\right)(a\xi) \left(h_{\alpha,\beta} f\right)(t) \left(h_{\alpha,\beta} g\right)(\xi)$$

$$\times \left(\int_0^\infty j_{\alpha-\beta}(y\xi)\, j_{\alpha-\beta}(yt)\, Q_a(y,z)\, d\sigma(y)\right) d\sigma(t)\, d\sigma(\xi)$$

$$= \int_0^\infty \int_0^\infty \left(h_{\alpha,\beta}\psi\right)(at) \left(h_{\alpha,\beta}\psi\right)(a\xi) \left(h_{\alpha,\beta} f\right)(t) \left(h_{\alpha,\beta} g\right)(\xi)$$

$$\times\, L_a(t,\xi,z)\, d\sigma(t)\, d\sigma(\xi):$$

Therefore

$$(f \otimes g)(z)$$

$$= \int_0^\infty \int_0^\infty \left(h_{\alpha,\beta}\psi\right)(at) \left(h_{\alpha,\beta}\psi\right)(a\xi) \left(\int_0^\infty j_{\alpha-\beta}(xt)\, f(x)\, d\sigma(x)\right)$$

$$\times \left(\int_0^\infty j_{\alpha-\beta}(y\xi)\, g(y) d\sigma(y)\right) L_a(t,\xi,z)\, d\sigma(t)\, d\sigma(\xi)$$

$$= \int_0^\infty \int_0^\infty f(x)\, g(y) \left(\int_0^\infty \int_0^\infty j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(y\xi) \left(h_{\alpha,\beta}\psi\right)(at) \left(h_{\alpha,\beta}\psi\right)(a\xi)\, L_a(t,\xi,z)\, d\sigma(t)\, d\sigma(\xi)\right) d\sigma(x)\, d\sigma(y)$$

$$= \int_0^\infty \int_0^\infty f(x)\, g(y)\, D_a(x,y,z)\, d\sigma(x) d\sigma(y),$$

where

$$D_a(x,y,z) = \int_0^\infty \int_0^\infty j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(y\xi) \left(h_{\alpha,\beta}\psi\right)(at) \left(h_{\alpha,\beta}\psi\right)(a\xi)\, L_a(t,\xi,z)\, d\sigma(t)\, d\sigma(\xi).$$

If we define the generalized translation by

$$F_a(z,y) = \left(\tau_{z,a} f\right)(y) = \int_0^\infty D_a(x,y,z)\, f(x)\, d\sigma(x)\,,$$

then

$$(f \otimes g)(z) = \int_0^\infty \left(\tau_{z,a} f\right)(y)\, g(y)\, d\sigma(y).$$

Thus proof is completed.

Theorem 2.3: Assume that

$$\inf_w \left|\left(h_{\alpha,\beta}\, \psi\right)(aw)\right| = B_\psi(a) > 0.$$

Then

$\|D_a(x,y,z)\| \le \frac{1}{B_\psi(a)}\, a^{4\beta-2}\, \|\psi\|_{1,\sigma}^2$.

Proof: From (2.2), we have

$$D_a(x,y,z) = \int_0^\infty \int_0^\infty j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(y\xi) \left(h_{\alpha,\beta}\, \psi\right)(at) \left(h_{\alpha,\beta}\, \psi\right)(a\xi)\, L_a(t,\xi,z)\, d\sigma(t)\, d\sigma(y)$$

$$= \int_0^\infty \int_0^\infty j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(y\xi) \left(h_{\alpha,\beta}\, \psi\right)(at) \left(h_{\alpha,\beta}\, \psi\right)(a\xi)$$

$$\times \left(\int_0^\infty j_{\alpha-\beta}(\eta\xi)\, j_{\alpha-\beta}(\eta t)\, Q_a(\eta,z)\, d\sigma(\eta)\right) d\sigma(t)\, d\sigma(\xi)$$

$$= \int_0^\infty \int_0^\infty j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(y\xi) \left(h_{\alpha,\beta}\, \psi\right)(at) \left(h_{\alpha,\beta}\, \psi\right)(a\xi)$$

$$\times \left(\int_0^\infty j_{\alpha-\beta}(\eta\xi)\, j_{\alpha-\beta}(\eta t) \left(\int_0^\infty \frac{j_{\alpha-\beta}(wz)\, j_{\alpha-\beta}(\eta w)}{\left(h_{\alpha,\beta}\, \psi\right)(aw)}\, d\sigma(w)\right) d\sigma(\eta)\right) d\sigma(t)\, d\sigma(\xi)$$

$$= \int_0^\infty \left(\int_0^\infty j_{\alpha-\beta}(xt)\, j_{\alpha-\beta}(\eta t) \left(h_{\alpha,\beta}\, \psi\right)(at)\, d\sigma(t)\right)$$

$$\times \left(\int_0^\infty j_{\alpha-\beta}(y\xi)\, j_{\alpha-\beta}(\eta\xi) \left(h_{\alpha,\beta}\, \psi\right)(a\xi)\, d\sigma(\xi)\right) Q_a(z,\eta)\, d\sigma(\eta)$$

$$= \int_0^\infty h_{\alpha,\beta}\left[j_{\alpha-\beta}(xt) \left(h_{\alpha,\beta}\, \psi\right)(at)\right](\eta)\, h_{\alpha,\beta}\left[j_{\alpha-\beta}(y\xi) \left(h_{\alpha,\beta}\, \psi\right)(a\xi)\right](\eta)\, Q_a(z,\eta)\, d\sigma(\eta)$$

$$= \int_0^\infty \int_0^\infty h_{\alpha,\beta}\left[j_{\alpha-\beta}(xt) \left(h_{\alpha,\beta}\, \psi\right)(at) \,\#\, j_{\alpha-\beta}(y\xi) \left(h_{\alpha,\beta}\, \psi\right)(a\xi)\right](\eta)$$

$$\times\, j_{\alpha-\beta}(w\eta)\, j_{\alpha-\beta}(wz) \left[\left(h_{\alpha,\beta}\, \psi\right)(aw)\right]^{-1} d\sigma(w)\, d\sigma(\eta)$$

$$= \int_0^\infty \left[j_{\alpha-\beta}(x\ \cdot) \left(h_{\alpha,\beta}\, \psi\right)(a\ \cdot) \,\#\, j_{\alpha-\beta}(y\ \cdot) \left(h_{\alpha,\beta}\, \psi\right)(a\ \cdot)\right](w)$$

$$\times\, j_{\alpha-\beta}(wz) \left[\left(h_{\alpha,\beta}\, \psi\right)(aw)\right]^{-1} d\sigma(w).$$

Now, set $F_a(t) = j_{\alpha-\beta}(xt)\, h_{\alpha,\beta}\, \psi(at)$ and assume that

$$\inf_{w}\left|\left(h_{\alpha,\beta}\,\psi\right)(aw)\right| = B_{\psi}(a) > 0.$$

Since $\left|j_{\alpha-\beta}(z)\right| \leq 1,\ [2,\ p.\ 336],$ we have

$$|D_a(x,y,z)| \leq \frac{1}{B_\psi(a)} \int_0^\infty |(F_a \# F_a)(w)|\, d\sigma(w).$$

Using (1.6), we have

$$\begin{aligned} |D_a(x,y,z)| &\leq \frac{1}{B_\psi(a)} \int_0^\infty \|F\|_{1,\sigma}\ \|F\|_{1,\sigma} \\ &\leq \frac{1}{B_\psi(a)} \left[\int_0^\infty \left|j_{\alpha-\beta}(xv)\left(h_{\alpha,\beta}\,\psi\right)(av)\right| d\sigma(v)\right]^2 \\ &\leq \frac{1}{B_\psi(a)} \left[\int_0^\infty |\psi(av)|\, d\sigma(v)\right]^2 \\ &\leq \frac{1}{B_\psi(a)} \left[\|\psi_a\|_{1,\sigma}\right]^2 \\ &\leq \frac{a^{4\beta-2}}{B_\psi(a)} \left[\|\psi_a\|_{1,\sigma}\right]^2. \end{aligned}$$

In order to obtain Plancheral formula for the Bessel type wavelet transform, we define the space

$$W^2(I \times I) = \left\{ g(b,a) : \|g\|_{W^2} = \left(\int_0^\infty \int_0^\infty |g(b,a)|^2 \, \frac{d\sigma(b)\, d\sigma(a)}{a^{4\alpha}} \right)^{\frac{1}{2}} < \infty \right\}.$$

Theorem 2.3: Let $f \in L^2_\sigma(I)$, $\psi \in L^2_\sigma(I)$. Then

$$\left\|\left(B_\psi f\right)(b,a)\right\|_{W^2} = \sqrt{C_\psi}\, \|f\|_{2,\sigma}.$$

Proof: Putting $f = g$ in (1.15), we prove the above theorem.

Theorem 2.4: Let $f, g \in L^2_\sigma(I)$ and let $\psi \in L^2_\sigma(I)$ be a Bessel wavelet which satisfies

$$C_\psi = \int_0^\infty \left|\left(h_{\alpha,\beta}\,\psi\right)(aw)\right|^2 \frac{d\sigma(a)}{a^{4\alpha}} > 0.$$

Then

$$\|f \otimes g\|_{2,\sigma} \leq \|f\|_{2,\sigma}\, \|g\|_{2,\sigma}\, \|\psi\|_{2,\sigma}.$$

Proof: Using Theorem 2.3 and (2.1)

$$\begin{aligned} \sqrt{C_\psi}\, \|f \otimes g\|_{2,\sigma} &= \left\|B_\psi(f \otimes g)\right\|_{W^2} \\ &= \left\|B_\psi f(b,a)\, B_\psi g(b,a)\right\|_{W^2} \\ &= \left(\int_0^\infty \int_0^\infty \left|B_\psi f(b,a)\, B_\psi g(b,a)\right|^2 \frac{d\sigma(a) d\sigma(b)}{a^{4\alpha}}\right)^{\frac{1}{2}}. \end{aligned} \tag{2.7}$$

From (1.14) and (1.9), we have

$$\left|B_\psi g(b,a)\right| \leq \left|\left(g(a.) \# \psi(\cdot)\right)(b/a)\right| \leq \|g\|_{2,\sigma}\, \|\psi\|_{2,\sigma}. \tag{2.8}$$

Applying (2.7) and (2.8), we get

$$\sqrt{C_\psi}\, \|f \otimes g\|_{2,\sigma} \leq \|g\|_{2,\sigma}\, \|\psi\|_{2,\sigma} \left(\int_0^\infty \int_0^\infty \left|B_\psi f(b,a)\right|^2 \frac{d\sigma(a) d\sigma(b)}{a^{4\alpha}}\right)^{\frac{1}{2}}.$$

From Theorem 2.3, we obtain

$$\sqrt{C_\psi}\ \|f \otimes g\|_{2,\sigma} \leq \|g\|_{2,\sigma}\ \|\psi\|_{2,\sigma} \sqrt{C_\psi}\ \|f\|_{2,\sigma}\ .$$

Hence

$$\|f \otimes g\|_{2,\sigma} \leq \|g\|_{2,\sigma}\ \|\psi\|_{2,\sigma}\ \|f\|_{2,\sigma}\ .$$

Thus proof is completed

Weighted Sobolev Space:

In this section we study certain properties of the Bessel type wavelet convolution on a weighted Sobolev space defined below.

Definition 3.1: The Zemanian space $H_{\alpha,\beta}\ (I),\ I = (0,\infty)$ is the set of all infinitely differentiable functions ϕ on $(0,\infty)$ such that

$$\rho_{m,k}^{\alpha,\beta}\ (\phi) = \operatorname*{Sup}_{x\ \epsilon\ (0,\infty)} \left| x^m \left(x^{-1}\ \frac{d}{dx}\right)^k\ x^{2\beta-1}\ \phi(x)\right| < \infty$$

for all $m, k \in \mathbb{N}_0$. Then $f \in H'_{\alpha,\beta}\ (I)$ is defined by the following way:

$$\langle f, \phi\rangle = \int_0^\infty f(x)\ \phi(x)\ dx\ , \qquad \phi \in H_{\alpha,\beta}\ (I)\ .$$

Definition 3.2: Let $k(w)$ be an arbitrary weight function. Then a function $\Phi \in \left[H_{\alpha,\beta}\ (I)\right]'$ is said to belong to the weighted Sobolev space

$G_{\alpha,\beta,k}^p\ (I)$ for $(\alpha-\beta) \in \mathbb{R},\ 1 \leq p < \infty,$ if it satisfies

$$\|\Phi\|_{p,\alpha,\beta,\sigma,k} = \left(\int_0^\infty \left|k(w)\left(h_{\alpha,\beta}\ \Phi\right)(w)\right|^p\ d\sigma(w)\right)^{\frac{1}{p}}, \tag{3.1}$$

where $a > 0$ and $\Phi \in L_\sigma^p\ (I)$.

In what follows we shall assume that $k(w) = \left|\left(h_{\alpha,\beta}\ \psi\right)(aw)\right|$ for fixed $a > 0$.

Theorem 3.1: Let $f \in G_{\alpha,\beta,k}^1(I)\ and\ g \in G_{\alpha,\beta,k}^p(I)\ ,\ p \geq 1.$ Then

$$\|f \otimes g\|_{p,\alpha,\beta,\sigma,k} \leq \|f\|_{1,\alpha,\beta,\sigma,k}\ \|g\|_{p,\alpha,\beta,\sigma,k}\ .$$

Proof: In view of (3.1) we have

$$\|f \otimes g\|_{p,\alpha,\beta,\sigma,k} = \left(\int_0^\infty \left|k(w)\ h_{\alpha,\beta}\ (f \otimes g)\ (w)\right|^p\ d\sigma(w)\right)^{\frac{1}{p}}$$

By (1.8) and (2.6) we have

$$\begin{aligned} \|f \otimes g\|_{p,\alpha,\beta,\sigma,k} &\leq \|F_a\ (w)\|_{1,\alpha,\beta,\sigma,k}\ \|G_a\ (w)\|_{p,\alpha,\beta,\sigma,k} \\ &\leq \left\|\left(h_{\alpha,\beta}\ \psi\right)(aw)\left(h_{\alpha,\beta}\ f\right)(w)\right\|_{1,\alpha,\beta,\sigma,k} \qquad (3.2) \\ &\times \left\|\left(h_{\alpha,\beta}\ \psi\right)(aw)\left(h_{\alpha,\beta}\ g\right)(w)\right\|_{p,\alpha,\beta,\sigma,k} \end{aligned}$$

From Definition 3.2, we get

$$\|f \otimes g\|_{p,\alpha,\beta,\sigma,k} \leq \|f\|_{1,\alpha,\beta,\sigma,k}\ \|g\|_{p,\alpha,\beta,\sigma,k} \tag{3.3}$$

This completes the proof.

Theorem 3.1: Let $f \in G_{\alpha,\beta,k}^p\ (I)\ and\ g \in G_{\alpha,\beta,k}^q\ (I)$ with $1 \leq p, q < \infty$ and $1/r = \frac{1}{p} + \frac{1}{q} - 1$. Then

$$\|f \otimes g\|_{r,\alpha,\beta,\sigma,k} \leq \|f\|_{p,\alpha,\beta,\sigma,k}\ \|g\|_{q,\alpha,\beta,\sigma,k}\ . \tag{3.4}$$

Proof: Using (1.10) and (3.1) we get (3.4).

Approximation properties of the Bessel type wavelet convolution are given next.

Theorem 3.2: Let $\psi_{n,a}\ (w) = \psi_n(aw)\ ,\ n = 0,1,2, \ldots \ldots$ be the sequence of basic wavelet functions such that

1. $\psi_{n,a}\ (w) \geq 0,\ \ 0 < w < \infty,$

2. $\int_0^\infty \psi_{n,a}\,(w)\,d\sigma\,(w) = 1,$

3. $\lim_{n\to\infty} \int_\varepsilon^\infty \psi_{n,a}\,d\sigma(w) = 0\;,\; for\ each\ \ \varepsilon > 0\,,$

4. $\left(h_{\alpha,\beta}\,\psi_{n,a}\right)(w) \in L^1_\sigma\,(I)$

5. $h^{-1}_{\alpha,\beta}\left[\left(h_{\alpha,\beta}\,\psi_{n,a}\right)(w)\right] = \psi_{n,a}\,(w)\,.$

Then

$$\lim_{n\to\infty} \left\| f(b) - \left(B_{\psi_n} f\right)(b,a)\right\|_{1,\sigma} = 0\,.$$

Proof: Proof can be completed by following [3, pp. 315-316]

Theorem 3.3: Let $k_n\,(w) = \left(h_{\alpha,\beta}\,\psi\right)(aw)\left(h_{\alpha,\beta}\,g_n\right)(w)$ for fixed $a > 0,\ n \in \mathbb{N},\ and\ \phi(w) = \left(h_{\alpha,\beta}\,\psi\right)(aw)\left(h_{\alpha,\beta}\,f\right)(w)$ satisfy.

1. $k_n\,(w) \geq 0\;,\; 0 < w < \infty,$

2. $\int_0^\infty k_n\,(w)\,d\sigma\,(w) = 1\;,\; w = 0,1,2,3,\dots\dots,$

3. $\lim_{n\to\infty}\int_\delta^\infty k_n(w)\,d\sigma\,(w) = 0\quad,\; for\ each\ \delta > 0,$

4. $\phi(w) \in L^\infty_\sigma(I),$

5. ϕ is continuous at w_0 and $\left(h_{\alpha,\beta}\,\psi\right)(aw_0) \neq 0\ \ for\ w_0 \in [w-\delta,\ w+\delta],\ \delta > 0.$

Then

$$\lim_{n\to\infty} h_{\alpha,\beta}\,(f \otimes g_n)\,(w_0) = \left(h_{\alpha,\beta}\,f\right)(w_0).$$

Proof : In view of relation (2.6), we have

$$\left(h_{\alpha,\beta}\,\psi\right)(aw)\,h_{\alpha,\beta}\,(f\otimes g_n)\,(w) = (\phi \# k_n)\,(w):$$

Now using Theorem 1.1, we have

$$\begin{aligned}\lim_{n\to\infty}\left(h_{\alpha,\beta}\,\psi\right)(aw_0)\,h_{\alpha,\beta}\ \ (f\otimes g_n)(w_0) &= \lim_{n\to\infty}(\phi \# k_n)\,(w_0)\\ &= \phi\,(w_0)\\ &= \left(h_{\alpha,\beta}\,\psi\right)(aw_0)\ \left(h_{\alpha,\beta}\,f\right)(w_0)\,.\end{aligned}$$

This implies that

$$\lim_{n\to\infty} h_{\alpha,\beta}\ \,(f\otimes g_n)(w_0) = \ \left(h_{\alpha,\beta}\,f\right)(w_0)\,.$$

Thus proof is completed.

Theorem 3.4: Let $f,\psi \in L^1_\sigma\,(I)\,,\ and\ \ k_n\,(w)$ be the same as Theorem 3.3 which satisfies all the four properties of Theorem 3.2. Then

$$\lim_{n\to\infty}\left\|\left(h_{\alpha,\beta}\,\psi\right)(aw_0)\left(h_{\alpha,\beta}\,f\right)(w_0) - \left(h_{\alpha,\beta}\,\psi\right)(aw_0)\,h_{\alpha,\beta}\,(f\otimes g_n)\,(w_0)\right\|_{1,\sigma} = 0\,.$$

Proof: Using (2.6), we have

$$\lim_{n\to\infty}\left\|\left(h_{\alpha,\beta}\,\psi\right)(aw_0)\left(h_{\alpha,\beta}\,f\right)(w_0) - \left(h_{\alpha,\beta}\,\psi\right)(aw_0)\,h_{\alpha,\beta}\,(f\otimes g_n)(w_0)\right\|_{1,\sigma}$$

$$= \lim_{n\to\infty}\left\|\left(h_{\alpha,\beta}\,\psi\right)(aw_0)\ \left(h_{\alpha,\beta}\,f\right)(w_0) - \begin{bmatrix}\left(h_{\alpha,\beta}\,\psi\right)(a\ \cdot)\left(h_{\alpha,\beta}\,f\right)(\cdot)\\ \times \#\left(h_{\alpha,\beta}\,\psi\right)(a\ \cdot)\left(h_{\alpha,\beta}\,g_n\right)(a\ \cdot)\end{bmatrix}(w_0)\right\|_{1,\sigma}$$

$$= \lim_{n\to\infty}\left\|\phi\,(w_0) - (\phi\,\#\,k_n)\,(w_0)\right\|_{1,\sigma}\,.$$

Since $f,\psi_a \in L^1_\sigma\,(I)\,,\ \phi\,(w) = \left(h_{\alpha,\beta}\,f\right)\left(h_{\alpha,\beta}\,\psi_a\right) = h_{\alpha,\beta}\,(f\,\#\,\psi_a) \in L^1_\sigma\,(I).$

Therefore using the tools of $[3,\ Corollary\ 2\,c,\ pp. 313-314],$ we have

$$\lim_{n\to\infty}\left\|\left(h_{\alpha,\beta}\,\psi\right)(aw_0)\left(h_{\alpha,\beta}\,f\right)(w_0) - \left(h_{\alpha,\beta}\,\psi\right)(aw_0)\ h_{\alpha,\beta}\,(f\otimes g_n)\,(w_0)\right\|_{1,\sigma} = 0.$$

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