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N – dimensional generalized heat equation and its heat polynomial

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ABSTRACT
In this paper we consider the generalized heat equation of n^th order

$$\frac{\partial^2 u}{\partial r^2}+\frac{n-1}{r}\frac{\partial u}{\partial r}-\frac{d^2}{r^2}u=\frac{\partial u}{\partial t}.$$

If the initial temperature is an even power function, then the heat transform with the source solution as the kernel gives the heat polynomials. We discuss various properties of the heat polynomial and its Appell type transform. Also, we give series representation of the heat transform when the initial temperature is a power function.

Introduction

In this paper we shall establish various properties of the polynomial solutions and its Appell transforms of the generalized heat equation of the n^th order.

$$\frac{\partial^2 u}{\partial r^2}+\frac{n-1}{r}\frac{\partial u}{\partial r}-\frac{d^2}{r^2}u=\frac{\partial u}{\partial t},$$

where $r^2=x_1^2+x_2^2+\ldots\ldots+x_n^2$. We shall also give a series expansion of the generalized temperature in terms of Laguerre polynomials and confluent hypergeometric functions. Most of the results derived here are similar to the ones found in [6, 7], which are for the less general equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{4a}{x}\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t}$$

which in turn is a generalization of the ordinary heat equation [8]

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}.$$

These known results can be considered as special cases of our more general results when $d=0$ *and* $n=1$.

Preliminaries:

Consider the equation

$$\Delta_n\,\psi\,(r,\theta)=\frac{\partial\psi}{\partial t}\qquad,$$

where $r^2=x_1^2+x_2^2+\cdots\ldots+x_n^2$ and $\theta=\tan^{-1}\left(\frac{r}{x_n}\right)$. Then we have

$$\frac{\partial^2\psi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2\sin^{n-2}\theta}\frac{\partial}{\partial\theta}\left[\sin^{n-2}\theta\,\frac{\partial\psi}{\partial\theta}\right]=\frac{\partial\psi}{\partial t}.$$

If the solution is of the type

$$\psi(r,\theta)=u(r,t)\,p(\theta),$$

then

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$$p(\theta)\left[\frac{\partial^2 u}{\partial r^2}+\frac{n-1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2 sin^{n-2}\theta}\frac{d}{d\theta}\left[sin^{n-2}\theta\ \frac{dp}{d\theta}\right]u\right]=\frac{\partial u}{\partial t}\ p(\theta).$$

Letting

$$\frac{1}{p(\theta)sin^{n-2}\theta}\frac{d}{d\theta}\left[sin^{n-2}\theta\ \frac{dp}{d\theta}\right]=-d^2\ , \qquad (2.1)$$

we finally have

$$\frac{\partial^2 u}{\partial r^2}+\frac{4a}{r}\frac{\partial u}{\partial r}-\frac{d^2}{r^2}\ u=\frac{\partial u}{\partial t}\ , \qquad (2.2)$$

where $n = 4a + 1$, the generalized heat equation.

Now from (2.1), we have

$$\frac{1}{sin^{n-2}\theta}\frac{d}{d\theta}\left[sin^{n-2}\theta\ \frac{dp}{d\theta}\right]=-d^2\ p(\theta),$$

or

$$\frac{d^2 p}{d\theta^2}+(n-2)\cot\theta\ \frac{dp}{d\theta}=-d^2 p.$$

Let $\xi = \cos\theta$, then from above, we obtain

$$(1-r^2)\frac{d^2 p}{d\xi^2}-(n-1)\ \xi\ \frac{dp}{d\xi}=-d^2 p,$$

which has a solution

$$p(\xi)=(\xi^2-1)^{-m/2}\ P_{a,b}^m\ (\xi)\ ,$$

where $m=\frac{1}{2}(n-3)$, $d^2 = 2a(4a+2b) = 4a(2a+b)$ and

$P_{a,b}^m\ (\xi)$ is the Legendre function of the first kind $[2, p. 122]$.

Also by elementary methods $(see[8])$, we can find the solution of (2.2) as

$$u(r,t)=\int_0^\infty \mathrm{U}(s,r\colon t)\ u(s,0)ds\ ,$$

where

$$U(s,r\colon t)=\frac{1}{2t}\ s^{3a+b}\ r^{-(a-b)}\ e^{-\frac{1}{4t}(s^2+r^2)}\ I_{\alpha-\beta}\left[\frac{sr}{2t}\right]\ , \qquad (2.3)$$

where $(\alpha-\beta)^2 = (a-b)^2 + d^2$ and $I_{\alpha-\beta}\ (z)$, the usual modified Bessel type function of the first kind. We shall call the function U to be the source solution of the heat equation (2.2). If U is considered as the kernel, then for a suitable f, its heat transform F is defined by

$$r^k\ F(r,t)=\int_0^\infty U(s,r\colon t)\ s^k\ f(s)\ ds\ ,$$

where $k = \alpha-\beta-a+b\ and\ F(r,0)=f(r)$, the initial temperature. Numerous properties of the heat transform have been given in [8]. We note that its inversion is given by

$$r^k\ f(r)=\int_0^\infty U(s,ir\colon t)\left(\frac{s}{i}\right)^k\ F(is,t)ds. \qquad (2.4)$$

Suppose now that the initial temperature is the power function $f(r) = r^m$, m real and positive, then from (2.3), its heat transform,

$$P_{m,\alpha,\beta}\ (r,t)=\int_0^\infty U(s,r\colon t)\ s^{k+m}\ ds \qquad (2.5)$$

$$=\frac{\Gamma\left(3\alpha+\beta+\frac{m}{2}\right)}{\Gamma(3\alpha+\beta)}\ (4t)^{m/2}\ r^k\ 1F_1\ \left[-\frac{m}{2};3\alpha+\beta;\frac{r^2}{4t}\right]$$

$(\alpha-\beta) > -1,\ t > 0,\ [9, p. 394]$. Thus giving a solution of (2.2) involving the hypergeometric function $1F_1$. As a special case if

$$m = 2n, \qquad n = 0,1,2, \dots .…$$

then

$$P_{2n,\alpha,\beta}\ (r,t) = n!\ (4t)^n\ r^k\ L_n^{\alpha,\beta}\ (-r^2/4t)$$
$$= (4t)^n\ r^k\ \sum_{p=0}^{n} \frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta+p)} \binom{n}{p} \left[\frac{x^2}{4t}\right]^p \qquad (2.6)$$

defining the heat polynomial of degree $2n$ in r and of degree n in t, involving the Laguerre polynomial. If we let $k = 0$, we have the special case given in [6].

Next we define the Appell type transform of $P_{m,\alpha,\beta}(r,t)$, m real and positive as

$$W_{m,\alpha,\beta}\ (r,t) = A_p\left[P_{m,\alpha,\beta}\ (r,t)\right]$$
$$= H_{\alpha,\beta}\ (0,r{:}\,t)\ P_{m,\alpha,\beta}\left(\frac{r}{t},\frac{1}{t}\right),$$

where $H_{\alpha,\beta}$, the Green's function is defined by

$$U(s,r{:}\,t) = s^{\alpha-\beta+3a+b}\left(\frac{r}{t}\right)^k H_{\alpha,\beta}\ (s,r{:}\,t),$$

and

$$H_{\alpha,\beta}(s,r{:}\,t) = \frac{t^{k-1}}{2(sr)^{\alpha-\beta}}\ e^{-\left(\frac{s^2+r^2}{4t}\right)}\ I_{\alpha-\beta}\left(\frac{sr}{2t}\right). \qquad (2.7)$$

It can be readily seen that

$$W_{m,\alpha,\beta}\ (r,t) = H_{\alpha,\beta}\ (0,r{:}\,t)\ t^{-m-k}\ P_{m,\alpha,\beta}(r,-t). \qquad (2.8)$$

Now, $H_{\alpha,\beta}\ (0,r{:}\,t) = \frac{1}{2^{2r+1}\,\Gamma(3\alpha+\beta)}\ t^{-(3a+b)}\ e^{-r^2/4t}$, (2.9)

therefore we can write

$$W_{m,\alpha,\beta}(r,t) = \frac{1}{2^{4\alpha}\Gamma(3\alpha+\beta)}\ t^{-(m+3\alpha+\beta)}\ e^{-r^2/4t}\ P_{m,\alpha,\beta}(r,-t),$$

where $k = \alpha - \beta + a - b$.

3. Properties of $P_{n,\alpha,\beta}(r,t)$ and $W_{n,\alpha,\beta}(r,t)$:

In this section we shall establish various results involving the function $P_{2n,\alpha,\beta}(r,t)$and its Appell type transform $W_{2n,\alpha,\beta}(r,t)$. Using the asymptotic expansions, it is an easy matter to calculate the following estimates:

$$U(s,r{:}\,t) = 0\left(|s|^{a-b+\frac{1}{2}}\,e^{-\frac{1}{4t}(s-r)^2}\right)\ as\ |s|\ \to\infty$$
$$P_{2n,\alpha,\beta}(r,t) = 0(r^{2n+k})\ as\ r\to\infty$$
$$P_{2n,\alpha,\beta}(r,t) = 0\left(\frac{4nt}{e}\right)\ as\ n\to\infty.$$

Lemma 3.1: For $0 \le x < \infty,\ \ t > 0,$

$$\int_0^\infty U(s,r{:}\,t)\ P_{2n,\alpha,\beta}(s,-t)ds = r^{k+2n}. \qquad (3.1)$$

Proof: With the help of above estimates, one can easily conclude that the integral converges. By using (2.6) the definition of $P_{2n,\alpha,\beta}$ twice, we have,

$$\int_0^\infty U(s,r{:}\,t)\ P_{2n,\alpha,\beta}(s,-t)ds$$
$$= \sum_{p=0}^{n} \frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta+p)}\ (-4t)^{n-p}\ \binom{n}{p} \int_0^\infty U(s,r{:}\,t)\ s^{k+2p}\ ds$$
$$= \sum_{p=0}^{n} \frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta+p)}\ (-4t)^{n-p}\ \binom{n}{p}\ P_{2p,\alpha,\beta}(r,t)$$

$$= \sum_{p=0}^{n} \frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta+p)} (-4t)^{n-p} \binom{n}{p} \sum_{m=0}^{p} \frac{\Gamma(3\alpha+\beta+p)}{\Gamma(3\alpha+\beta+m)} (4t)^{p-m} \binom{p}{m} r^{k+2m}$$

$$= \sum_{m=0}^{n} (-1)^n \frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta+m)} (4t)^{n-m} r^{k+2m} \sum_{p=m}^{n} (-1)^p \binom{n}{p} \binom{p}{m}. \qquad (3.2)$$

Now consider the inner sum

$$\sum_{p=m}^{n} (-1)^p \binom{n}{p} \binom{p}{m} = \sum_{i=0}^{n-m} (-1)^{i+m} \binom{n}{i+m} \binom{i+m}{m}$$

$$= (-1)^m \frac{n!}{m!} \sum_{i=0}^{n-m} \frac{(-1)^i}{i!\,(n-m-i)!}$$

$$= \frac{(-1)^m n!}{m!\ l!} \sum_{i=0}^{l} (-1)^i \binom{l}{i}.$$

Thus the inner sum is 0 if $l \neq 0$ and 1 if $l = 0$ i.e. if $m = n$. Therefore (3.2) reduces to $r^{k+2n}$and hence

$\int_0^\infty U(s, r\colon t)\, P_{2n,\alpha,\beta}(s, -t) ds = r^{k+2n}$as desired.

This completes the proof.

The equation (3.1) gives us an inversion formula of (2.5) with $m = 2n$. We now derive a generating function for $P_{2n,\alpha,\beta}(r, t)$.

Lemma 3.2: For $0 \leq x < \infty,\ -\infty < t < \infty,\ \ y < \frac{1}{4t}$,

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} P_{2n,\alpha,\beta}(r, t) = \frac{r^k}{(1 - 4\,yt)^{3\alpha+\beta}} e^{r^2 y/(1-4yt)},$$

$k = \alpha - \beta - a + b$.

Proof: Let $t > 0$ using (2.5) and (2.3), we have

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} P_{2n,\alpha,\beta}(r, t) = \sum_{n=0}^{\infty} \frac{y^n}{n!} \int_0^\infty U(s, r\colon t)\, s^{k+2n} ds$$

$$= \int_0^\infty U(s, r\colon t)\, s^k \sum_{n=0}^{\infty} \frac{(s^2 y)^n}{n!} ds$$

$$= \int_0^\infty U(s, r\colon t)\, s^k\, e^{s^2 y}\ ds$$

$$= \frac{1}{2t} r^{-(a-b)} e^{-r^2/4t} \int_0^\infty s^{k+3a+b} e^{-s^2\left(\frac{1}{4t}-y\right)} I_{\alpha-\beta}\left[\frac{sr}{2}\right] ds$$

$$= \frac{r^k}{(1-4t)^{3\alpha+\beta}} e^{\frac{r^2 y}{1-4yt}},$$

$[8, p. 394]$ as required. The interchange of summation and integration is valid since

$$\int_0^\infty s^{k+3a+b} e^{-s^2\left(\frac{1}{4t}-y\right)} I_{\alpha-\beta} \left[\frac{sr}{2t}\right] ds < \int_0^\infty s^{k+3a+b} e^{-s^2\left(\frac{1}{4t}-y\right)} e^{\frac{sr}{2t}} ds < \infty.$$

If $t = 0$, the result can easily be computed, since $P_{2n,\alpha,\beta}(r, 0) = r^{k+2n}$.

For $t < 0$, we use the fact that

$$P_{2n,\alpha\beta}(r, -t) = i^{2n-k} P_{2n,\alpha,\beta}(ir, t) \qquad (3.3)$$

from its representation given is (2.6). The lemma is then proved on the similar lines as for the case $t > 0$.

Now we give a generating function for $W_{2n,\alpha,\beta}(r, t)$, the Appell type transform of $P_{2n,\alpha,\beta}(r, t)$.

Lemma 3.3: For $t \geq 0$, $|z| < \frac{1}{4}t$ and $k = \alpha - \beta - a + b$,

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} W_{2n,\alpha,\beta}(r,t) = \left[\frac{r}{t+4z}\right]^k H_{\alpha,\beta}(0,r:t+4z). \tag{3.4}$$

Proof: We note that $W_{2n,\alpha,\beta}(r,t) = 0\left[\frac{4n}{et}\right]^n$, $as \ n \to \infty$, and hence the series converges absolutely when $|z| < \frac{1}{4}t$. Using (2.8), we have

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} W_{2n,\alpha,\beta}(r,t) = H_{\alpha,\beta}(0,r:t)\, t^{-k} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{z}{t^2}\right)^n P_{2n,\alpha,\beta}(r,-t)$$

$$= H_{\alpha,\beta}(0,r:t)\, t^{-k} x^k \left[\frac{t}{t+4z}\right]^{3\alpha+\beta} e^{\frac{r^2 z}{t(t+4z)}},$$

$$= \left[\frac{r}{t+4z}\right]^k H_{\alpha,\beta}(0,r:t+4z),$$

due to Lemma 3.2 and making use of the definition of $H_{\alpha,\beta}$ given by (2.9).

If we expand the right hand side of (3.4) by Taylor series in powers of z, we have

$$\left[\frac{r}{t+4z}\right]^k H_{\alpha,\beta}(0,r:t+4z) = \sum_{n=0}^{\infty} \frac{(4z)^n}{n!}\left(\frac{\partial}{\partial t}\right)^n \left[\left(\frac{r}{t}\right)^k H_{\alpha,\beta}(0,r:t)\right].$$

On comparing this series with the series on the left hand side of (3.4), we obtain

$$W_{2n,\alpha,\beta}(r,t) = 2^{2n}\left(\frac{\partial}{\partial t}\right)^n \left[\left(\frac{r}{t}\right)^k H_{\alpha,\beta}(0,r:t)\right]$$

$$= 2^{2n}\left(\frac{\partial}{\partial t}\right)^n \left[\left(\frac{r}{t}\right)^k \cdot \frac{t^{-(3a+b)}}{2^{4\alpha}\,\Gamma(3\alpha+\beta)}\, e^{-r^2/4t}\right],$$

$$= \frac{2^{2n-\alpha+\beta}}{\Gamma(3\alpha+\beta)}\left[\frac{\partial}{\partial t}\right]^n r^{k-\alpha+\beta} \int_0^{\infty} J_{\alpha-\beta}(ru)\, e^{-tu^2} u^{3\alpha+\beta}\, du,$$

$$= \frac{2^{2n-\alpha+\beta}}{\Gamma(3\alpha+\beta)} r^{k-\alpha+\beta} \int_0^{\infty} J_{\alpha-\beta}(ru)\, u^{3\alpha+\beta}\left[\frac{\partial}{\partial t}\right]^n \left[e^{-tn^2}\right] du$$

$$= \frac{(-1)^n\, 2^{2n-\alpha+\beta}}{\Gamma(3\alpha+\beta)} r^{-(a-b)} \int_0^{\infty} J_{\alpha-\beta}(ru)\, u^{2n+3\alpha+\beta} e^{-tu^2}\, du, \tag{3.5}$$

giving us an integral representation for $W_{2n,\alpha,\beta}(r,t)$.

Also we give other generating functions for the function $P_{2n,\alpha,\beta}(r,t)$ and its Appell type transform $W_{2n,\alpha,\beta}(r,t)$. We shall simply write down the results, which can be proved following a similar analysis as used for the Lemmas 3.2 and 3.3 above.

Lemma 3.4: For $-\infty < t < \infty$ and all complex z,

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{n!\,\Gamma(3\alpha+\beta+n)} P_{2n,\alpha,\beta}(r,t) = z^{-(\alpha-\beta)} r^{-(a-b)} e^{4tz^2} I_{\alpha-\beta}(2(z)).$$

Lemma 3.5: For $-\infty < t < \infty$ and all complex z,

$$\sum_{n=0}^{\infty} \frac{\Gamma(3\alpha+\beta)}{n!\,\Gamma(3\alpha+\beta+n)}\left(\frac{z}{4}\right)^{2n} W_{2n,\alpha,\beta}(r,t) = \left(\frac{r}{t}\right)^k H_{\alpha,\beta}(z,r:t).$$

Now we shall prove an important property of the sets of functions $P_{2n,\alpha,\beta}(r,t)$ and $W_{2n,\alpha,\beta}(r,t)$ and show that they form a biorthogonal system.

Theorem 3.1: For $t > 0$,

$$\int_0^{\infty} P_{2m,\alpha,\beta}(x,-t)\, W_{2n,\alpha,\beta}(x,-t)\, x^{4a}\, dx = \frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta)}\, n!\, 4^{2n}\, \delta_{m,n},$$

where $\delta_{m,n}$ is the Dirac-delta function.

Proof: Using (2.8),

$$\int_0^\infty P_{2m,\alpha,\beta}\,(x,-t)\, W_{2n,\alpha,\beta}(x,-t)\, x^{4a}\, dx$$

$$= \int_0^\infty H_{\alpha,\beta}\,(0,x;t)\, t^{-2n-\alpha+\beta}\, P_{2n,\alpha,\beta}(x,-t)\, P_{2m,\alpha,\beta}\,(x,-t)\, x^{4a}\, dx$$

$$= \frac{1}{2^{4\alpha}\Gamma(3\alpha+\beta)}\, t^{m-n-(3\alpha+\beta)}\, n!\, m!\, (-4)^{m+n} \int_0^\infty x^{4\alpha}\, e^{-x^2/4t}\, L_n^{\alpha,\beta}\left[\frac{x^2}{4t}\right] L_m^{\alpha,\beta}\left[\frac{x^2}{4t}\right] dx, \qquad (3.6)$$

due to (2.6).

The integral on the right hand side of (3.6) with a change of variable can be written as, $[3, p. 188]$.

$$2^{4\alpha}\, t^{3\alpha+\beta} \int_0^\infty y^{\alpha-\beta}\, e^{-y}\, L_n^{\alpha,\beta}\,(y)\, L_m^{\alpha,\beta}\,(y)\, dy = 2^{4\alpha}\, \frac{\Gamma(3\alpha+\beta+n)}{\Gamma(n+1)}\, t^{3\alpha+\beta}\, \delta_{mn}.$$

Hence the right hand side of (3.6) gives

$$\frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta)}\, t^{m-n}\, m!\, (-4)^{m+n}\, \delta mn$$

$$= \frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta)}\, n!\, 4^{2n}\, \delta mn\,,$$

as required.

Next we shall establish a generating function for the biorthogonal set $P_{2m,\alpha,\beta}\,(x,t)\, W_{2n,\alpha,\beta}\,(x,t)$.

Lemma 3.6: For x, y, s and $t > 0$ and $|z^2 t| < s$,

$$\sum_{n=0}^{\infty} \frac{\Gamma(3\alpha+\beta)}{n!\,\Gamma(3\alpha+\beta+n)} \left[\frac{z}{4}\right]^{2n} P_{2n,\alpha,\beta}(x,t)\, W_{2n,\alpha,\beta}\,(y,s) = \left[\frac{xy}{s+z^2 t}\right]^k H_{\alpha,\beta}\,(xz, y\colon s+z^2 t).$$

Proof: Note that the series converges for $|z^2 t| < s$, using the asymptotic estimates of the functions $P_{2n,\alpha,\beta}$ *and* $W_{2n,\alpha,\beta}$, therefore

$$\sum_{n=0}^{\infty} \frac{\Gamma(3\alpha+\beta)}{n!\,\Gamma(3\alpha+\beta+n)} \left[\frac{z}{4}\right]^{2n} P_{2n,\alpha,\beta}(x,t) W_{2n,\alpha,\beta}\,(y,s)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, y^{-(a-b)}}{2^n\, n!\,\Gamma(3\alpha+\beta+n)}\, P_{2n,\alpha,\beta}(x,t) \int_0^\infty (2\mu)^{3\alpha+\beta+2n}\, e^{-su^2}\, J_{\alpha-\beta}(yu) du,$$

due to (3.5),

$$= 2y^{-(a-b)} \int_0^\infty u^{3\alpha+\beta}\, e^{-su^2}\, J_{\alpha-\beta}(yu)\, du \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(3\alpha+\beta+n)} \left[\frac{uz\,i}{2}\right]^{2n} P_{2n,\alpha,\beta}(x,t)$$

$$= 2^{3\alpha+\beta}\, z^{-(\alpha-\beta)}(xy)^{-(a-b)} \int_0^\infty u\, e^{-u^2(s+z^2 t)} J_{\alpha-\beta}(xuz) J_{\alpha-\beta}\,(yu)\, du$$

$$= \frac{2^{\alpha-\beta}}{s+z^2 t}\, (xy)^{-(a-b)} z^{-(\alpha-\beta)}\, e^{\frac{y^2+x^2z^2}{4(s+z^2t)}}\, I_{\alpha-\beta}\left[\frac{xyz}{2(s+z^2 t)}\right],$$

$[1, p. 51]$, due to the definition in (2.7).

Now two results on finite sums involving the functions $P_{2n,\alpha,\beta}$ and $W_{2n,\alpha,\beta}$.

Lemma 3.7: For $t > 0$, $(\alpha-\beta) > 0$, and a complex z,

$$\sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n+\alpha-\beta}{n-m} z^m\, P_{2n,\alpha,\beta}(r,t) = r^k\, (1-4tz)^n\, L_n^{\alpha,\beta}\left[\frac{zr^2}{1-4tz}\right].$$

Proof: By (2.5), we have

$$\sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n+\alpha-\beta}{n-m} z^m \, P_{2n,\alpha,\beta}(r,t)$$

$$= \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n+\alpha-\beta}{n-m} \, z^m \int_0^\infty U(s,r:t)\, s^{k+2m}\, ds$$

$$= \int_0^\infty U(s,r:t)\, s^k\, ds. \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n-\alpha+\beta}{n-m} \, (zs^2)^m$$

$$= \int_0^\infty U\,(s,r:t)\, s^k \, L_n^{\alpha,\beta}\,(zs^2)\, ds,$$

$$= \frac{1}{2t}\, r^{-(a-b)}\, e^{-\frac{r^2}{4t}} \int_0^\infty s^{3\alpha+\beta}\, e^{-s^2/4t}\, I_{\alpha-\beta}\left[\frac{sr}{2t}\right] L_n^{\alpha,\beta}\,(zs^2)\, ds$$

$$= r^k\,(1-4t\,z)^n\, L_n^{\alpha,\beta}\left[\frac{z\,r^2}{1-4\,zt}\right], \qquad [1, p.43],$$

as required

A similar result can also be proved involving $W_{2n,\alpha,\beta}$.

Lemma 3.8: For $t > 0$, $(\alpha-\beta) > 0$ and a complex z,

$$\sum_{m=0}^{n} \frac{(-1)^m}{m\,!} \binom{n+\alpha-\beta}{n-m} \, z^m \, W_{2m,\alpha,\beta}(r,t)$$

$$= \frac{r^k}{2^{3\alpha+\beta}\Gamma(3\alpha+\beta)}.\, t^{-(n+3\alpha+\beta)}\, e^{-r^2/4t}\,(t+4z)^n\, L_n^{\alpha,\beta}\left[\frac{zr^2}{t(t+4z)}\right].$$

4. Series expansion:

In this section we shall establish a series representation of the heat transform $F(r,t)$in terms of Laguerre polynomials and confluent hypergeometric functions.

As mentioned earlier, for a suitable f, its heat transform F is given by

$$r^k\, F(r,t) = \int_0^\infty U(s,r:t)\, s^k\, f(s)\, ds \quad , \quad t > 0,$$

where $F(r,0) = f(r)$ and $r^k\, F(r,t)$ is a solution of the generalized heat equation.

Theorem 4.1: If $f(x) = \sum_{n=0}^{\infty} a_n\, x^n$, has a growth $\left[1, \frac{e}{4\sigma}\right], \sigma > 0,$

then

$$r^k\, F(r,t) = \begin{cases} \int_0^\infty U(s, ir: -t)\,(s/i)^k\, f(is)\, ds\,, & -\sigma < t < 0 \\ \int_0^\infty U\ (s,r:t)\, s^k\, f(s)\, ds\,, & 0 < t < \sigma \end{cases}$$

where $k = \alpha-\beta-(a-b)$.

Proof: If $0 < t < \delta$, we have

$$r^k\, F(s,t) = \int_0^\infty U(s,r:t)\, s^k \sum_{n=0}^{\infty} a_n\, s^n\, ds$$

$$= \sum_{n=0}^{\infty} a_n \int_0^{\infty} U(s,r:t)\, s^{k+n}\, ds$$

$$= \sum_{n=0}^{\infty} a_n\, P_{n,\alpha,\beta}(r,t),$$

due to (2.5). The interchange of summation and integration is valid since

$$\int_0^{\infty} |U(s,r:t)\, s^{k+n}|\, ds \;<\; \int_0^{\infty} e^{-\frac{1}{4t}(s+r)^2}\, s^{k+n+3a+b}\, ds < \infty.$$

Also, if $-\delta < t < 0$,

$$\int_0^{\infty} U(s,ir:-t)\,(s/i)^k f(s)\; ds = \int_0^{\infty} U(s,ir:-t)\,(s/i)^k \sum_{n=0}^{\infty} a_n\,(is)^n\, ds$$

$$= \sum_{n=0}^{\infty} a_n\, i^{n-k} \int_0^{\infty} U(s,ir:-t)\, s^{k+n}\, ds$$

$$= \sum_{n=0}^{\infty} a_n\; i^{n-k}\, P_{n,\alpha,\beta}\,(ir,-t)$$

$$= \sum_{n=0}^{\infty} a_n\; P_{n,\alpha,\beta}\,(r,t),$$

due to (3.3). Hence the result.

Next, for $0 < |t| < \delta$,

$$r^k\, F(r,t) = \sum_{n=0}^{\infty} a_n\, P_{n,\alpha,\beta}(r,t).$$

or

$$r^k\, F(r,t) = \sum_{n=0}^{\infty} a_{2n}\, P_{2n,\alpha,\beta}(r,t) + \sum_{n=0}^{\infty} a_{2n+1}\; P_{2n+1,\alpha,\beta}\,(r,t).$$

Now making use of the definitions given in (2.5) and (2.6), we obtain

$$F(r,t) = \sum_{n=0}^{\infty} a_{2n}\, n!\; (4t)^n\, L_n^{\alpha,\beta}\,(-r^2/4t)$$
$$+ \sum_{n=0}^{\infty} a_{2n+1}\, \frac{\Gamma(4\alpha+2\beta+n)}{\Gamma(3\alpha+\beta)}\, (4t)^{n+\frac{1}{2}}\, 1F_1\left[-n-\frac{1}{2} : r+1 : \frac{r^t}{4t}\right],$$

giving us a representation involving Laguerre polynomial and confluent hypergeometric function.

If we set $d = 0$ i.e. $\alpha-\beta = a-b$ and $k = 0$, throughout most of the results derived here, reduce to known results given in [6] and [7]. Further, if we set $a-b = -\frac{1}{2}$ i.e. $n = 1$, the results coincide with those derived in [5].

Remarks:

1. If we set $\alpha = \frac{1}{4}+\frac{\mu}{2}$, $\beta = \frac{1}{4}-\frac{\mu}{2}$, $a = \frac{v}{2}$, $b = \frac{1}{2}-\frac{v}{2}$ throughout this paper, then the results derived here reduce to known results given in [7] and thereby in [4] and [5].

2. If we set $\alpha = \frac{1}{4}+\mu/2$, $\beta = \frac{1}{4}-\mu/2$, $a-b = -\frac{1}{2}$, then results coincide with those derived in [8].

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