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Discrete Mathematics

Elixir Dis. Math. 74 (2014) 27137-27150

ISSN: 2229-712X

Complexity of products of K_m with some special graphs

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ARTICLE INFO

Article history:
Received: 13 August 2014;
Received in revised form:
15 September 2014;
Accepted: 23 September 2014;

Keywords
Number of spanning trees,
Chebyshev polynomials,
Cartesian product, Tensor product,
Composition product, Normal product,
Strong sum.

ABSTRACT
In this paper we derive new formulas for the number of spanning trees, of some product graphs such as cartesian product, tensor product, composition product, normal product, and strong sum using linear algebra, Chebyshev polynomials and matrix theory techniques.

1. Introduction

The problem of calculating the number of spanning trees on the graph G is an important, well- studied problem in graph theory. Deriving formulas for different types of graphs can be prove to be helpful in identifying those graphs that contain the maximum number of spanning trees. Such an investigation has practical consequences related to network. Thus for both theoretical and practical consequences related to network. Thus for both theoretical and practical purpose, we interested to deriving formulas for the number of spanning trees of a graph based on its time complexity in order to calculate the formula. Many cases have been examined depending on the choice of G . It has been studied when G is labeled molecular graph [18], when G is a circulant graph [21], when G is a complete multipartite graph [20], when G is a cubic cycle and quadruple cycle graph [20], when G is a threshold graph [11] and so on. A spanning tree of G is a minimal connected subgraph of G that has the same vertex set as G . The number of spanning trees in G , also called, the complexity of the graph, denoted by $\tau(G)$. A classical result of Kirchhoff [5], can be used to determine the number of spanning trees for $G=(V,E)$. Let $V=\{v_1,v_2,.....,v_n\}$, then the Kirchhoff matrix H defined as $n\times n$ characteristic matrix $H=D-A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G, $H=[a_{ij}]$ defined as follows: (i) $a_{ij}=-1$ v_i and v_j are adjacent and $i\neq j$, (ii) a_{ij} equals the degree of vertex v_i if $i=j$, and (iii) $a_{ij}=0$ otherwise. All of cofactors of H are equal to $\tau(G)$.

There are other methods for calculating $\tau(G)$. Let, denote the eignvalues of H matrix of a p point graph. Then it is easily shown that $\mu_p=0$. Furthermore, Kelmans $\mu_1\geq\mu_1\geq.......\geq\mu_p$ and Chelnokov [1] shown that, $\tau(G)=\frac{1}{p}\prod_{k=1}^{p-1}\mu_k$. The formula for the number of spanning trees in a d-regular graph G can be expressed as $\tau(G)=\frac{1}{p}\prod_{k=1}^{p-1}(d-\lambda_k)$ where $\lambda_0=d,\lambda_1,\lambda_2,.......,\lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding

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spanning trees, especially when these numbers are very large. One of the first such results is due to Cayley [4] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that He showed $\tau(K_n) = n^{n-2}, n \geq 2$. Another result, $\tau(K_{p,q}) = p^{q-1}q^{p-1}, p,q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in e.g., [9, 10]. Another result is due to Sedlacek [7] who derived a formula for the wheel on $n+1$ vertices, W_{n+1}, He showed that $\tau(W_{n+1}) = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2$, for $n \geq 3$. Sedlacek [8] also later derived a formula for the number of spanning trees in a Mobius ladder, M_n, $\tau(M_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2]$ for $n \geq 2$. Another class of graphs for which an explicit formula has been derived is based on a prism. Boesch, et .al. [2, 3].

Now we introduce following lemma which describe a way to calculate the number of spanning trees by an extension of Kirchhoff formula.

Lemma 1.1 [14] Let G be a graph with n vertices. Then

$$\tau(G) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A})$$

where $\bar{A}$, $\bar{D}$ are the adjacency and degree matrices of $\bar{G}$, the complement of G , respectively, and I is the $n \times n$ unit matrix.

The advantage of these formula in Lemma1.1 is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, Yuanping, et. al. [22].

Let $A_n(x)$ be $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \mathrm{L} & 0 \\ -1 & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{M} \\ 0 & \mathrm{O} & \mathrm{O} & \mathrm{O} & 0 \\ \mathrm{M} & \mathrm{O} & \mathrm{O} & \mathrm{O} & -1 \\ 0 & \mathrm{L} & 0 & -1 & 2x \end{pmatrix},$$

where all other elements are zeros.

Further we recall that the Chebyshev polynomials of the first kind defined by:

$$T_n(x) = \cos(n \arccos x) \qquad (1)$$

The Chebyshev polynomials of the second kind are defined by:

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)} \qquad (2)$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \qquad (3)$$

It can then be shown from this recursion that by expanding $\det A_n(x)$ one gets

$$U_n(x) = \det(A_n(x)),\ \ n \geq 1 \qquad (4)$$

Furthermore by using standard methods for solving the recursion (3), one obtains the explicit formula

$$U_n(x)=\frac{1}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^{n+1}-(x-\sqrt{x^2-1})^{n+1}],\quad n\geq 1 \tag{5}$$

where the identity is true for all complex x (except at $x=\pm 1$ where the function can be taken as the limit).

The definition of $U_n(x)$ easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x)=2^{n-1}\prod_{j=1}^{n-1}(x-\cos\frac{j\pi}{n}) \tag{6}$$

One further notes that:

$$U_{n-1}(-x)=(-1)^{n-1}U_{n-1}(x) \tag{7}$$

These two results yield another formula for $U_n(x)$,

$$U_{n-1}{}^2(x)=4^{n-1}\prod_{j=1}^{n-1}(x^2-\cos^2\frac{j\pi}{n}) \tag{8}$$

Finally, simple manipulation of the above formula yields the following, which also will be extremely useful to us latter:

$$U_{n-1}{}^2(\sqrt{\frac{x+2}{4}})=\prod_{j=1}^{n-1}(x-2\cos\frac{2j\pi}{n}) \tag{9}$$

Furthermore one can show that:

$$U_{n-1}{}^2(x)=\frac{1}{2(1-x^2)}[1-T_{2n}]=\frac{1}{2(1-x^2)}[1-T_n(2x^2-1)], \tag{10}$$

and

$$T_n(x)=\frac{1}{2}[(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n]. \tag{11}$$

Lemma 2.1 [15] Let $B_n(x)$ be $n\times n$ matrix such that:

$$B_n(x)=\begin{pmatrix} x & -1 & 0 & \cdots & \cdots & 0\\ -1 & 1+x & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 1+x & -1\\ 0 & \cdots & \cdots & 0 & -1 & x \end{pmatrix}$$

Then: $$\det(B_n(x))=(x-1)U_{n-1}(\frac{1+x}{2})$$

Lemma 2.2 [16] Let $C_n(x)$ be $n\times n$ matrix, $n\geq 3,\ x>2$ such that:

$$C_n(x)=\begin{pmatrix} x & 0 & 1 & \cdots & \cdots & 1\\ 0 & x+1 & \ddots & \ddots & \ddots & \vdots\\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 1\\ \vdots & \ddots & \ddots & \ddots & x+1 & 0\\ 1 & \cdots & \cdots & 1 & 0 & x \end{pmatrix}$$

Then:

$$\det(C_n(x)) = (n + x - 2)U_{n-1}(\frac{x}{2}).$$

Lemma 2.3 [13] Let $D_n(x)$ be $n \times n$ matrix, $n \geq 3,\ x \geq 4$ such that:

$$D_n(x) = \begin{pmatrix} x & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & x \end{pmatrix}$$

Then:

$$\det(D_n(x)) = \frac{2(x+n-3)}{x-3}[T_n(\frac{x-1}{2}) - 1].$$

Lemma 2.4 [17] Let $E_n(x)$ be $n \times n$ matrix, $x \geq 2$ such that:

$$E_n(x) = \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix}$$

Then

$$\det(E_n) = (x + n - 1)(x - 1)^{n-1}.$$

We can generalize the above Lemma as follows:

Lemma 2.5 Let $A, B \in F^{n\times n}$ and $\mathrm{F} \in F^{kn\times kn}$ such that:

$$\mathrm{F} = \begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix}$$

Then:

$$\det \mathrm{F} = [\det(A - B)]^{k-1} \det[A + (k-1)B].$$

Formula in Lemma 2.5 gives some sort of symmetry in some matrices which facilitates our calculation of determinants.

3. Number of Spanning Trees of Cartesian Product of Some Graphs

The Cartesian product, $G_1 \times G_2$, of two graphs G_1 and G_2 is the simple graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$, and edge set $E(G_1 \times G_2) = [(E_1 \times V_2) \cup (V_1 \times E_2)]$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ iff, either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$ [6].

Theorem 3.1 For $m \geq 1,$ and $n \geq 2$, we have:

$$\tau(K_m \times P_n) = m^{m-2}\left[\frac{1}{2^n\sqrt{m^2+4m}}\left((m+2+\sqrt{m^2+4m})^n-(m+2-\sqrt{m^2+4m})^n\right)\right]^{m-1}.$$

Proof: Applying Lemma 1.1, we have:

$$\tau(K_m \times P_n) = \frac{1}{(mn)^2}\det(mn\,I - \overline{D} + \overline{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & \cdots & B \\ B & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B & \cdots & B & A \end{pmatrix}_{m\times m} ; \quad A = \begin{pmatrix} m+1 & 0 & 1 & \cdots & 1 \\ 0 & m+2 & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & m+2 & 0 \\ 1 & \cdots & 1 & 0 & m+1 \end{pmatrix}_{n\times n} , \quad B = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}_{n\times n}$$

Using Lemma 2.5, we get:

$$\tau(K_m \times P_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$= \frac{1}{(mn)^2}\left(\det\begin{pmatrix} m+1 & -1 & 0 & \cdots & 0 \\ -1 & m+2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & m+2 & -1 \\ 0 & \cdots & 0 & -1 & m+1 \end{pmatrix}\right)^{m-1} \times \det\begin{pmatrix} m+1 & m-1 & m & \cdots & m \\ m-1 & m+2 & \ddots & \ddots & \vdots \\ m & \ddots & \ddots & \ddots & m \\ \vdots & \ddots & \ddots & m+2 & m-1 \\ m & \cdots & m & m-1 & m+1 \end{pmatrix}$$

Using Lemma 2.1, we obtain:

$$\tau(K_m \times P_n) = \frac{1}{(m\,n)^2}\left(mU_{n-1}\left(\frac{m+2}{2}\right)\right)^{m-1} \times mn^2 = m^{m-2}\left(U_{n-1}\left(\frac{m+2}{2}\right)\right)^{m-1}$$

$$= m^{m-2}\left[\frac{1}{2^n\sqrt{m^2+4m}}\left((m+2+\sqrt{m^2+4m})^n-(m+2-\sqrt{m^2+4m})^n\right)\right]^{m-1}.$$

Theorem 3.2 For $m \geq 1$, and $n \geq 3$, we have:

$$\tau(K_m \times C_n) = \frac{n}{m\;2^{n(m-1)}}\left[(m+2+\sqrt{m^2+4m})^n+(m+2-\sqrt{m^2+4m})^n-2^{n+1}\right]^{m-1}.$$

Proof: Applying Lemma 1.1, we have:

$$\tau(K_m \times C_n) = \frac{1}{(mn)^2}\det(mn\,I - \overline{D} + \overline{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & \cdots & B \\ B & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B & \cdots & B & A \end{pmatrix}_{m\times m} ; \quad A = \begin{pmatrix} m+2 & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & m+2 \end{pmatrix}_{n\times n} , \quad B = \begin{pmatrix} 0 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix}_{n\times n}$$

Using Lemma 2.5, we get:

$$\tau(K_m \times C_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$= \frac{1}{(mn)^2}\left(\det\begin{pmatrix} m+2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & m+2 \end{pmatrix}\right)^{m-1} \times \det\begin{pmatrix} m+2 & m-1 & m & \cdots & m & m-1 \\ m-1 & \ddots & \ddots & \ddots & \ddots & m \\ m & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & m \\ m & \ddots & \ddots & \ddots & \ddots & m-1 \\ m-1 & m & \cdots & m & m-1 & m+2 \end{pmatrix}$$

$$= \frac{1}{(mn)^2} \times \left(\frac{m}{n+m}\det\begin{pmatrix} m+3 & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & m+3 \end{pmatrix}\right)^{m-1} \times mn^3.$$

Using Lemma 2.3, we obtain :

$$\begin{aligned}\tau(K_m \times C_n) &= \frac{1}{m^2 n^2}\left(\frac{m}{m+n}\right)^{m-1} \times \frac{2^{m-1}(m+n)^{m-1}}{m^{m-1}} \times [T_n(\frac{m+2}{2})-1]^{m-1} \times m n^3 \\ &= \frac{n}{m} \times 2^{m-1} \times [T_n(\frac{m+2}{2})-1]^{m-1} \\ &= \frac{n}{m\, 2^{n(m-1)}}\left[(m+2+\sqrt{m^2+4m})^n + (m+2-\sqrt{m^2+4m})^n - 2^{n+1}\right]^{m-1}.\end{aligned}$$

Theorem 3.3 For $m \geq 2$, and $n \geq 2$, we have:

$$\tau(K_m \times K_n) = m^{m-2}\ n^{n-2}\ (m+n)^{(m-1)(n-1)}.$$

Proof: Applying Lemma 1.1, we have:

$$\tau(K_m \times K_n) = \frac{1}{(mn)^2}\det(mn\,I - \overline{D} + \overline{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & \cdots & B \\ B & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B & \cdots & B & A \end{pmatrix};\ A = \begin{pmatrix} m+n-1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & m+n-1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}$$

Using Lemma 2.5, we get:

$$\tau(K_m \times K_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$= \frac{1}{(mn)^2}\left(\det\begin{pmatrix} m+n-1 & -1 & \cdots & \cdots & -1 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ -1 & \cdots & \cdots & -1 & m+n-1 \end{pmatrix}\right)^{m-1} \times \det\begin{pmatrix} m+n-1 & m-1 & \cdots & \cdots & m-1 \\ m-1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & m-1 \\ m-1 & \cdots & \cdots & m-1 & m+n-1 \end{pmatrix}$$

Using Lemma 2.4, yields :

$$\tau(K_m \times K_n) = \frac{1}{(mn)^2}\,[m\,(m+n)^{n-1}]^{m-1}\,[(m-1)^n\,\frac{mn}{m-1}\,(\frac{n}{m-1})^{n-1}]$$

$$= m^{m-2}\;n^{n-2}\;(m+n)^{(m-1)(n-1)}.$$

4. Number of Spanning Trees of Tensor Product of Some Graphs

The tensor product, or Kronecher product, $G_1 \otimes G_2$, of two graphs G_1 and G_2 is the simple graph with $V\,(G_1 \otimes G_2) = V_1 \times V_2$ where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ iff u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in, G_2 [12].

Theorem 4.1 For $m \geq 2$, and $n \geq 2$, we have:

$$\tau(K_m \otimes P_n) = \frac{(m-1)^{n-1}}{m}[\frac{m\,(m-2)}{2\sqrt{(m-1)^2-1}}((m-1+\sqrt{(m-1)^2-1}\,)^{n-1} - (m-1+\sqrt{(m-1)^2-1}\,)^{n-1}\,)]^{m-1}.$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m \otimes P_n) = \frac{1}{(mn)^2}\det(mn\,I - \bar{D} + \bar{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & \cdots & B \\ B & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B & \cdots & B & A \end{pmatrix}_{m\times m};\quad A = \begin{pmatrix} m & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & 2m-1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2m-1 & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & m \end{pmatrix}_{n\times n},\; B = \begin{pmatrix} 1 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & 0 & 1 \end{pmatrix}_{n\times n}$$

Using Lemma 2.5, we obtain:

$$\tau(K_m \otimes P_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$= \frac{1}{(mn)^2}(\det\begin{pmatrix} m-1 & 1 & 0 & \cdots & 0 \\ 1 & 2m-2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2m-2 & 1 \\ 0 & \cdots & 0 & 1 & m-1 \end{pmatrix})^{m-1} \times \det\begin{pmatrix} 2m-1 & 1 & m & \cdots & m \\ 1 & 3m-2 & \ddots & \ddots & \vdots \\ m & \ddots & \ddots & \ddots & m \\ \vdots & \ddots & \ddots & 3m-2 & 1 \\ m & \cdots & m & 1 & 2m-1 \end{pmatrix}$$

Straightforward induction using properties of Chebyshev polynomials. We have:

$$\tau(K_m \otimes P_n) = \frac{1}{(m\,n)^2}\,[m\,(m-2)U_{n-2}(m-1)]^{m-1}\,[m\,n^2\,(m-1)^{n-1}]$$

$$= \frac{(m-1)^{n-1}}{m}[\frac{m\,(m-2)}{2\sqrt{(m-1)^2-1}}((m-1+\sqrt{(m-1)^2-1}\,)^{n-1} - (m-1+\sqrt{(m-1)^2-1}\,)^{n-1})]^{m-1}.$$

Theorem 4.2 For $m \geq 2$, we have:

$$\tau(K_m \otimes C_n) = \begin{cases} \dfrac{n\,(m-1)^{n-1}}{m}[(m-1+\sqrt{(m-1)^2-1}\,)^n + (m-1-\sqrt{(m-1)^2-1}\,)^n + 2\,]^{m-1};\; n = 3,5,7,........ \\ \dfrac{n\,(m-1)^{n-1}}{m}[(m-1+\sqrt{(m-1)^2-1}\,)^n + (m-1-\sqrt{(m-1)^2-1}\,)^n - 2\,]^{m-1};\; n = 4,6,8,........ \end{cases}$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m \otimes C_n) = \frac{1}{(mn)^2}\det(mn\,I - \bar{D} + \bar{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & \cdots & B \\ B & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B & \cdots & B & A \end{pmatrix}_{m\times m};\quad A = \begin{pmatrix} 2m-1 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & 2m-1 \end{pmatrix}_{n\times n},\ B = \begin{pmatrix} 1 & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix}_{n\times n}$$

Using Lemma 2.5, we obtain:

$$\tau(K_m \otimes C_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$= \frac{1}{(mn)^2}\left(\det\begin{pmatrix} 2m-2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 1 & 2m-2 \end{pmatrix}\right)^{m-1} \times \det\begin{pmatrix} 3m-2 & 1 & m & \cdots & m & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & m \\ m & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & m \\ m & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & m & \cdots & m & 1 & 3m-2 \end{pmatrix}$$

Using Lemma 2.3. We get:

$$\tau(K_m \otimes C_n) = \begin{cases} \dfrac{1}{(mn)^2}\,(2[T_n(m-1)+1])^2\,(mn^3(m-1)^{n-1});\ m\ge 2, n=3,5,7,\ldots\ldots.. \\ \dfrac{1}{(mn)^2}\,(2[T_n(m-1)-1])^2\,(mn^3(m-1)^{n-1});\ m\ge 2, n=4,6,8,\ldots\ldots.. \end{cases}$$

Thus:

$$\tau(K_m \otimes C_n) = \begin{cases} \dfrac{n(m-1)^{n-1}}{m}[(m-1+\sqrt{(m-1)^2-1})^n + (m-1-\sqrt{(m-1)^2-1})^n + 2]^{m-1};\ m\ge 2, n=3,5,7,\ldots\ldots.. \\ \dfrac{n(m-1)^{n-1}}{m}[(m-1+\sqrt{(m-1)^2-1})^n + (m-1-\sqrt{(m-1)^2-1})^n - 2]^{m-1};\ m\ge 2, n=4,6,8,\ldots\ldots.. \end{cases}$$

Theorem 4.3 For $m \ge 2,$ and $n \ge 2,$ we have:

$$\tau(K_m \otimes K_n) = m^{m-2}\, n^{n-2}\, (n-1)^{m-1}(m-1)^{n-1}\, (mn-m-n)^{(m-1)(n-1)}.$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m \otimes K_n) = \frac{1}{(mn)^2}\det(mn\,I - \bar{D} + \bar{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & \cdots & B \\ B & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B & \cdots & B & A \end{pmatrix}_{m\times m};\ A = \begin{pmatrix} (m-1)n-(m-2) & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & (m-1)n-(m-2) \end{pmatrix}_{n\times n},\ B = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n}$$

Using Lemma 2.5, we obtain:

$$\tau(K_m \otimes K_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$=\frac{1}{(mn)^2}(\det\begin{pmatrix}(m-1)(n-1) & 1 & \cdots & \cdots & 1\\ 1 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 1\\ 1 & \cdots & \cdots & 1 & (m-1)(n-1)\end{pmatrix})^{m-1}\times\det\begin{pmatrix}(m-1)n+1 & 1 & \cdots & \cdots & 1\\ 1 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 1\\ 1 & \cdots & \cdots & 1 & (m-1)n+1\end{pmatrix}$$

Using Lemma 2.4, we have :

$$\tau(K_m\otimes K_n)=\frac{1}{(mn)^2}[((m-1)(n-1)+n-1)((m-1)(n-1)-1)^{n-1}]^{m-1}(n(m-1)+1+n-1)(n(m-1))^{n-1}$$

$$=m^{m-2}\ n^{n-2}(n-1)^{m-1}(m-1)^{n-1}(m\,n-m-n)^{(m-1)(n-1)}.$$

5. Number of Spanning Trees of Composition Product of Some Graphs

The composition, or lexicographic product, $G_1[G_2]$, of two graphs G_1 and G_2 is the simple graph with $V_1\times V_2$ as the vertex set in which the vertices (u_1,u_2) and (v_1,v_2) are adjacent if either u_1 is adjacent to v_1 or $u_1=v_1$ and u_2 is adjacent to v_2 in G_2 [12].

Theorem 5.1 For $m\geq 1$, and $n\geq 2$, we have:

$$\tau(K_m[P_n])=m^{m-2}n^{m-2}[\frac{1}{2\sqrt{(\frac{(m-1)n+2}{2})^2-1}}((\frac{(m-1)n+2}{2}+\sqrt{(\frac{(m-1)n+2}{2})^2-1})^n-(\frac{(m-1)n+2}{2}-\sqrt{(\frac{(m-1)n+2}{2})^2-1})^n)]^m$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m[P_n])=\frac{1}{(mn)^2}\det(mn\,I-\overline{D}+\overline{A})$$

$$=\frac{1}{(mn)^2}\det\begin{pmatrix}A & B & \cdots & B\\ B & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & B\\ B & \cdots & B & A\end{pmatrix}_{m\times m};\ A=\begin{pmatrix}(m-1)n+2 & 0 & 1 & \cdots & 1\\ 0 & (m-1)n+3 & \ddots & \ddots & \vdots\\ 1 & \ddots & \ddots & \ddots & 1\\ \vdots & \ddots & \ddots & (m-1)n+3 & 0\\ 1 & \cdots & 1 & 0 & (m-1)n+2\end{pmatrix}_{n\times n},\ B=\begin{pmatrix}0 & \cdots & \cdots & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & \cdots & 0\end{pmatrix}_{n\times n}$$

$$=\frac{1}{(mn)^2}(\det\begin{pmatrix}(m-1)n+2 & 0 & 1 & \cdots & 1\\ 0 & (m-1)n+3 & \ddots & \ddots & \vdots\\ 1 & \ddots & \ddots & \ddots & 1\\ \vdots & \ddots & \ddots & (m-1)n+3 & 0\\ 1 & \cdots & 1 & 0 & (m-1)n+2\end{pmatrix})^m$$

Using Lemma 2.2, we obtain:

$$\tau(K_m[P_n])=\frac{1}{(mn)^2}(mnU_{n-1}(\frac{(m-1)n+2}{2}))^m=m^{m-2}\ n^{m-2}U_{n-1}^m(\frac{(m-1)n+2}{2})$$

$$=m^{m-2}\ n^{m-2}[\frac{1}{2\sqrt{(\frac{(m-1)n+2}{2})^2-1}}((\frac{(m-1)n+2}{2}+\sqrt{(\frac{(m-1)n+2}{2})^2-1})^n-(\frac{(m-1)n+2}{2}-\sqrt{(\frac{(m-1)n+2}{2})^2-1})^n)]^m.$$

Theorem 5.2 For $m\geq 1$, and $n\geq 3$, we have:

$$\tau(K_m[C_n])=\frac{m^{m-2}}{n^2(m-1)^m}[(\frac{(m-1)n+2}{2}+\sqrt{(\frac{(m-1)n+2}{2})^2-1})^n+((\frac{(m-1)n+2}{2}-\sqrt{(\frac{(m-1)n+2}{2})^2-1})^n-2]^m.$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m[C_n]) = \frac{1}{(mn)^2}\det(mn\,I - \overline{D} + \overline{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & L & B \\ B & O & O & M \\ M & O & O & B \\ B & L & B & A \end{pmatrix}_{m\times m} ;\ A = \begin{pmatrix} (m-1)n+3 & 0 & 1 & L & 1 & 0 \\ 0 & O & O & O & O & 1 \\ 1 & O & O & O & O & M \\ M & O & O & O & O & 1 \\ 1 & O & O & O & O & 0 \\ 0 & 1 & L & 1 & 0 & (m-1)n+3 \end{pmatrix}_{n\times n},\ B = \begin{pmatrix} 0 & L & L & L & L & 0 \\ M & O & O & O & O & M \\ M & O & O & O & O & M \\ M & O & O & O & O & M \\ M & O & O & O & O & M \\ 0 & L & L & L & L & 0 \end{pmatrix}_{n\times n}$$

$$= \frac{1}{(mn)^2}(\det\begin{pmatrix} (m-1)n+3 & 0 & 1 & L & 1 & 0 \\ 0 & O & O & O & O & 1 \\ 1 & O & O & O & O & M \\ M & O & O & O & O & 1 \\ 1 & O & O & O & O & 0 \\ 0 & 1 & L & 1 & 0 & (m-1)n+3 \end{pmatrix})^m$$

Using Lemma 2.3. We get:

$$\tau(K_m[C_n]) = \frac{1}{(mn)^2}(\frac{2m}{m-1}[T_n(\frac{(m-1)n+2}{2})-1])^m = \frac{2^m m^{m-2}}{n^2(m-1)^m}([T_n(\frac{(m-1)n+2}{2})-1])^m$$
$$= \frac{m^{m-2}}{n^2(m-1)^m}[((\frac{(m-1)n+2}{2}+\sqrt{(\frac{(m-1)n+2}{2})^2-1})^n+((\frac{(m-1)n+2}{2}-\sqrt{(\frac{(m-1)n+2}{2})^2-1})^n-2]^m.$$

6. Number of Spanning Trees of Normal Product of Some Graphs

The normal product, or the strong product, $G_1 \mathrm{o} G_2$, of two graphs G_1 and G_2 is the simple graph with $V(G_1 \mathrm{o} G_2) = V_1 \times V_2$ where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \mathrm{o} G_2$ iff either $u_1 = v_1$ and u_2 is adjacent to v_2, or u_1 is adjacent to v_1 and $u_2 = v_2$, or u_1 is adjacent to v_1 and u_2 is adjacent to v_2 [12].

Theorem 6.1 For $m \geq 1$, and $n \geq 2$, we have:

$$\tau(K_m \mathrm{o} P_n) = 2^{2m-2} \times 3^{(m-1)(n-2)} \times m^{mn-2}.$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m \mathrm{o} P_n) = \frac{1}{(mn)^2}\det(mn\,I - \bar{D} + \bar{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & L & B \\ B & O & O & M \\ M & O & O & B \\ B & L & B & A \end{pmatrix};\ A = \begin{pmatrix} 2m & 0 & 1 & L & L & 1 \\ 0 & 3m & O & O & O & M \\ 1 & O & O & O & O & M \\ M & O & O & O & O & 1 \\ M & O & O & O & 3m & 0 \\ 1 & L & L & 1 & 0 & 2m \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 1 & L & L & 1 \\ 0 & O & O & O & O & M \\ 1 & O & O & O & O & M \\ M & O & O & O & O & 1 \\ M & O & O & O & O & 0 \\ 1 & L & L & 1 & 0 & 0 \end{pmatrix}$$

Using Lemma 2.5, we get:

$$\tau(K_m \mathrm{o} P_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$= \frac{1}{(mn)^2}(\det\begin{pmatrix} 2m & 0 & L & L & 0 \\ 0 & 3m & O & O & M \\ M & O & O & O & M \\ M & O & O & 3m & 0 \\ 0 & L & L & 0 & 2m \end{pmatrix})^{m-1}\ \det\begin{pmatrix} 2m & 0 & m & L & m \\ 0 & 3m & O & O & M \\ m & O & O & O & m \\ M & O & O & 3m & 0 \\ m & L & m & 0 & 2m \end{pmatrix}$$

$$= \frac{1}{(mn)^2}\ [(2m)^2\,(3m)^{n-2}]^{m-1} \times m^n\ \ n^2 = 2^{2m-2} \times 3^{(m-1)(n-2)} \times m^{mn-2}.$$

Theorem 6.2 For $m \geq 1$, and $n \geq 3$, we have:

$$\tau(K_m \circ C_n) = 3^{n(m-1)} \times n\ m^{mn-2}.$$

Proof : Applying Lemma 1.1, we get:

$$\tau(K_m \circ C_n) = \frac{1}{(mn)^2}\det(mn\,I - \bar{D} + \bar{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & \cdots & B \\ B & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B & \cdots & B & A \end{pmatrix}_{m\times m} ;\ A = \begin{pmatrix} 3m & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 3m \end{pmatrix}_{n\times n},\ B = \begin{pmatrix} 0 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & 0 & 0 \end{pmatrix}_{n\times n}$$

Using Lemma 2.5, we obtain:

$$\tau(K_m \circ C_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$= \frac{1}{(mn)^2}(\det\begin{pmatrix} 3m & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 3m \end{pmatrix})^{m-1} \times \det\begin{pmatrix} 3m & 0 & m & \cdots & m & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & m \\ m & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & m \\ m & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & m & \cdots & m & 0 & 3m \end{pmatrix}$$

$$= \frac{1}{(mn)^2}((3m)^n)^{m-1} \times m^n\ n^3 = 3^{n(m-1)} \times n\ \ m^{mn-2}.$$

Theorem 6.3 For $m \geq 1$, and $n \geq 2$, we have:

$$\tau(K_m \circ K_n) = \tau(K_m[K_n]) = (m\,n)^{m\,n-2}.$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m \circ K_n) = \tau(K_m[K_n]) = \frac{1}{(mn)^2}\det(mn\,I - \bar{D} + \bar{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & \cdots & B \\ B & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B & \cdots & B & A \end{pmatrix}_{m\times m} ;\ A = \begin{pmatrix} mn & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & mn \end{pmatrix}_{n\times n},\ B = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n}$$

$$= \frac{1}{(mn)^2}(\det\begin{pmatrix} mn & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & mn \end{pmatrix})^m = \frac{1}{(mn)^2} \times (mn)^{mn} = (mn)^{mn-2}.$$

7. Number of spanning trees of Strong Sum of Graphs

The Strong, $G_1 \oplus G_2$, of two graphs G_1 and G_2 is the simple graph with $V\ (G_1 \oplus G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \oplus G_2$ iff u_2 is adjacent to v_2 in G_2 and either u_1 is adjacent to v_1 in G_1, or $u_1 = v_1$[12].

Theorem 7.1 For $m \geq 1$ and $n \geq 2,$ we have:

$$\tau(K_m \oplus P_n) = 2^{(m-1)(n-2)} \times m^{mn-2}.$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m \oplus P_n) = \frac{1}{(mn)^2}\det(mn\,I - \bar{D} + \bar{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & L & B \\ B & O & O & M \\ M & O & O & B \\ B & L & B & A \end{pmatrix}_{m\times m} ;\ A = \begin{pmatrix} m+1 & 0 & 1 & L & L & 1 \\ 0 & 2m+1 & O & O & O & M \\ 1 & O & O & O & O & M \\ M & O & O & O & O & 1 \\ M & O & O & O & 2m+1 & 0 \\ 1 & L & L & 1 & 0 & 2m \end{pmatrix}_{n\times n},\ B = \begin{pmatrix} 1 & 0 & 1 & L & L & 1 \\ 0 & O & O & O & O & M \\ 1 & O & O & O & O & M \\ M & O & O & O & O & 1 \\ M & O & O & O & O & 0 \\ 1 & L & L & 1 & 0 & 1 \end{pmatrix}_{n\times n}$$

Using Lemma 2.5, we get:

$$\tau(K_m \oplus P_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$= \frac{1}{(mn)^2}(\det\begin{pmatrix} m & 0 & L & L & 0 \\ 0 & 2m & O & O & M \\ M & O & O & O & M \\ M & O & O & 2m & 0 \\ 0 & L & L & 0 & m \end{pmatrix})^{m-1}\ \det\begin{pmatrix} 2m & 0 & m & L & m \\ 0 & 3m & O & O & M \\ m & O & O & O & m \\ M & O & O & 3m & 0 \\ m & L & m & 0 & 2m \end{pmatrix}$$

$$= \frac{1}{(mn)^2}\,[m^2\,(2m)^{n-2}]^{m-1} \times m^n\ \ n^2 = 2^{(m-1)(n-2)} \times m^{mn-2}.$$

Theorem 7.2 For $m \geq 1,$ and $n \geq 3,$ we have:

$$\tau(K_m \oplus C_n) = 2^{(m-1)n} \times n\,m^{mn-2}.$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m \oplus C_n) = \frac{1}{(mn)^2}\det(mn\,I - \bar{D} + \bar{A})$$

$$= \frac{1}{(mn)^2}\det\begin{pmatrix} A & B & L & B \\ B & O & O & M \\ M & O & O & B \\ B & L & B & A \end{pmatrix}_{m\times m} ;\ A = \begin{pmatrix} 2m+1 & 0 & 1 & L & 1 & 0 \\ 0 & O & O & O & O & 1 \\ 1 & O & O & O & O & M \\ M & O & O & O & O & 1 \\ 1 & O & O & O & O & 0 \\ 0 & 1 & L & 1 & 0 & 2m+1 \end{pmatrix}_{n\times n},B = \begin{pmatrix} 1 & 0 & 1 & L & 1 & 0 \\ 0 & O & O & O & O & 1 \\ 1 & O & O & O & O & M \\ M & O & O & O & O & 1 \\ 1 & O & O & O & O & 0 \\ 0 & 1 & L & 1 & 0 & 1 \end{pmatrix}_{n\times n}$$

Using Lemma 2.5, we get:

$$\tau(K_m \oplus C_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$=\frac{1}{(mn)^2}(\det\begin{pmatrix} 2m & 0 & L & L & L & 0 \\ 0 & O & O & O & O & M \\ M & O & O & O & O & M \\ M & O & O & O & O & M \\ M & O & O & O & O & 0 \\ 0 & L & L & L & 0 & 2m \end{pmatrix})^{m-1} \det\begin{pmatrix} 3m & 0 & m & L & m & 0 \\ 0 & O & O & O & O & m \\ m & O & O & O & O & M \\ M & O & O & O & O & m \\ m & O & O & O & O & 0 \\ 0 & m & L & m & 0 & 3m \end{pmatrix}$$

$$=\frac{1}{(mn)^2}\ (2m)^{n(m-1)}\times m^n\ n^3 = 2^{(m-1)n}\times nm^{mn-2}.$$

Theorem 7.3 For $m, n \geq 2,$ we have:

$$\tau(K_m \oplus K_n) = \frac{1}{(mn)^2}\ (n(m-1))^{n(m-1)}\ (n(m-1)+m)^n.$$

Proof: Applying Lemma 1.1, we get:

$$\tau(K_m \oplus K_n) = \frac{1}{(mn)^2}\det(mn\,I - \overline{D} + \overline{A})$$

$$=\frac{1}{(mn)^2}\det\begin{pmatrix} A & B & L & B \\ B & O & O & M \\ M & O & O & B \\ B & L & B & A \end{pmatrix}_{m\times m};\ A=\begin{pmatrix} n(m-1)+1 & 0 & L & L & 0 \\ 0 & O & O & O & M \\ M & O & O & O & M \\ M & O & O & O & 0 \\ 0 & L & L & 0 & n(m-1)+1 \end{pmatrix}_{n\times n},\ B=\begin{pmatrix} 1 & 0 & L & L & 0 \\ 0 & O & O & O & M \\ M & O & O & O & M \\ M & O & O & O & 0 \\ 0 & L & L & 0 & 1 \end{pmatrix}_{n\times n}$$

Using Lemma 2.5, we get:

$$\tau(K_m \oplus C_n) = \frac{1}{(mn)^2}[\det(A-B)]^{m-1}[\det(A+(m-1)B)]$$

$$=\frac{1}{(mn)^2}(\det\begin{pmatrix} n(m-1) & 0 & L & L & 0 \\ 0 & O & O & O & M \\ M & O & O & O & M \\ M & O & O & O & 0 \\ 0 & L & L & 0 & n(m-1) \end{pmatrix})^{m-1} \det\begin{pmatrix} n(m-1)+m & 0 & L & L & 0 \\ 0 & O & O & O & M \\ M & O & O & O & M \\ M & O & O & O & 0 \\ 0 & L & L & 0 & n(m-1)+m \end{pmatrix}$$

$$=\frac{1}{(mn)^2}\ (n(m-1))^{n(m-1)}\ (n(m-1)+m)^n.$$

8. Conclusion

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and their proofs.

9. References

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