Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 74 (2014) 26806-26820

ISSN: 2229-712X

On the theory and applications of Hardy and Bergman spaces

Elhadi Elnour Elniel* and M. E. Hassan

Department of Mathematics, College of Science and Arts, Taif University, Saudi Arabia.

ARTICLE INFO

Article history:

Received: 20 April 2014;

Received in revised form:

21 August 2014;

Accepted: 29 August 2014;

Keywords

Hardy Spaces, Bergman Spaces, Composition operators, boundedness, Compactness, Nevanlinna counting functions.

ABSTRACT

We study composition operators between higher order weighted Bergman spaces. Certain growth conditions for generalized Nevanlinna counting functions of the inducing map are shown to be necessary and sufficient for such operators to be bounded or compact. Under a mind condition we show that a composition operators C_φ is compact on the higher order weighted Bergman spaces and Hardy spaces of the open unit ball in C^n if and only if $\frac{1-|z|^2}{1-|\varphi(z)|^2} \to 0$ as $|z| \to 1$-.

Introduction

Let D be the open unit disk in the complex plane and denote Lebesgue measure on D by dA, normalized so that A(D) = 1. The Hardy space H^p is the space of functions f that are analytic on D and satisfy

$$\|f\|_{H^p}^p = \lim_{r\to 1-} \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta < \infty,$$

and the Bergman space A^p consists of those analytic functions such that

$$\|f\|_{A^p}^p = \int_D |f|^p \, dA < \infty.$$

Let $\varphi : D \to D$ be an analytic self-map of D. It is a well known consequence of Littlewood's subordination principle [1], [2] that φ induces through composition a bounded linear operator on the classic hardy and Bergman spaces (see for example [3], [4] , [5] ,[6] or [7]. That is, if we define C_φ by $C_\varphi(f) = f \circ \varphi$, then

$C_\varphi : H^p \to H^q$ and $C_\varphi : A^p \to A^p$ are bounded operators. Such operators are called composition operators.

The open unit ball in n- dimensional complex Euclidean Spaces

$C^n = C \times C \times \cdots \times C$ is the set $B_n = \{ z \in \mathbb{C}^n : |z| < 1 \}$.

The space of holomorphic functions in B_n will be denoted by H (B_n). Let dv be Lebesque volume measure on B_n, normalized so that $v(B_n) = 1$. For any $\alpha > -1$ we let $dv_\alpha(z) = C_\alpha (1 - |z|^2)^\alpha \, dv(z)$,Where C_α is a positive constant chosen so that $v_\alpha(B_n) = 1$. The Weighted Bergman space A_α^p , where p > 0, consists of functions $f \in H(B_n)$ such that $\int_{B_n} |f(z)|^p dv_\alpha(z) < \infty$.

The space A_α^2 is a Hilbert space with inner product $\langle f, g\rangle = \int_{B_n} f(z)\overline{g(z)} dv_\alpha(z)$. Every holomorphic $\varphi : B_n \to B_n$ induces a composition operator

$C_\alpha : H(B_n) \to H(B_n)$ namely, $C_\alpha \, f = f \circ \varphi$. When n = 1, it is well known that C_α is always bounded on A_α^p ; and C_φ is compact on A_α^p if and only if

$\lim \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0$

See [8], [9],[10] and [11]. $|z| \to$

When n > 1, not every composition operator is bounded on A_α^p .

Tele:

E-mail addresses: elhadielniel_2003@hotmail.com

Theorem1.1. Suppose $p > 0$ and $\alpha > -1$. If the composition operator C_α is bounded on $\mathbf{A}^{\mathbf{p}}_{\boldsymbol{\beta}}$ for some $q > 0$ and $-1 < \boldsymbol{\beta} < \boldsymbol{\alpha}$, then C_α is compact on $\mathbf{A}^{\mathbf{p}}_{\boldsymbol{\alpha}}$ if and only if

$$\lim \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0 \qquad (1)$$

Theorem1.2. Let $0 < p \le q$ and suppose $\boldsymbol{\varphi}$ is an analytic self-map of D. Then

a) C_φ: $A^p \to A^q$ is bounded if and only if

$N_{\varphi,2}(w) = O([\log(1/|w|)]^{2q/p})$ $(|w| \to 1)$;

b) C_φ : $A^p \to H^q$ is bounded if and only if

$N_{\varphi,1}(w) = O([\log(1/|w|)]^{2q/p})$ $(|w| \to 1)$;

c) C_φ : $H^p \to A^q$ is bounded if and only if

$N_{\varphi,2}(w) = O([\log(1/|w|)]^{q/p})$ $(|w| \to 1)$;

Theorem1.3. [R, Theorem IV.3 and Theorem IV.4]. Let $0 < p \le q$ and suppose $\boldsymbol{\varphi}$ is an analytic self-map of D. Then C_φ : $H^p \to H^q$ is bounded if and only if

$N_{\varphi,1}(w) = O([\log(1/|w|)]^{q/p})$ $(|w| \to 1)$,

and moreover C_φ is compact if and only if

$N_{\varphi,1}(w) = O([\log(1/|w|)^{q/p})$ $(|w| \to 1)$.

Corollary1.4. Let $\eta \ge 2$ and suppose φ is an analytic self-map of D. Then the following are equivalent.

a) There exists $p > 0$ such that C_φ : $H^p \to H^{\eta q}$ is bounded;

b) C_φ : $H^p \to H^{\eta p}$ is bounded for all $p > 0$;

c) There exists $p > 0$ such that C_φ : $A^p \to H^{\eta p/2}$ is bounded;

d) C_φ: $A^p \to H^{\eta p/2}$ is bounded for all $p > 0$.

Moreover, these four statements remain equipment when “bounded” is replaced by “compact”

Corollary1.5. Let $\eta \ge 1$ and suppose φ is an analytic self-map of D. Then the following are equivalent.

a) There exists $p > 0$ such that C_φ : $A^p \to H^{\eta p}$ is bounded;

b) C_φ : $A^p \to A^{\eta p}$ is bounded for all $p > 0$;

c) There exists $p > 0$ such that C_φ: $H^p \to A^{2\eta p}$ is bounded;

d) C_φ : $H^p \to A^{2\eta p}$ is bounded for all $p > 0$.

Moreover, these four statements remain equipment when “bounded” is replaced by “compact” .

Corollary1.6. Let $\eta \ge 1$ and suppose φ is an analytic self-map of D.

If C_φ : $H^p \to H^{\eta p}$ is bounded for some (and hence all) $p > 0$, then C_φ : $A^p \to A^{\eta p}$ is bounded for all $p > 0$. Moreover, this remains true when “bounded” is replaced by “compact”.

Background

Definition2.1 We introduce a family of weighted Bergman type spaces that allows us to handle the classical Bergman and Hardy spaces in a unified manner.

For $\alpha > -1$ define the measure dA_α on D by $dA_\alpha(w) = [\log(1/|w|)]^\alpha dA(w)$.

For $0 < p < \infty$ and $\alpha > -1$ we define the weighted Bergman space $\mathbf{A}^{\mathbf{p}}_{\boldsymbol{\alpha}}$ to be those functions f analytic on D and satisfying.

$$\|\mathbf{f}\|^{\mathbf{p}}_{\mathbf{A}^{\mathbf{p}}_{\boldsymbol{\alpha}}} = \int_{\mathbf{D}} |\mathbf{f}(\mathbf{w})|^{\mathbf{p}} \mathbf{dA}_{\boldsymbol{\alpha}}(\mathbf{w}) < \infty.$$

In this definition, the measure dA_α can be replaced by the measure $(1 - |w|)^\alpha\, dA(w)$, as in [3], [12] and [13] . This result in the same space of functions and an equivalent norm, since $(1 - |w|)^\alpha$ and $[\log (1/|w|)]^\alpha$ are comparable for $\frac{1}{2} \le |w| < 1$, and the singularity of dA_α at the origin is integrable.

Definition2.2 Let dA (z) be the area measure on D normalized so that area of D is 1. For each $\alpha \in (-1,\infty)$, we set $dA_\alpha(z)= (\alpha+1)\,(1 - |z|^2)^\alpha\, dA(z)$, $z \in D$.

Then dA_α is a probability measure on D. For $0 < p < \infty$ the weighted Bergman space $\mathbf{A^p_\alpha}$ is defined as

$$\mathbf{A^p_\alpha} = \left\{ \mathbf{f} \in \mathbf{H(D)}: \left|\left|\mathbf{f}\right|\right|_{\mathbf{A^p_\alpha}} = \left(\int_{\mathbf{D}} |\mathbf{f(z)}|^{\mathbf{p}} \mathbf{dA}_\alpha(\mathbf{z}) \right)^{\mathbf{1/p}} < \infty \right\}.$$

Note that $\left|\left|\mathbf{f}\right|\right|_{\mathbf{A^p_\alpha}}$ is a true norm only if $1 \le p < \infty$ and in this case $\mathbf{A^p_\alpha}$ is a Branch space.

Definition2.3 For any $\alpha > 0$, the space $A^{-\alpha}$ consists of Analytic functions f in D such that $||\mathbf{f}||_{\mathbf{A}^{-\alpha}} = \mathbf{sup}\{(\mathbf{1} - |\mathbf{z}|^{\alpha})^{\alpha} |\mathbf{f(z)}| : \mathbf{z} \in \mathbf{D}\} < \infty.$

Each $A^{-\alpha}$ is a non-separable Branch space with the norm defined above and contains all bounded analytic functions on D. The closure in $A^{-\alpha}$ of the set of polynomials will be denoted by $\mathbf{A_0^{-\alpha}}$, which is a separable Banach space and consists of exactly those functions f in $A^{-\alpha}$ with

$$\lim_{\mathbf{z} \to \mathbf{1-}} (\mathbf{1} - |\mathbf{z}|^2)^{\alpha} |\mathbf{f(z)}| = \mathbf{0}$$

For general background on weighted Berman spaces $\mathbf{A^p_\alpha}$ and Bergman type spaces, $A^{-\alpha}$ and $\mathbf{A_0^{-\alpha}}$, one may consult [14] and [15] and the references therein.

Lemma2.4. If $0 < p < \infty$, then

$$||\mathbf{f}||^{\mathbf{p}}_{\mathbf{A^p_\alpha}} \approx |\mathbf{f(0)}|^{\mathrm{P}} + \int_{\mathbf{D}} |\mathbf{f(w)}|^{\,\mathrm{P}-2} |\mathbf{f'(w)}|^{\mathbf{2}} \mathbf{dA}_{\alpha+\mathbf{2}}(\mathbf{w}).$$

Here the symbol "≈" means that the left hand side is bounded above and below by constant multiples of the right hand side, where the constants are positive and independent of f.

Proposition 2.5. Let $\boldsymbol{\varphi}$ be an analytic self-map of D and let f be analytic on D. Then, for $\boldsymbol{\alpha} \ge -1$, $||\mathbf{fo}\boldsymbol{\varphi}||^{\mathbf{p}}_{\mathbf{A^p_\alpha}} \approx |\mathbf{f}(\boldsymbol{\varphi}(\mathbf{0}))|^{\mathrm{P}} + \int_{\mathbf{D}} |\mathbf{f}|^{\,\mathrm{P}-2} |\mathbf{f}'|^{\mathbf{2}} \mathbf{N}_{\varphi,\alpha+\mathbf{2}}\, \mathbf{dA}$

Lemma2.6. Let $0 < p < \infty$ and $\alpha \ge -1$. If $f \in \mathbf{A^p_\alpha}$ and $w \in D$, then

$| f(w)| \le C\, ||\mathbf{f}||_{\mathbf{A^p_\alpha}} (\mathbf{1} - |\mathbf{w}|)^{-(\alpha+2)/\mathbf{p}}.$

Lemma2.7. Suppose $p > 0$ and $\alpha > -1$. Then the following conditions are equivalent for any positive Borel measure μ on B_n .

(i) μ is a Carleson measure for $\mathbf{A^p_\alpha}$, that is, there exists a constant $C > 0$ such that

$\int_{\mathbf{B_n}} |\mathbf{f}|^{\mathbf{p}} \mathbf{d}\mu \le \mathbf{C} \int_{\mathbf{B_n}} |\mathbf{f(z)}|^{\mathbf{p}} \mathbf{dv}_{\alpha}$ for all $f \in \mathbf{A^p_\alpha}$

(ii) For some (or each) $R > 0$ there exists a constant $C > 0$ (depending on R and α but independent of a) such that

$\mu\,(D\,(a,R)) \le Cv_\alpha\,(D\,(a,R))$

For all $a \in B_n$ where D (a,R) is the Bergman metric ball at a with radius R.

Proof. See [14] for example.

Corollary 2.8. Suppose $p > 0$, $q > 0$, and $\alpha > -1$. Then C_α is compact on $\mathbf{A^p_\alpha}$ if and only if C_φ is compact on $\mathbf{A^p_\alpha}$.

We need two more technical lemmas. The first of which is called Schur's test and concerns the boundedness of integral operators on L^p spaces. Thus we consider a measure space (X ,μ) and an integral operator

$$\mathbf{T\,f(x)} = \int_{\mathbf{X}} \mathbf{H(x,y)f(y)d\mu(y)}, \qquad (2)$$

Where H is a nonnegative measerable function on X x X.

Lemma 2.9. Suppose there exist a positive measurable function h on X such that

$$\int_X \mathbf{H(x,y)h(y)d\mu(y) \le Ch(x)}$$

for almost all x and

$$\int_X \mathbf{H(x,y)h(x)d\mu(x) \le Ch(y)}$$

for almost all y , where C is a positive constant. Then the integral operator T defined in (2) is bounded on $L^2(X, d\mu)$. Moreover, the norm of T on $L^2(X, d\mu)$ is less than or equal to the constant C.

proof. See [16], [17]and [18].

Lemma 2.10. Suppose $\alpha > -1$ and $t > 0$. Then there exists a constant $C > 0$ such that

$$\int_{B_n} \frac{dv_\alpha(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+t}} \le \frac{C}{(1-|z|^2)^t} \quad \text{for all } z \in B_n$$

Proof. See [2].

Theorem 2.11. Suppose $p > 0$, $\alpha > -1$ and $t > 0$. Then the composition operator C_α is bounded on $\mathbf{A}^p_\alpha$ if and only if

$$\sup_{a\in B_n} (1-|a|^2)^t \int_{B_n} \frac{dv_\alpha(z)}{|1-\langle \alpha,\varphi(z)\rangle|^{n+1+\alpha+t}} < \infty. \qquad (3)$$

Proof. It follows from Lemma2.10 that the boundedness of C_φ on $\mathbf{A}^p_\alpha$ implies condition (3).

Next we assume that condition (3) holds. Then by the change of variables formula there exists a constant $C > 0$ such that

$$(1-|a|^2)^t \int_{B_n} \frac{d\mu_{\varphi,\alpha}(z)}{|1-\langle a,z\rangle|^{n+1+\alpha+t}} \le C$$

For all $a \in B_n$. For any fixed positive radius R we have

$$(1-|a|^2)^t \int_{D(a,R)} \frac{d\mu_{\varphi,\alpha}(z)}{|1-\langle a,z\rangle|^{n+1+\alpha+t}} \le C$$

for all $a \in B_n$. It is well known that

$$|\mathbf{1}-\langle a,z\rangle| \sim \mathbf{1}-|a|^2$$

for $z \in D(a,R)$, and it is also well known that

$$(\mathbf{1}-|a|^2)^{n+1+\alpha} \sim v_\alpha(D(a,R));$$

see [14]. It follows that there exists another positive constant C (independent of a) such that

$$\mu_{\varphi,\alpha}(D(a,R)) \le Cv_\alpha(D(a,R))$$

for all $a \in B_n$. By Lemma 2.7, the measure $\mu_{\varphi,\alpha}$ is Carleson for $\mathbf{A}^p_\alpha$ and so the composition operator C_φ is bounded on $\mathbf{A}^p_\alpha$.

Theorem 2.12. Suppose $p > 0$, $\alpha > -1$, and $t > 0$. Then C_φ is compact on $\mathbf{A}^p_\alpha$ if and only if

$$\lim \ (1-|a|^2)^t \int_{B_n} \frac{dv_a(z)}{|1-\langle \alpha,\varphi(z)\rangle|^{n+1+\alpha+t}} = 0 \qquad (4)$$

$|a| \to$

Necessary and Sufficient Conditions for Bounded

In this section we justify Necessary and sufficient conditions for $C_\varphi : A^p_{(\alpha+1)^2} \to A^q_{(\beta+1)^2}$ to be bounded(see [2]and [10]).

For any holomorphic $\varphi : B_n \to B_n$ we can define a positive Borel measure $\mu_{\varphi,(\alpha+1)_2}$ on B_n as follows. Given a Borel set E in B_n , we set

$$\mu_{\varphi,(\alpha+1)_2}(E) = v_{(\alpha+1)_2}(\varphi^{-1}(E)) = C_\varphi \int_{\varphi^{-1}(E)} (1-|(z)|^2)^{(\alpha+1)}{}_2 \, dv(z) .$$

Obviously, $\mu_{\varphi,(\alpha+1)^2}$ is the pullback measure of $dv_{(\alpha+1)^2}$ under the map φ . Therefore, we have the following change of variables formula: $\int_{B_n} f\,(\varphi)dv_{(\alpha+1)^2} = \int_{B_n} fd\,\mu_{\varphi,(\alpha+1)^2}$,

Where f is either nonnegative or belongs to $L^1(B_n, d\mu_{\varphi,(\alpha+1)^2})$.

A sufficient condition for C_φ : $A^p_{(\alpha+1)^2} \to A^q_{(\beta+1)^2}$ to be bounded

Theorem3.1. Let $0 < p \le q$ suppose $\boldsymbol{\varphi}$ is an analytic self-map of D satisfying.

$\mathbf{N}_{(\beta+1)^2+2}(w) = O([\log(1/|w|)]^{((\alpha+1)^2+2)q/p})$ $(|w| \to 1)$.

Then C_φ : $A^p_{(\alpha+1)^2} \to A^q_{(\beta+1)^2}$ is bounded .

Proof.

By the Closed Graph Theorem, it suffices to show that $C_\varphi(f) \in A^q_{(\beta+1)^2}$ if

$f \in A^p_{(\alpha+1)^2}$.

Let $f \in A^p_{(\alpha+1)^2}$. Then, by Proposition 2.5,

$$\|\mathbf{f o\varphi}\|^q_{A^q_{(\alpha+1)^2}} \approx |\mathbf{f}(\boldsymbol{\varphi}(\mathbf{0}))|^q + \int_{\mathbf{D}} |\mathbf{f}|^{p-2}|\mathbf{f}'|^2 \mathbf{N}_{(\beta+1)^2+2}\, \mathbf{dA}$$

and it is clear that $C_\varphi(f) \in A^q_{(\beta+1)^2}$ if and only if there is $r \in (0,1)$ such that

$$\int_{\mathbf{D}\backslash r\mathbf{D}} |\mathbf{f}|^{q-2}|\mathbf{f}'|^2 \mathbf{N}_{(\beta+1)^2+2}\, \mathbf{dA} < \infty.$$

By our hypothesis on the growth of $\mathbf{N}_{(\beta+1)^2+2}$ there is a constant $K < \infty$ and

$r_0 \in (0,1)$ such that

$\mathbf{N}_{(\beta+1)^2+2}(w) \le K(\log(1/|w|))^{((\alpha+1)^2+2)q/p}$, $w \in \boldsymbol{D}\backslash \boldsymbol{r_0 D}$

Using this and the growth estimate for $|f|$ from Lemma 2.6,

$$\int_{D\backslash r_0 D} |f|^{q-2}|f'|^2 N_{(\beta+1)^2+2}\, dA \le$$

$$CK\|f\|^{q-p}_{A^p_{(\alpha+1)^2}} \int_{D\backslash r_0 D} |f(w)|^{p-2}|f'(w)^2|(1-|w|)^{(p-q)\left((\alpha+1)^2+2\right)} dA^{((\alpha+1)^2+2)q/p}(w).$$

We may assume $r_0 \ge ½$, so that $\log(1/|w|) \le 2(1-|w|)$. Using this estimate in our upper bound above, we see that

$$\begin{aligned}\int_{D\backslash r_0 D} |f|^{q-2}|f'|^2 N_{(\beta+1)^2+2}\, dA &\le C\|f\|^{q-p}_{A^p_{(\alpha+1)^2}} \int_{D\backslash r_0 D} |f(w)|^{p-2}|f'(w)|^2\, dA_{((\alpha+1)^2+2)}(w)\\ &\le C\|f\|^q_{A^p_{(\alpha+1)^2}}\end{aligned}$$

which completes the proof.

Theorem3.2. Let $0 < q \le p$ and suppose $\boldsymbol{\varphi}$ is an analytic self-map of D satisfying.

$\mathbf{N}_{(\beta+1)^2+2}(w) = O([\log(1/|w|)]^{\eta})$ $(|w| \to 1)$,

for some

$$\eta > \frac{\left((\alpha+1)_2 + 1\right)q + p}{p}$$

Then

$C_\varphi : A^p_{(\alpha+1)^2} \to A^q_{(\beta+1)^2}$

is bounded.

Proof.

Let f $\in A^p_{(\alpha+1)^2}$. As in the proof of Theorem 3.1, it suffices to show that f $C_\varphi(f) \in A^q_{(\beta+1)^2}$ and for this it suffices to show there is $r \in (0,1)$ such that

$$\int_{D\backslash rD} |f|^{q-2}|f'|^2 N_{(\beta+1)_2+2}\, dA < \infty.$$

By our hypothesis on the growth of $N_{(\beta+1)^2+2}$, there is a constant $K < \infty$ and

$r_0 \in (0,1)$ such that

$N_{(\beta+1)^2+2}(w) \leq K(\log(1/|w|))^\eta$, $\quad w \in D\backslash r_0 D$

where

$$\eta > \frac{\left((\alpha+1)_2 + 1\right)q + p}{p}$$

Now, $(\alpha+1)^2 + 1 \geq 0$ and so $\eta > 1$. Thus, by Lemma 2.4, it suffices to show that

$$\int_{D\backslash r_0 D} |f(w)|^q (1-|w|)^{\eta-2} dA(w) < \infty.$$

By Hölder's inequality, this integral is bounded by

$$\left(\int_D [|f(w)|^q (1-|w|)^{(\alpha+1)_2\, q/p}]^{p/q}\right)^{q/p} \left(\int_D [(1-|w|)^{\eta-2-\frac{(\alpha+1)}{p}\frac{q}{2}}]^{\frac{p}{p-q}} dA(w)\right)^{(p-q)/p}$$

The first factor is bounded by $\|f\|^q_{A^p_{(\alpha+1)^2}}$ and so is finite, while the second factor is finite because the assumed lower bound for η is equivalent to the exponent in the integral being strictly greater than -1. Thus the proof is complete.

A necessary condition for $C_\varphi : A^p_{(\alpha+1)^2} \to A^q_{(\beta+1)^2}$ to be bounded

Lemma3.3. [5, Corollary 6,7]. Let ψ be an analytic self-map of D and let $\gamma \geq 1$. If $\psi(0)$ 0 and $0 < r < |\psi(0)|$, then

$$N_{\psi,\gamma}(0) \leq \frac{1}{r^2}\int_{rD} N_{\psi,\gamma}\, dA.$$

The next lemma shows how the counting functions transform under composition.

The case $\gamma = 1$ of this lemma can be found in [5].

Lemma3.4. Let ψ be an analytic self-map of D let $\alpha \in$ D and let

$$\sigma_a(w) \leq \frac{a-w}{1-\bar{a}w}$$

By the Möbius self-map of D that interchanges 0 and a. Then

$$(N_{\psi,\gamma}) \circ \sigma_a = N_{\sigma_a \circ \psi,\gamma}.$$

Theorem3.5. suppose that φ is an analytic self-map of D that induces a bounded composition operator $C_\varphi : A^p_{(\alpha+1)^2} \to A^q_{(\beta+1)^2}$.

Then

$\mathbf{N}_{(\beta+1)^2+2}$ (w) = O ((log (1/|w|))$^{((\alpha+1)\ 2\ +2)q/p}$) (|w| → 1).

Proof: By Proposition 2.5, there is a constant C_1 such that

$$C_1 \,||C_\varphi (k_a)||^q_{A^q_{(\beta+1)^2}} \ge \int_D |k_a(w)|^{q-2} |k'_a(w)|^2 \mathbf{N}_{(\beta+1)^2+2}(w)\, dA(w)$$

$$= \frac{4((\alpha+1)\ \ +2)^2}{p^{2^2}} |a|^2 (1-|a|^2)^{((\alpha+1)\ 2\ +2)q/p} \int_D \frac{\mathbf{N}_{(\beta+1)^2+2}(w)}{|1-\bar{a}w|^{2+2((\alpha+1)_2\ +2)q/p}}\, dA(w)$$

$$= \frac{4((\alpha+1)\ \ +2)^2}{p^{2^2}} |a|^2 (1-|a|^2)^{((\alpha+1)\ 2\ +2)q/p-2} \int_D \frac{\mathbf{N}_{(\beta+1)^2+2}(w)}{|1-\bar{a}w|^{2((\alpha+1)_2\ +2)q/p-2}} |\sigma'_a(w)|^2\, dA(w)$$

$$= \frac{4((\alpha+1)\ \ +2)^2}{p^{2^2}} |a|^2 (1-|a|^2)^{((\alpha+1)\ 2\ +2)q/p-2} \int_D \frac{\mathbf{N}_{(\beta+1)^2+2}(\sigma_a(z))}{|1-\bar{a}\sigma_a(z)|^{2((\alpha+1)_2\ +2)q/p-2}}\, dA(z)$$

Here $\sigma_a = \sigma_\alpha^{-1}$ is the Möbius self-map of D that interchanges 0 and a, as in Lemma 3.4 and the change of variable z = σ_a (w) was made in the last line. Now,

$$\frac{1}{|1-\bar{a}\sigma_a(z)|} = \frac{|1-\bar{a}z|}{1-|a|^2} \ge \frac{1}{2}\frac{1}{1-|a|^2}\ , |z| \le \frac{1}{2}$$

and so

$$C_1 \,||C_\varphi (k_a)||^q_{A^q_{(\beta+1)^2}} \ge \frac{4^{2-\left((\alpha+1)_2\ +2\right)q/p} ((\alpha+1)_2\ \ +2)^2 |a|^2}{p^2(1-|a|^2)^{((\alpha+1)_2\ +2)q/p}} \int_{\frac{1}{2}D} \mathbf{N}_{(\beta+1)^2+2}(\sigma_a(z))\, dA(z)\,.$$

We now apply first Lemma 3.4, then Lemma 3.3 and then Lemma 3.4 again to see, provided that σ_a o φ (0) > ½, the integral in the line above is at least

$\frac{1}{4}\ \mathbf{N}_{(\beta+1)^2+2}$, σ_a o φ (0) = $\frac{1}{4}\ \mathbf{N}_{(\beta+1)^2+2}$ (a).

Since σ_a o φ (0) > ½ if |a| is sufficiently close to 1, this provides the estimate that

$$\mathbf{N}_{(\beta+1)^2+2}(a) \le C_1 \,||C_\varphi (k_a)||^q_{A^q_{(\beta+1)^2}} \frac{4^{\left((\alpha+1)_2\ +2\right)q/p-1}\ p^2}{((\alpha+1)_2\ \ +2)^2|a|^2} (1-|a|^2)^{((1+\alpha)\ 2\ +2)q/p} \tag{5}$$

For all such a. Since $||k_a||_{A^p_{(\alpha+1)^2}} \approx 1$ and log (1/|a|) is comparable to (1-|a|^2) for ½ < |a| < 1, the assumption that C_φ is bounded provides the asserted bound for $\mathbf{N}_{(\beta+1)^2+2}$ and the proof is complete.

As noted at the beginning of this section, the necessary condition in Theorem 3.5 for C_φ to be pounded agrees with the sufficient condition from Theorem 3.1 that holds for q ≥ p, and so the following corollary results.

Corollary3.6. Let 0 < p ≤ q and let φ be an analytic self-map of D. Then

C_φ : $A^p_{(\alpha+1)^2} \to A^q_{(\beta+1)^2}$

is bounded if and only if

$\mathbf{N}_{(\beta+1)^2+2}$ (w) = O ((log (1/|w|))$^{((\alpha+1)\ 2\ +2)q/p}$) (|w| → 1).

Compatness of C_φ

A bounded linear operator T from a Branch space X to a Branch space Y is said to be compact provided the closure of T (B) is a compact subset of Y, where B is the unit ball of X. Equivalently, T is compact if and only if some subsequence of {T (x_n)} converges in Y, whenever {x_n} is a bounded sequence in X.

Our goal here is to characterize those analytic functions $\varphi : D \to D$ that induce bounded composition operators from one higher order weighted Bergman to another such spaces (see [2]and [4]).

Theorem4.1. Let $0 < p \le q$ and let φ be an an analytic self-map of D. Then

$C_\varphi : A^p_{(\alpha+1)^2} \to A^q_{(\beta+1)^2}$

is compact if and only if

$\mathbf{N}_{(\beta+1)^2+2}(w) = O\left((\log(1/|w|))^{((\alpha+1)^2+2)q/p}\right) \qquad (|w| \to 1).$

Proof. We first show that C_φ is compact, assuming that $\mathbf{N}_{(\beta+1)^2+2}$ satisfying the given growth condition. Let $||f_n||_{A^p_{(\alpha+1)^2}} \le 1$ for $n \ge 1$. We must show that $\{f_n \circ \varphi\}$ has a subsequence that converges in $A^q_{(\beta+1)^2}$. By Lemma 2.6, the f_n are uniformly bounded on compact subsets of D, and hence $\{ f_n \}$ is a normal family there. Thus there is a subsequence, which for simplicity we continue to denote by $\{ f_n \}$, that converges uniformly on compact subsets of D to an analytic function f. Now, for each $r \in (0,1)$,

$$\int_{rD} | f |^{p-2}|f'_n|^2 dA_{(\alpha+1)^2+2} = \lim_{n\to\infty} \int_{rD} | f_n|^{p-2}|f'_n|^2 dA_{(\alpha+1)^2+2}$$

$$\le \limsup_{n\to\infty} C||f_n||_{A^p_{(\alpha+1)^2}} \le C,$$

by Lemma 2.4. It follows that $f \in A^p_{(\alpha+1)^2}$, and so $f o\varphi \in A^q_{(\beta+1)^2}$ by Theorem 3.1. To complete the proof, it suffices to show $||f_n o\varphi - f o\varphi||_{A^q_{(\beta+1)^2}} \to \mathbf{0}$ as $n \to \infty$.

To establish this, note that from Proposition 2.5 it suffices to show that

$$| f_n (\varphi (0))- f (\varphi (0))|^p + \int_{rD} | f_n - f |^{q-2}| f'_n - f'|^2 \, \mathbf{N}_{(\beta+1)^2+2}\, dA$$
$$+ \int_{D\setminus rD} | f_n - f |^{q-2}|f'_n - f|^2 \, \mathbf{N}_{(\beta+1)^2+2} dA \qquad (6)$$

can be made arbitrarily small by choosing n large. For any fixed $r \in (0,1)$, the uniform convergence of f_n to f on compact subsets of D shows that the first two terms in the display above converge to 0 as $n \to \infty$. Thus it suffices to show that the third term in (6) tends to zero, uniformly of n, as $r \to 0$. To this end, let $\varepsilon > 0$, and note that by hypothesis we can choose $r \in (0,1)$ so that

$$\int_{D\setminus rD} | f_n - f |^{q-2}|f'_n - f'|^2 \, \mathbf{N}_{(\beta+1)^2+2} dA \le \varepsilon \int_{D\setminus rD} | f_n - f |^{q-2}|f'_n - f'|^2 \, dA_{((\alpha+1)^2+2)q/p} .$$

We are now in exactly the same situation that occurred at the end of the proof of Theorem 3.1. Without providing all the details from that proof, the bound for $|f_n - f|$ from Lemma 2.6 leads to the estimate

$$\int_{D\setminus rD} | f_n - f |^{q-2}|f'_n - f'|^2 \, \mathbf{N}_{(\beta+1)^2+2} dA \le \varepsilon C||f_n - f ||^q_{A^q_{(\beta+1)^2}}$$
$$\le \varepsilon C \left(||f||_{A^p_{(\alpha+1)^2}} +1\right)^q.$$

Since $\varepsilon > 0$ was arbitrary , C_φ is compact and the first part of the proof is complete.

We now finish the proof by assuming that C_φ is compact and proving that $\mathbf{N}_{(\beta+1)^2+2}$ satisfies the stated growth conditions. For a $\in$ D, let K_a be as defined in ξ 3,

$$k_a(w) = \frac{(1 - |a|^2)^{((\alpha+1)^2+2)/p}}{(1 - \bar{a}w)^{2((\alpha+1)^2+2)/p}}$$

and recall that $||k_a||_{A^p_{(\alpha+1)^2}} \approx 1$. Let $\{a_n\} \subset D$ satisfy $|a_n| \to 1$ as $n \to \infty$. From the definition of k_a, it is clear that this implies $k_{a_n}(z)$ converges uninformly to 0 on compact subsets of D as $n \to \infty$. Hence the zero element of $A^q_{(\beta+1)^2}$ is the only possible limit point of $\{ k_{a_n}(z) \circ \varphi \}$. The compactness of C_φ therefore yields that

$$\lim_{|a| \to 1} || C_\varphi(k_a)||_{A^q_{(\beta+1)^2}} = 0.$$

The required growth condition for $N_{(\beta+1)^2+2}$ is an immediate consequence of this and (5), the bound for $N_{(\beta+1)^2+2}$ that was derived in the proof of theorem 3.5, and the proof is complete.

Theorem4.2. Let $0 < q < p$ and suppose φ is an analytic self-map of D satisfying the conditions of Theorem 3.3. Then

$C_\varphi : A^p_{(\alpha+1)^2} \to A^q_{(\beta+1)^2}$ is compact.

Applications of Compactness on the higher order weighted Bergman spaces and Hardy spaces

We show the conditions that a composition operators $C\varphi$ is compact on the higher order weighted Bergman spaces and Hardy spaces of the open unit ball in C^n (see [18] and [19] .

Applications of Compactness on the higher order weighted Bergman spaces

Theorem 5.1. Suppose $p > 0$ and $(\alpha+1)^2+1 > 0$. If C_φ is bounded on $A^q_{(\beta+1)^2}$ for some $q > 0$ and $-1 < (\beta+1)^2 < (\alpha+1)^2$, then C_φ is compact on $A^p_{(\alpha+1)^2}$ if and only if

$$\lim_{|z| \to 1-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0 \tag{7}$$

Proof. According to Corollary 2.8, we may assume that p=2.

The normalized reproducing kernels of $A^2_{(\alpha+1)^2}$ are given by

$$k_z(w) = \frac{(1-|z|^2)^{\left((\alpha+1)^2+n+1\right)/2}}{|1-\langle w,z\rangle|^{(\alpha+1)^2+n+1}}.$$

Each k_z is a unit vector in $A^2_{(\alpha+1)^2}$ and it is clear that

$\lim_{|z| \to 1-} k_z(w) = 0 \qquad w \in B_n$

Furthermore, the coverage is uniform when w is restricted to any compact subset of B_n. A standard computation shows that

$$\int_{B_n} |C^*_\varphi k_z|^2 dv_{(\alpha+1)^2} = \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{(\alpha+1)^2+n+1}$$

So the compactness of C_φ on $A^2_{(\alpha+1)^2}$ (which is the same as the compactness of C^*_φ on $A^2_{(\alpha+1)^2}$) implies condition (7).

We proceed to show that condition (7) implies the compactness of C_φ on $A^2_{(\alpha+1)^2}$, provided that C_φ is bounded on $A^q_{(\beta+1)^2}$ for some $(\beta+1)^2 \in (-1, (\alpha+1)^2)$.

An easy computation shows that the operator

$$C_\varphi C^*_\varphi : A^2_{(\alpha+1)^2} \to A^2_{(\alpha+1)^2}$$

admits the following integral representation:

$$C_\varphi C_\varphi^* f(z) = \int_{B_n} \frac{f(w)dv_{(\alpha+1)^2}(w)}{(1-\langle\varphi(z),\varphi(w)\rangle)^{(\alpha+1)^2+n+1}},\ f \in A^2_{(\alpha+1)^2} \qquad (8)$$

We will actually prove the compactness of $C_\varphi C_\varphi^*$ on $A^2_{(\alpha+1)^2}$, which is equivalent to the compactness of C_φ on $A^2_{(\alpha+1)^2}$. In fact, our arguments will prove the compactness of the following integral operator on L^2 (B_n, $dv_{(\alpha+1)^2}$):

$$Tf(z) = \int_{B_n} \frac{f(w)dv_{(\alpha+1)^2}(w)}{(1-\langle\varphi(z),\varphi(w)\rangle)^{(\alpha+1)^2+n+1}}. \qquad (9)$$

For any $r \in (0,1)$ let X_r denote the characteristic function of the set $\{z \in \mathbb{C}^n: r < |z| < 1\}$. Consider the following integral operator on L^2 (B_n, $dv_{(\alpha+1)^2}$):

$$T_r f(z) = \int_{B_n} H_r(z,w) f(w) dv_{(\alpha+1)^2}(w), \qquad (10)$$

where

$$H_r(z,w) = \frac{\chi_r(z)\chi_r(w)}{(1-\langle\varphi(z),\varphi(w)\rangle)^{(\alpha+1)^2+n+1}}$$

is a nonnegative integral kernel. We are going to estimate the norm of T_r on L^2 (B_n, $dv_{(\alpha+1)^2}$) in terms of the quantity.

$$M_r = \sup_{r<|z|<1} \frac{1-|z|^2}{1-|\varphi(z)|^2}.$$

We do this with the help of Schur's test.

Let $(\alpha+1)^2 = (\beta+1)^2 + \sigma$, where $\sigma > 0$, and consider the function

$$h(z) = (1-|z|^2)^{-\sigma}, \quad z \in B_n$$

We have

$$\int_{B_n} H_r(z,w)h(w)dv_{(\alpha+1)^2}(w)$$

$$= \frac{C_{(\alpha+1)^2}}{C_{(\beta+1)^2}} \int_{B_n} \frac{\chi_r(z)\chi_r(w)dv_{(\beta+1)^2}(w)}{(1-\langle\varphi(z),\varphi(w)\rangle)^{(\beta+1)^2+n+\sigma+1}} \le \frac{C_{(\alpha+1)^2}}{C_{(\beta+1)^2}} \int_{B_n} \frac{\chi_r(z)dv_{(\beta+1)^2}(w)}{(1-\langle\varphi(z),\varphi(w)\rangle)^{(\beta+1)^2+n+\sigma+1}}$$

By the boundedness of C_φ on $A^q_{(\beta+1)^2}$, there exists a constant $C_1 > 0$, independent of r and z , such that

$$\int_{B_n} H_r(z,w)h(w)dv_{(\alpha+1)^2}(w) \le C_1\chi_r(z) \int_{B_n} \frac{dv_{(\beta+1)^2}(w)}{|1-\langle\varphi(z),w\rangle|^{(\beta+1)^2+n+\sigma+1}}$$

We apply Lemma 2.10 to find another positive constant C_2 , independent of r and z, such that

$$\int_{B_n} H_r(z,w)h(w)dv_{(\alpha+1)^2}(w) \le \frac{C_2\chi_r(z)}{(1-|\varphi(z)|^2)^\sigma}$$

$$= C_2\chi_r(z)\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\sigma h(z)$$

$$\le C_2 M_r^\sigma h(z)$$

For all $z \in B_n$. By the symmetry of $H_r(z,w)$, we also have

$$\int_{B_n} H_r(z,w)h(z)dv_{(\alpha+1)^2}(z) \le C_2 M_r^\sigma h(w)$$

for all $w \in B_n$. It follows from Lemma 2.9 that the operator T_r is bounded on L^2 $(B_n, dv_{(\alpha+1)^2})$ and the norm of T_r on L^2 $(B_n, dv_{(\alpha+1)^2})$ does not exceed the constant $C_2 M_r^{\sigma}$.

Now fix some $r \in (0,1)$ and fix a bounded sequence $\{f_k\}$ in $A^2_{(\alpha+1)^2}$ that converges to 0 uniformly on every compact subset of B_n. In particular, $\{f_k\}$ converges uniformly to 0 on $|z| \le r$. We use (8) to write

$C_\varphi C_\varphi^* f_k(z) = F_k(z) + G_k(z), \quad z \in B_n,$

where

$$F_k(z) = \int_{|w|\le r} \frac{f_k(w)\,dv_{(\alpha+1)^2}(w)}{(1-\langle \varphi(z), \varphi(w)\rangle)^{(\alpha+1)_2 + n+1}},$$

and

$$G_k(z) = \int_{B_n} \frac{\chi_r(w) f_k(w)\,dv_{(\alpha+1)^2}(w)}{(1-\langle \varphi(z), \varphi(w)\rangle)^{(\alpha+1)_2 + n+1}}$$

Since $\{ f_k(w) \}$ converges to 0 uniformly for $|w| \le r$, we have

$$\lim_{k\to\infty} \int_{B_n} |f_k((z)|^2 dv_{(\alpha+1)^2}(z) = 0$$

For any fixed $z \in B_n$, the weak convergence of $\{f_k\}$ to 0 in L^2 $(B_n, dv_{(\alpha+1)^2})$ implies that $G_k(z) \to 0$ as $k \to \infty$. In fact, by splitting the ball into $|z| \le \delta$ and

$\delta < |z| < 1$, it is easy to show that

$$\lim_{k\to\infty} G_k(z) = 0$$

uniformly for z in any compact subset of B_n.

It follows from the definition of T_r that

$$\int_{B_n} |G_k|^2 dv_{(\alpha+1)^2} \le \int_{|z|\le r} |G_k|^2 dv_{(\alpha+1)^2} + \int_{B_n} |T_r(|f_k|)|^2 dv_{(\alpha+1)^2}.$$

Since $\{f_k\}$ is bounded in L^2 $(B_n, dv_{(\alpha+1)^2})$, and since the norm of the operator T_r on L^2 $(B_n, dv_{(\alpha+1)^2})$ does not exceed $\mathbf{C_2 M_r^{\sigma}}$ we can find a constant $C_3 > 0$, independent of r and k, such that

$$\int_{B_n} |T_r(|f_k|)|^2 dv_{(\alpha+1)^2} \le C_3 M_r^{2\sigma}$$

for all k. Combining this with

$$\lim_{k\to\infty} \int_{|z|\le r} |G_k|^2 dv_{(\alpha+1)^2}(z) = 0,$$

we obtain

$$\limsup_{k\to\infty} \int_{B_n} |G_k|^2 dv_{(\alpha+1)^2} \le C_3 M_r^{2\sigma}.$$

This along with the estimates for F_k in the previous paragraph gives

$$\limsup_{k\to\infty} \int_{B_n} |C_\varphi C_\varphi^* f_k|^2 dv_{(\alpha+1)^2} \le C_3 M_r^{2\sigma}$$

Since r is arbitrary and $M_r \to 0$ as $r \to 1-$ (which is equivalent to the condition in (7)), we conclude that

$$\lim_{k\to\infty} |C_\varphi C_\varphi^* f_k|^2 \le dv_{(\alpha+1)^2} = 0.$$

So C_φ is compact on $A^2_{(\alpha+1)^2}$, and the proof of the theorem is complete.

Applications of Compactness on the Hardy spaces

Theorem5.2. Suppose $p > 0$. If C_φ is bounded on H^q for some $q > 0$, then C_φ is compact on H^p if and only if $\lim_{|z|\to 1-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0$

Proof. According to Corollary 2.8, we may assume that p=2.

The normalized reproducing kernels of H^2 are given by

$$k_z(\theta) = \frac{(1-|re^{i\theta}|^2)^{(n+1)/2}}{|1-\langle re^{i\theta}, z\rangle|^{n+1}}.$$

Each k_z is a unit vector in H^2 and it is clear that

$$\lim_{|z|\to} k_z\left(re^{i\theta}\right) = 0 \qquad w \in B_n$$

Furthermore, the coverage is uniform when $\boldsymbol{\theta}$ is restricted to any compact subset of B_n. A standard computation shows that

$$\frac{1}{2\pi}\int_{B_n} |C_\varphi^* k_z|^2 d\theta = \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{n+1}$$

So the compactness of C_φ on H^2 (which is the same as the compactness of C_φ^* on H^2) implies condition (7).

We proceed to show that condition (7) implies the compactness of C_φ on H^2 , provided that C_φ is bounded on H^q . An easy computation shows that the operator

$$C_\varphi C_\varphi^* : H^2 \to H^2$$

admits the following integral representation:

$$C_\varphi C_\varphi^* f(z) = \frac{1}{2\pi}\int_{B_n} \frac{f\left(re^{i\theta}\right)d\theta}{(1-\langle\varphi(z),\varphi(re^{i\theta})\rangle)^{n+1}}, \quad f \in H^2 . \tag{11}$$

We will actually prove the compactness of $C_\varphi C_\varphi^*$ on H^2 , which is equivalent to the compactness of C_φ on H^2. In fact, our arguments will prove the compactness of the following integral operator on L^2 (B_n, $\mathbf{d\theta}$):

$$Tf(z) = \frac{1}{2\pi}\int_{B_n} \frac{f\left(re^{i\theta}\right)d\theta}{(1-\langle\varphi(z),\varphi(re^{i\theta})\rangle)^{n+1}} . \tag{12}$$

For any $r \in (0,1)$ let X_r denote the characteristic function of the set $\{z \in \mathbb{C}^n: r < |z| < 1\}$. Consider the following integral operator on L^2 (B_n, $\mathbf{d\theta}$):

$$T_r f(z) = \frac{1}{2\pi}\int_{B_n} H_r\left(z, re^{i\theta}\right) f\left(re^{i\theta}\right) d\theta, \tag{13}$$

where

$$H_r\left(z, re^{i\theta}\right) = \frac{\chi_r(z)\chi_r(re^{i\theta})}{(1-\langle\varphi(z),\varphi(re^{i\theta})\rangle)^{n+1}}$$

is a nonnegative integral kernel. We are going to estimate the norm of T_r on L^2 (B_n, $\mathbf{d\theta}$) in terms of the quantity.

$$M_r = \sup_{r<|z|<1} \frac{1-|z|^2}{1-|\varphi(z)|^2}.$$

We do this with the help of Schur's test.

Let $\sigma > 0$, and consider the function

$$h(z) = (1 - |z|^2)^{-\sigma}, \quad z \in B_n$$

We have

$$\int_{B_n} H_r\left(z, re^{i\theta}\right) h\left(re^{i\theta}\right) \mathrm{d\theta} \leq C \int_{B_n} \frac{\chi_r(z)\chi_r\left(re^{i\theta}\right)\mathrm{d\theta}}{(1-\langle\varphi(z),\varphi(re^{i\theta})\rangle)^{n+\sigma+1}}$$

$$\leq C \int_{B_n} \frac{\chi_r(z)\mathrm{d\theta}}{(1-\langle\varphi(z),\varphi(re^{i\theta})\rangle)^{n+\sigma+1}}$$

By the boundedness of C_φ on H^q, there exists a constant $C_1 > 0$, independent of r and z, such that

$$\int_{B_n} H_r\left(z, re^{i\theta}\right) h\left(re^{i\theta}\right) \mathrm{d\theta} \leq C_1\chi_r(z) \int_{B_n} \frac{\mathrm{d\theta}}{|1-\langle\varphi(z), re^{i\theta}\rangle|^{n+\sigma+1}}$$

We apply Lemma 2.10 to find another positive constant C_2, independent of r and z, such that

$$\frac{1}{2\pi}\int_{B_n} H_r\left(z, re^{i\theta}\right) h\left(re^{i\theta}\right) \mathrm{d\theta} \leq \frac{C_2\chi_r(z)}{(1-|\varphi(z)|^2)^\sigma}$$

$$= C_2\chi_r(z)\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\sigma h(z)$$

$$\leq C_2 M_r^\sigma h(z)$$

For all $z \in B_n$. By the symmetry of $H_r(z, re^{i\theta})$, we also have

$$\frac{1}{2\pi}\int_{B_n} H_r\left(z, re^{i\theta}\right) h(z) \mathrm{d\theta} \leq C_2 M_r^\sigma h(re^{i\theta})$$

for all $re^{i\theta} \in B_n$. It follows from Lemma 2.9 that the operator T_r is bounded on L^2 (B_n, $\mathbf{d\theta}$) and the norm of T_r on L^2 (B_n, $\mathbf{d\theta}$) does not exceed the constant $C_2 M_r^\sigma$.

Now fix some $r \in (0,1)$ and fix a bounded sequence $\{f_k\}$ in H^2 that converges to 0 uniformly on every compact subset of B_n. In particular, $\{f_k\}$ converges uniformly to 0 on $|z| \leq r$. We use (11) to write

$$C_\varphi C_\varphi^* f_k(z) = F_k(z) + G_k(z), \quad z \in B_n,$$

where

$$F_k(z) = \frac{1}{2\pi}\int_{|e^{i\theta}|\leq 1} \frac{f_k\left(re^{i\theta}\right)\mathrm{d\theta}}{(1-\langle\varphi(z),\varphi(re^{i\theta})\rangle)^{n+1}},$$

and

$$G_k(z) = \frac{1}{2\pi}\int_{B_n} \frac{\chi_r\left(re^{i\theta}\right) f_k(re^{i\theta})\mathrm{d\theta}}{(1-\langle\varphi(z),\varphi(re^{i\theta})\rangle)^{n+1}}$$

Since { $f_k(re^{i\theta})$ } converges to 0 uniformly for | $e^{i\theta}$ | ≤ 1 , we have

$$\lim_{k\to\infty} \frac{1}{2\pi}\int_{B_n} |f_k((z)|^2 d\theta = 0$$

For any fixed $z \in B_n$, the weak convergence of $\{f_k\}$ to 0 in $L^2(B_n, d\theta)$ implies that $G_k(z) \to 0$ as $k \to \infty$. In fact, by splitting the ball into $|z| \le \delta$ and

$\delta < |z| < 1$, it is easy to show that

$$\lim_{k\to\infty} G_k(z) = 0$$

uniformly for z in any compact subset of B_n.

It follows from the definition of T_r that

$$\int_{B_n} |G_k|^2 d\theta \le \int_{|z|\le r} |G_k|^2 d\theta + \int_{B_n} |T_r(|f_k|)|^2 d\theta.$$

Since $\{f_k\}$ is bounded in $L^2(B_n, d\theta)$, and since the norm of the operator T_r on $L^2(B_n, d\theta)$ does not exceed $C_2 M_r^{\sigma}$ we can find a constant $C_3 > 0$, independent of r and k, such that

$$\frac{1}{2\pi}\int_{B_n} |T_r(|f_k|)|^2 d\theta \le C_3 M_r^{2\sigma}$$

for all k. Combining this with

$$\lim_{k\to\infty} \frac{1}{2\pi}\int_{|z|\le r} |G_k|^2 d\theta = 0,$$

we obtain

$$\limsup_{k\to\infty} \frac{1}{2\pi}\int_{B_n} |G_k|^2 d\theta \le C_3 M_r^{2\sigma}.$$

This along with the estimates for F_k in the previous paragraph gives

$$\limsup_{k\to\infty} \frac{1}{2\pi}\int_{B_n} |C_\varphi C_\varphi^* f_k|^2 d\theta \le C_3 M_r^{2\sigma}$$

Since r is arbitrary and $M_r \to 0$ as $r \to 1-$ (which is equivalent to the condition in (7)), we conclude that

$$\lim_{k\to\infty} \frac{1}{2\pi}\int_{B_n} |C_\varphi C_\varphi^* f_k|^2 d\theta = 0.$$

So C_φ is compact on H^2, and the proof of the theorem is complete.

Acknowledgment:

The authors would like to express their sincere thanks to Taif university, Saudi Arabia for supported of this work..

References

[1]Littlewood, J.E., On Intequalities in the theory of functions, Proc, London Math. Soc, **23** (1925),, 481-519.

[2] Wayne Smith.,Composition operators between Bergman and Hardy spaces, Trans. Amer. Math. Soc. 348 (1996), 2331-2348.

[3] Duren, PL., Theory of Hp spaces. Academic Press, New York, 1970.

[4] Jordi Pau and Patricio A. Pérez, Composition operators acting on weighted Dirichlet spaces, J. Math. Anal. Appl. 401 (2013) 682–694

[5]Shapiro, J,H., The essential norm of a composition operator, Annals of Math. 127 (1987), 375-404.

[6] Alexey I. Popov and Adi Tcaciuc, Every operator has almost invariant subspaces, Journal of Functional Analysis 265 (2013) 257–265

[7]Shapiro, J.H., Composition Operators and Classical Functions Theory, Springer-Verlag, New York, 1993.

[8]C. Cowen and B. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1994.

[9]B. MacCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canadian J. Math. **38** (1986), 878-906.

[10] Ajay K. Sharma and Sei-ichiro Uek, Composition operators between weighted Bergman spaces with admissible Bekolle weights, Banach J. Math. Anal. Volume 8, Number 1 (2014), 1-334.

[11] Huiying Qu, Yongmin Liu, and Shulei Cheng, Weighted Differentiation Composition Operator from Logarithmic Bloch Spaces to Zygmund-Type paces, olume 2014 (2014), Article ID 832713, 14 pages

[12] Jasbir Singh Manhas and Ruhan Zhao, New estimates of essential norms of weighted composition operators between Bloch type spaces, J. Math. Anal. Appl. 389 (2012) 32–47

[13] D. Borgohain and S. Naik, Weighted Fractional Differentiation Composition Operators from Mixed-Norm Spaces to Weighted-Type Spaces, International ournal of Analysis Volume 2014 (2014), Article ID 301709, 9 pages

[14] K. Zhu, Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, J. Operator Theory 20 (1988), 329-357.

[15] Shawgy Hussein and El Hadi Elnour Elniel, Relations on Weighted Quadratic Factor of Hardy and Bergman Spaces with a Sharp Estimate, Australian Journal of Basic and Applied Sciences, 5(10)(2011) , pp. 1122-1129.

[16] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York, 1990.

[17] K. Zhu, Spaces of Holomorphic in the Unit Ball, Springer, New York, 2004.

[18] Benoît Sehba, Stevo Stevic: On some product-type operators from Hardy- Orlicz and Bergman-Orlicz spaces to weighted-type spaces, .Applied Mathematics and Computation 233: 565-581 (2014).

[19] K.Zhu , compact composition operators on Bergman Spaces of the unit ball, Houston Journal of Mathematics, 2007, pp. 1-11.